A Proposition of a Universal Coefficient of Growth of a Turbulent, Curved Wall-Jet

by

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Geometrical consideration of a space curve and fluidkinematical consideration of a curved wall jet with a similar velocity profile led to a conclusion that a coefficient of growth of a jet is a function of a ratio of width of a jet to a radius of curvature of trajectory of a jet. The effect of the past history of a jet seems to be negligible. A few experimental data on different geometries confirm the theory justifiable. Thus we propose a universal coefficient of growth of a curved wall jet.

1. Introduction

In case of calculating the characteristics of flow with tangential injection of air over a curved wall, it is in general necessary to predict the growth of a jet along the wall.

The prediction is made using its own experimental data which are determined for specific geometries such as a circular cylinder, logarithmic spiral, straight wall and so on. The data of a circular cylinder for instance is, however, applicable only to the flow over the wall with a similar geometry. This inconvenience seems to come mainly from facts that firstly the coordinate systems to express the flow are different from each other, just convenient to a specific problem, secondly it is not obviously known which one among many variables is essentially of vital importance and thirdly it is difficult to experimentally obtain a jet with an arbitrary trajectory.

However the first two problems are readily solved when the concept of natural equation in the differential geometry is applied to the problems.

The first problem is replaced with the one of finding a geometrical relation of trajectory of a jet with the wall. Considering the second problem geometrically, we deduced a concept of a universal coefficient of growth of a wall jet under the assumptions of a small ratio of a width of jet to a radius of curvature of the wall and a similar velocity profile.

This coefficient is a function of a ratio of a width of jet to a radius of curvature of a representative trajectory of jet, $y_{\frac{m}{2}}/\rho$, which was confirmed true by a few experimental data. The mixing length theory in a centrifugal field by Stratford, Jawor and Golesworthy seems to give an insight into the quantitative representation of that coefficient. It is desired to confirm our theory by a systematic experiment and find out a more reliable quantitative relation of the coefficient to the ratio $y_{\frac{m}{2}}/\rho$.

2. Theory

The fundamental theorem of the differential geometry requires only a curvature, torsion and arc length of a space curve to express itself, in which we need only two variables, i.e., curvature and arc length when the space is two dimensional.

Then the natural equation of a curve is given as follows,

$$\rho = \rho\ (s) \tag{1}$$

where ρ and s are a radius of curvature and arc length respectively.

By definition,

$$\frac{1}{\rho} = \frac{d\sigma}{ds} \tag{2}$$

where σ is an angle between two tangents at two points apart by ds on a curve. Thus we may use σ instead of ρ as an independent variable.

Therefore the total differential of the width of a jet $y_{\frac{m}{2}}$ is written as follows,

$$dy_{\frac{m}{2}} = \frac{\partial y_{\frac{m}{2}}}{\partial \sigma} d\sigma + \frac{\partial y_{\frac{m}{2}}}{\partial s} ds \tag{3}$$

where s is a trajectory of $y_{\frac{m}{2}}$.

From equations (2) and (3) we have

$$dy_{\frac{m}{2}} = \left(\frac{1}{\rho} \frac{\partial y_{\frac{m}{2}}}{\partial \sigma} + \frac{\partial y_{\frac{m}{2}}}{\partial s} \right) ds \tag{4}$$

The first term in the parentheses is an increment of width of a jet $y_{\frac{m}{2}}$ per unit variation of tangential direction multiplied by a curvature, the second term being an increment of $y_{\frac{m}{2}}$ per unit arc length.

Denoting the quantity in the parentheses by $C_1\ (s, \sigma, y_{\frac{m}{2}})$ such that

$$C_1\ (s, \sigma, y_{\frac{m}{2}}) = \frac{1}{\rho} \frac{\partial y_{\frac{m}{2}}}{\partial \sigma} + \frac{\partial y_{\frac{m}{2}}}{\partial s} \tag{4a}$$

equation (4) is reduced to

$$dy_{\frac{m}{2}} = C_1\ (s, \sigma, y_{\frac{m}{2}})\ ds \tag{4b}$$

where $C_1\ (s, \sigma, y_{\frac{m}{2}})$ is a function of σ, s and $y_{\frac{m}{2}}$ or a radius of curvature ρ, arc length s and $y_{\frac{m}{2}}$ only. The $C_1\ (s, \sigma, y_{\frac{m}{2}})$ in the form of equation (4a) seems to be convenient to experimentally find out the separate effect of each variable on $y_{\frac{m}{2}}$.

