On Neutrices and Neutrix Limits

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1. Introduction

J. G. van der Corput discussed the theory of neutrices and neutrix limits in [1] and [2]. The definitions of neutrices and neutrix limits are as follows:

DEFINITION 1.1. A domain N' is a non empty abstract set. A range N'' is a non-empty commutative additive group.

Consider a commutative additive group (N) formed by functions $\nu(\xi)$, defined at each element ξ of a given domain N' such that, for each ξ in N', $\nu(\xi)$ denotes an element of a given range N''. Where no misunderstanding is possible, we write simply N instead of (N).

If $\nu(\xi)$ denotes a function belonging to N this group contains also the function $\nu(\xi)-\nu(\xi)$ which is identically equal to zero. The group N is called a neutrix if the function which is identically equal to zero is the only constant function occuring in N.

If N saitsfies the neutrix condition it is called the neutrix with domain N' and range N'' and we call the functions belonging to N negligible in N.

DEFINITION 1.2. Consider a point set N' lying in a topological space with a limit point b which does not belong to N'. Consider furthermore a commutative additive group N of real or complex valued functions defined on N' with the following property: If N contains a function of ξ which tends to a finite limit γ as ξ tends on N' to b, then $\gamma=0$.

This group N is a neutrix.

If $f(\xi)$ is a function of ξ defined on N' and if it is possible to find a constant α such that $f(\xi)-\alpha$ is negligible in N', then we call α the N-limit of $f(\xi)$ as $\xi\to b$ and we write

$$N-\lim_{\xi\to b} f(\xi)=\alpha.$$

Using above definitions we could get several results on the product of distributions.

However, in the present paper, we would like to change the definitions of neutrices and neutrix limits so as to introduce a notion of order in the theory of neutrix limits.

As an application of the notion of order we would like to discuss the product of distributions.

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2. Definitions and Several Properties

Let K be the set $\boldsymbol{R}$ or the set $\boldsymbol{C}$ and let E be a metric space embedded in a topological space S with a limit point b in S which does not belong to E.

Consider the set K^E formed by functions $f(x)$, defined at each point x of the set E such that, for each x in E, $f(x)$ denotes an element of the set K. The set K^E is a commutative K-algebra.

DEFINITION 2.1. Let $f(x) \neq 0$ for all x in E. We consider submodules of the K-module K^E with the following condition:

(2.1) If a submodule contains a function $g(x)$ which satisfies $\lim_{x \to b} g(x)/f(x) = \gamma$ and γ is finite, then $\gamma = 0$.

We write these submodules as $M(f)$, $M_0(f)$, $M^*(f), \dots$. Where no misunderstanding is possible, we write M, M_0, $M^*, \dots$ instead of $M(f)$, $M_0(f)$, $M^*(f), \dots$.

Let $M(f)$ be a submodule satisfying the condition (2.1). Then we call the submodule $M(f)$ a neutrix and the functions belonging to $M(f)$ negligible.

Let $f(x) \neq 0$ for all x in E. We define a set $M_0(f)$ by

$$M_0(f) = \{g(x) \colon \lim_{x \to b} g(x)/f(x) = 0\}.$$

It is obvious that $M_0(f)$ is a submodule satisfying the condition (2.1).

THEOREM 2.2. *Let $f_1(x) \neq 0$, $f_2(x) \neq 0$ for all x in E and let $M_0(f_1)$, $M_0(f_2)$ be submodules defined as above. Then*

$$M_0(f_1) = M_0(f_2)$$

if and only if

$$0 < \liminf_{n \to \infty} |f_1(x_n)|/|f_2(x_n)| \leq \limsup_{n \to \infty} |f_1(x_n)|/|f_2(x_n)| < \infty \tag{2.2}$$

for every sequence $\{x_n\}$ in E with the limit point b.

PROOF. (i) Assume that condition (2.2) is satisfied. Then for any $g(x)$ in $M_0(f_2)$ we have

$$0 \leq \limsup_{n \to \infty} |g(x_n)|/|f_1(x_n)| \leq \frac{\limsup_{n \to \infty} |g(x_n)|/|f_2(x_n)|}{\liminf_{n \to \infty} |f_1(x_n)|/|f_2(x_n)|} = 0$$

for every sequence $\{x_n\}$ in E with the limit point b. Since E is a metric space. we have

$$\lim_{x \to b} |g(x)|/|f_1(x)| = 0$$

and so $g(x)$ is in $M_0(f_1)$.

Similariy, if $g(x)$ is in $M_0(f_1)$, then we have $g(x)$ is in $M_0(f_2)$.