If the quantity $C_1(s, \sigma, y_{\frac{m}{2}})$ is, however, to be handled as a coefficient in practical sense like a practical calculation of growth of a jet, then the trajectory of $y_{\frac{m}{2}}$, i.e., s in C_1 $(s, \sigma, y_{\frac{m}{2}})$ will be a draw-back of this theory as the relation of s to a wall surface l is in general not simply expressed.

So we use a variable ρ instead of σ, then we have $C_1(s, \rho, y_{\frac{m}{2}})$ for $C_1(s, \sigma, y_{\frac{m}{2}})$.

If the velocity profile is similar, the ratio between characteristic linear dimensions must be constant so that $s_1/\rho_1 = s_2/\rho_2$, $y_{\frac{m}{2}1}/\rho_1 = y_{\frac{m}{2}2}/\rho_2$ and so on when two trajectories s_1 and s_2 and two jets $y_{\frac{m}{2}1}$ and $y_{\frac{m}{2}2}$ are geometrically similar. Therefore C_1 $(s/\rho, y_{\frac{m}{2}}/\rho)$ must be constant for constant s/ρ and constant $y_{\frac{m}{2}}/\rho$. s/ρ in C_1 $(s/\rho, y_{\frac{m}{2}}/\rho)$ implies the history which $y_{\frac{m}{2}}$ has passed up to the point of interest.

Assuming $y_{\frac{m}{2}}/\rho$ to be small, l/ρ_l can approximate s/ρ where l and ρ_l are an arc length and radius of curvature of the wall surface respectively. What this means is that the history of a wall jet is calculable if $y_{\frac{m}{2}}/\rho$ is small. Furthermore, if $y_{\frac{m}{2}}/\rho$'s in two different jets are equal each other at a point of interest, $C_1(s/\rho, y_{\frac{m}{2}}/\rho)$'s too seem equal each other at the point no matter what kind of path two flows came up to that point through. This is seen in Fig. 1.

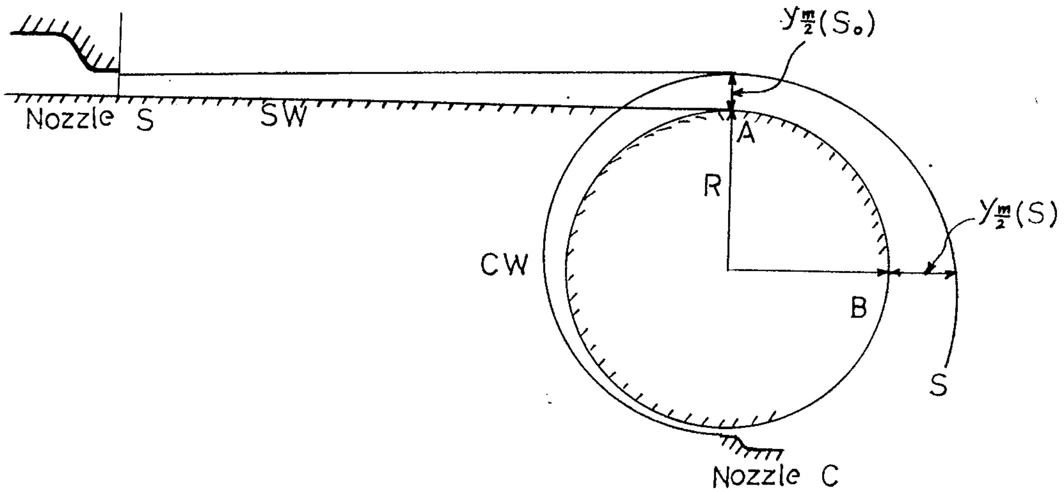

Fig. 1

There are shown two wall jets, SW along a straight wall to point A and CW over a curved wall in the figure. Both jets have walls with equal curvature after the point A. If both jets have equal $y_{\frac{m}{2}}(s_0)$'s at point A, then $y_{\frac{m}{2}}(s)$'s at point B in both jets are supposed to be equal as far as velocity profiles at point A are similar in both flows, ignoring a discontinuous change of curvature at point A in the jet SW. This is due to a similar profile assumption.

The above discussion means that $C_1(s/\rho, y_{\frac{m}{2}}/\rho)$ is a local quantity and a function of a variable $y_{\frac{m}{2}}/\rho$ only.

Thus we have instead of equation (4b)

$$dy_{\frac{m}{2}} = C_1 \, (y_{\frac{m}{2}}/\rho) \, ds \tag{4c}$$

where $y_{\frac{m}{2}}/\rho$ in the parentheses means C_1 is a function of $y_{\frac{m}{2}}/\rho$.

Stratford et al(1) showed the ratio, $\overline{L}$, of mixing length in a jet of a centrifugal field to that in a straight wall jet in still air was approximately related to b/ρ such that

$$\overline{L} = \frac{1}{1 - \frac{2}{3}\frac{b}{\rho}} \tag{5}$$

where b is a width of half jet. If $y_{\frac{m}{2}}$ is a distance, perpendicular to the wall, measured from the wall to the point at which the velocity is half the maximum velocity of jet, then we have $b \simeq 2\, y_{\frac{m}{2}}$ and

$$\overline{L} = \frac{1}{1 - \frac{4}{3}\frac{y_{\frac{m}{2}}}{\rho}} \tag{6}$$

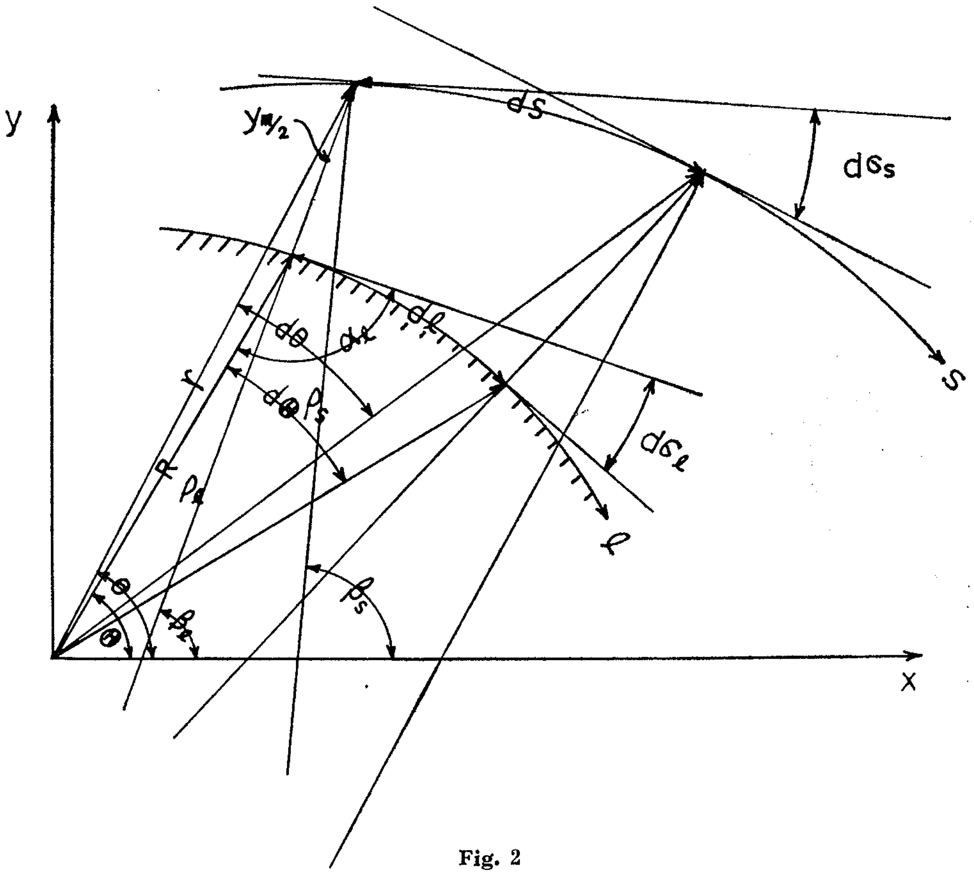

Fig. 2

It is known from the theory of jet that a width of jet is proportinal to the mixing length.

Therefore

$$C_1\left(\frac{y_{\frac{m}{2}}}{\rho}\right) = C\bar{L} = \frac{C}{1-\frac{4}{3}\frac{y_{\frac{m}{2}}}{\rho}} \tag{7}$$

where C is a constant of a straight jet.

Now the remaining problem is to relate a curve s to a certain fixed curve, of course in this case, to a wall l.

All the symbols necessary are shown in Fig. 2, in which (r, θ) and (R, Θ) are cylindrical coordinates of the curves s and l respectively. The origin of them must be appropriately fixed somewhere on the body, for instance to a center of a circle in case of a circular cylinder. β is an angle of the radius of curvature measured from x axis. The symbols with the suffixes s and l mean to be those of the curves s and l respectively. Since we have

$$ds = (r^2 + r'^2)^{1/2}\, d\theta \tag{8}$$

$$\rho_s = \frac{(r^2 + r'^2)^{3/2}}{|\, r^2 + 2r'^2 - rr''\, |} \tag{9}$$

$$\beta_l = \Theta - \tan^{-1}\frac{R'}{R} \tag{10}$$

$$r\cos\theta = R\cos\Theta + y_{\frac{m}{2}}\cos\beta_l \tag{11}$$

$$r\sin\theta = R\sin\Theta + y_{\frac{m}{2}}\sin\beta_l \tag{12}$$

$$\frac{dr}{d\theta} = \left(\frac{dr}{d\Theta}\right) \Big/ \left(\frac{d\theta}{d\Theta}\right) \tag{13}$$

where the primes on r and R indicate the derivatives of them with respect to θ and Θ respectively, we can express ds in equation (4c) by R, Θ and $y_{\frac{m}{2}}$.

We will not make an involved calculation for a general case, but will deal with the cases of a logarithmic spiral, circular cylinder and a straight wall. Comparing the results with the data by the other researchers, we will show our theory justifiable.

3. Logarithmic Spiral Wall

A logarithmic spiral wall is described by the following equation,

$$R = ke^{a\Theta} \tag{14}$$

where k and a are constants.

Using the relations (8) and (9) along with (14), we get

$$\rho_l = k(1 + a^2)^{\frac{1}{2}}\, e^{a\Theta} \tag{15}$$

$$l = \frac{k}{a} (1 + a^2)^{\frac{1}{2}} e^{a\Theta} \tag{16}$$

and

$$\frac{l}{\rho_l} = \frac{R}{R'} = \frac{1}{a} = K = constant \tag{17}$$

for a given K where K is a constant used by Kamemoto in his report.(3)

Since $y_{\frac{m}{2}}$ is, as the first approximation, proportional to l , $C_1 (y_{\frac{m}{2}}/\rho)$ in the form of equation (7) is written as follows,

$$C_1 \left(\frac{y_{\frac{m}{2}}}{\rho_l} \right) = \frac{C}{1 - \frac{4}{3} \frac{C'l}{\rho_l}} \tag{18}$$

where C' is a proportional constant.

We empirically know that $y_{\frac{m}{2}}/\rho$ is always non-zero and is a constant from the origin of the jet on for a given K. Therefore C', the proportional constant of $y_{\frac{m}{2}}$, is always a constant in a constant centrifugal field and has never been equal to C from the beginning of a jet for a given K, i.e., $C'>C$.

Equation (4c) is rewritten below by using equation (18),

$$d\, y_{\frac{m}{2}} = \frac{C}{1 - \frac{4}{3} C' \frac{l}{\rho_l}} ds \tag{19}$$

From equations (10), (11) and (12), we have

$$r^2 = R^2 + y_{\frac{m}{2}}^2 + 2 R y_{\frac{m}{2}} \cos \alpha_k \tag{20}$$

where

$$\alpha_k = \tan^{-1} \frac{1}{K} = \tan^{-1} \frac{R'}{R} \tag{21}$$

Experience for a logarithmic spiral wall indicates

$$\frac{y_{\frac{m}{2}}}{\rho_l} = K \frac{y_{\frac{m}{2}}}{l} = C_2 = constant \tag{22}$$

for a given K where $y_{\frac{m}{2}}/l$ is equal to C' in equation (18).

Introducing this relation and equations (14) and (15) into (20) gives the following equation for r,

$$r = k\, C_4 (K)\, e^{a\Theta} \tag{23}$$

where

$$C_4 (K) = \left\{ 1 + 2 C_2 \sqrt{1 + a^2} \cos \alpha_k + C_2^2 (1 + a^2) \right\}^{\frac{1}{2}} \tag{24}$$

From Fig. 2, we obtain

$$y_{\frac{m}{2}}^2 = r^2 + R^2 - 2rR \cos (\theta - \Theta) \tag{25}$$

therefore

$$\cos(\theta - \Theta) = \frac{1 + C_2 \sqrt{1+a^2} \cos \alpha_k}{C_4(K)} = \text{constant} \qquad (26)$$

for given K.

Consequently that $d\theta = d\Theta$ gives

$$r' = \frac{dr}{d\theta} = \frac{dr}{d\Theta} \qquad (27).$$

So equation (23) yields

$$r' = ka\, C_4(K)\, e^{a\Theta} \qquad (28).$$

Eliminating r and r' from equation (8), we have

$$ds = k\sqrt{1+a^2}\, C_4(K)\, e^{a\Theta}\, d\Theta \qquad (29).$$

Inasmuch as

$$dl = k\sqrt{1+a^2}\, e^{a\Theta}\, d\Theta \qquad (30)$$

we have

$$ds = C_4(K)\, dl \qquad (31).$$

Then equation (19) is written as follows,

$$\frac{dy_{\frac{m}{2}}}{dl} = \frac{C}{1 - \frac{4}{3} C' K} C_4(K) \qquad (32).$$

Picking values of K and $y_{\frac{m}{2}}/l$ off the figure 7 of the reference (1), we can make the Table I.

Table I

K	y/l	$C_4(K)$	$dy_{\frac{m}{2}}/dl$
0.50	0.155	1.09	0.155
0.75	0.20	1.17	0.187
1.0	0.29	1.32	0.276

$C = 0.128$ was used to make Table I. $y_{\frac{m}{2}}/l$ is value from the reference (3), $dy_{\frac{m}{2}} dl$ being by equation (32).

4. Circular Cylinder and Straight Wall.

Equation (4c) and (8) yield

$$\frac{dy_{\frac{m}{2}}}{d\theta} = C_1 \left(\frac{y_{\frac{m}{2}}}{\rho_s} \right) \left(r^2 + r'^2 \right)^{\frac{1}{2}} \qquad (33).$$

Introducing the relations for a circular cylinder

$$\left. \begin{array}{l} r = R + y_{\frac{m}{2}} \\ \theta = \Theta \end{array} \right\} \qquad (34)$$

into equation (33), we have

$$\frac{dy_{\frac{m}{2}}}{d\Theta} = C_1\left(\frac{y_{\frac{m}{2}}}{\rho_s}\right) R\left(1+\frac{y_{\frac{m}{2}}}{R}\right)\left\{1+\frac{\left(dy_{\frac{m}{2}}/d\Theta\right)^2}{R^2\left(1+y_{\frac{m}{2}}/R\right)^2}\right\}^{\frac{1}{2}} \quad (35).$$

Making the same approximation as in the boundary layer theory, i. e., letting $Rd\Theta \gg dy_{\frac{m}{2}}$ and assuming $y_{\frac{m}{2}}/R \ll 1$, we can ignore the second term in the braces of the above equation and we can approximate $y_{\frac{m}{2}}/\rho_s$ to $y_{\frac{m}{2}}/R$. Then we have

$$\frac{dy_{\frac{m}{2}}}{d\Theta} = C_1\left(\frac{y_{\frac{m}{2}}}{R}\right) R\left(1+\frac{y_{\frac{m}{2}}}{R}\right) \quad (36)$$

where $C_1\,(y_{\frac{m}{2}}/R)$ is given by equation (7), that is,

$$C_1\left(\frac{y_{\frac{m}{2}}}{R}\right) = \frac{C}{1-\frac{4}{3}\frac{y_{\frac{m}{2}}}{R}}, \quad (37)$$

$y_{\frac{m}{2}}$ on a circular cylinder is, as the first approximation, proportional to the arc length of the surface $R\,\Theta$.

In this case, different from the case of a logarithmic spiral wall, $y_{\frac{m}{2}}/R$ continuously changes its value starting from zero at the origin of a jet. Therefore the proportional constant as the first approximation is taken equal to that on a straight wall jet so that $y_{\frac{m}{2}} \doteqdot CR\,\Theta$.

Equation (37) becomes

$$C_1\left(\frac{y_{\frac{m}{2}}}{R}\right) = \frac{C}{1-\frac{4}{3}c\Theta} \quad (38).$$

Finally we get

$$\frac{dy_{\frac{m}{2}}}{d\Theta} = \frac{C}{1-\frac{4}{3}c\Theta} R\left(1+\frac{y_{\frac{m}{2}}}{R}\right) \quad (39).$$

Newman(4) empirically obtained a relation for $y_{\frac{m}{2}}$ and Θ as below,

$$\frac{y_{\frac{m}{2}}}{\Theta} = C\,R\left(1+k\frac{y_{\frac{m}{2}}}{R}\right) \quad (40).$$

Differentiating the above equation with respect to Θ, we get a very similar equation to (39),

$$\frac{dy_{\frac{m}{2}}}{d\Theta} = \frac{C}{1-c\,k\,\Theta} R\left(1+k\frac{y_{\frac{m}{2}}}{R}\right) \quad (41)$$

where C is a constant of growth of a jet and k a constant of order of unity.

Newman gave to them,

$$\left.\begin{aligned} C &= 0.11 \\ k &= 1.5 \end{aligned}\right\} \quad (42).$$

The relation (40) together with the values of C and k were later confirmed by Furuya.[5]

In case of a straight wall jet, ρ_s is infinity and the first term on the right hand side of equation (4a) vanishes and equation (7) merely gives a constant to $C_1\ (y_{\frac{m}{2}}/\rho_s)$,

$$C_1\left(\frac{y_{\frac{m}{2}}}{\rho^s}\right) = C.$$

Thus we get

$$\frac{dy_{\frac{m}{2}}}{d_s} = C \qquad (43).$$

5. Discussion and Concluding Remarks

Comparing $dy_{\frac{m}{2}}/dl$ with $y_{\frac{m}{2}}/l$ in Table I, we see that our theory seems to underestimate the value of $C_1\ (y_{\frac{m}{2}}/\rho)$ when K is large. This takes place also in case of a circular cylinder.

Integrating equation (39) with respect to Θ, we get

$$\frac{y_{\frac{m}{2}}}{R} = \frac{1}{\left(1-\frac{4}{3}c\,\Theta\right)^{3/4}} - 1 \qquad (44).$$

On the other hand, rearranging equation (40) yields

$$\frac{y_{\frac{m}{2}}}{R} = \frac{1/k}{1-c\,k\,\Theta} - \frac{1}{k} \qquad (45).$$

Letting $C=0.11$ and $k = 1.5$, we computed $y_{\frac{m}{2}}/R$ from both equaitons and show them in Table II.

Table II Value of $y_{\frac{m}{2}}/R$

Θ		0	1	2	3
$\frac{y_{\frac{m}{2}}}{R}$	eqn (44)	0	0.127	0.298	0.547
	eqn (45)	0	0.132	0.328	0.654

Table II indicates that $C_1\ (y_{\frac{m}{2}}/\rho)$ is underestimated in our theory.

The explicit expression of our theory, equation (7), is based on the mixing length theory by Stratford et al, the model of which seems to be over simplified. The discrepancy of the value of $C's$ from that of a straight wall jet may originate in the mechanism of turbulence itself or in thc way of an expression of $C_1\ (y_{\frac{m}{2}}/\rho)$, eqn (7) or in a difference of Reynolds number.

Though we ignored an effect on $C_1\ (s/\rho,\ y_{\frac{m}{2}}/\rho)$ of the past history of a jet, this must have some influence on $C_1\ (s/\rho,\ y_{\frac{m}{2}}/\rho)$ because turbulence does not instantly decay, nor is developed instantly.

Nevertheless, we believe that the data discussed in the sections 3 and 4 confirms our theory justifiable, that is, $y_{\frac{m}{2}}/\rho$ plays a role of vital importance in $C_1\ (y_{\frac{m}{2}}/\rho)$

and other apparent factors in many an experimental equation come merely from difference of coordinate systems between each experiment. Stratford et al showed that a mixing length in a centrifugal field is a function of $y_{\frac{m}{2}}/\rho$ only but never showed what else was geometrically ignored. On the other hand, we showed that there are two variables $y_{\frac{m}{2}}/\rho$ and s/ρ and the former is much more important than the latter also in case where we consider the accumulated effect of a centrifugal field on the growth of a jet by applying Stratford et al 's theory and a few existing experimental data. The advantage of our theory is that we can predict how a jet grows along its own trajectory.

The theory assumes the small $y_{\frac{m}{2}}/\rho$ and a similar velocity profile of a turbulent jet. Thus we propose the universal coefficient of growth of a curved wall-jet.

In order to find out the unforseen effect on a constant C and the more reliable form of C_1 ($y_{\frac{m}{2}}/\rho$), it is desired to carry out a systematic experiment on this subject.

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