(ii) Assume that there exists a sequence $\{x_n\}$ in E satisfying the conditions

$$\lim_{n\to\infty} x_n = b, \quad \text{and} \quad \limsup_{n\to\infty} |f_1(x_n)|/|f_2(x_n)| = \infty.$$

We put $|f_1(x)|/|f_2(x)| = k_0(x)$. Then there exists a subsequence $\{x_{n(j)}\}$ such that

$$k_0(x_{n(j)}) \geq j \qquad \text{for} \quad j = 1, 2, \ldots.$$

Let

$$k(x) = \begin{cases} [k_0(x_{n(j)})]^{-1/2} & \text{for} \quad j = 1, 2, \ldots. \\ 0 & \text{otherwise,} \end{cases}$$

and let $f_0(x) = k(x) f_1(x)$. Then

$$|f_0[x_{n(j)}]|/|f_2[x_{n(j)}]| = [k_0(x_{n(j)})]^{1/2} \to \infty \qquad \text{as } j \to \infty,$$

and so $f_0(x)$ is not in $M_0(f_2)$.

Since

$$0 \leq |f_0(x_n)|/|f_1(x_n)| = k(x_n)$$
$$\to 0 \qquad \text{as } n \to \infty,$$

for every sequence $\{x_n\}$ in E with the limit point b, we have $f_0(x)$ is in $M_0(f_1)$.

Thus $M_0(f_1) \neq M_0(f_2)$.

Assume that there exists a sequence $\{x_n\}$ in E satisfying the conditions $\lim_{n\to\infty} x_n = b$, and $\liminf_{n\to\infty} |f_1(x_n)|/|f_2(x_n)| = 0$. then similarly we have $M_0(f_1) \neq M_0(f_2)$.

We have immediately the following lemma.

LEMMA 2.3. *Let $M(f)$ be a submodule satisfying the condition* (2.1) *and let $f_0(x) \neq 0$ for all x in E. Then the set*

$$f_0 M(f) = \{f_0(x) g(x) : g(x) \in M(f)\}$$

is a submodule satisfying the condition (2.1) *using $f_0(x) f(x)$ instead of $f(x)$.*

Therefore we can define a submodule $M(f_0 f)$ satisfying the condition (2.1) *by*

$$f_0 M(f) = M(f_0 f).$$

We use the notations $M(1)$, $M_0(1), \ldots$ instead of $M(f)$, $M_0(f), \ldots$ when $f(x) = 1$ for all x in E.

DEFINITION 2.4. Let $M(1)$ be a submodule satisfying the condition (2.1) and let $f(x) \neq 0$ for all x in E. If

$$g(x)/f(x) - \alpha \in M(1) \qquad (\alpha \in K)$$

then we call α the neutrix limit of order $f(x)$ of $g(x)$ for $x \to b$, and we write

$$NL(f, M, b)g(x) = \alpha.$$

THEOREM 2.5. *The limit α, if it exists, is uniquely defined.*

PROOF. Let $g(x)/f(x) - \alpha_1$, $g(x)/f(x) - \alpha_2$ be in $M(1)$. Then

$$[g(x)/f(x) - \alpha_1] - [g(x)/f(x) - \alpha_2] = -\alpha_1 + \alpha_2 \in M(1)$$

and so $\alpha_1 = \alpha_2$.

We have immediately the following theorem.

THEOREM 2.6. *Let $M(f)$ and $M(1)$ be submodules satisfying the condition (2.1) and $fM(1) = M(f)$. Then*

$$NL(f, M, b)g(x) = \alpha$$

if and only if

$$g(x) - \alpha f(x) \in M(f).$$

Let $\{0\} = \{g(x): g(x) = 0 \text{ for all } x \in E\}$. Then $\{0\}$ is a submodule, the trivial module and satisfies the condition (2.1) for every function $f(x)$ which does not vanish on E.

We have the following theorem and corollary immediately.

THEOREM 2.7. *$NL(f, \{0\}, b)g(x) = \alpha$ if ans only if*

$$g(x) = \alpha f(x) \qquad \text{for all} \quad x \in E.$$

COROLLARY 2.8. *$NL(1, \{0\}, b)g(x) = \alpha$ if and only if*

$$g(x) = \alpha \qquad \text{for all} \quad x \in E,$$

where 1 means the function $f(x) = 1$ for all x in E.

THEOREM 2.9. *Let*

$$NL(f, M, b)g_1(x) = \alpha_1, \qquad NL(f, M, b)g_2(x) = \alpha_2,$$

then for any a_1, a_2 in K we have

$$NL(f, M, b)[a_1 g_1(x) + a_2 g_2(x)] = a_1\alpha_1 + a_2\alpha_2.$$

PROOF. Since

$$g_1(x)/f(x) - \alpha_1, \quad g_2(x)/f(x) - \alpha_2 \in M(1)$$

we have

$$a_1[g_1(x)/f(x)-\alpha_1]+a_2[g_2(x)/f(x)-\alpha_2]$$
$$=[a_1g_1(x)+a_2g_2(x)]/f(x)-(a_1\alpha_1+a_2\alpha_2)\in M(1)$$

and so

$$NL(f,\ M,\ b)[a_1g_1(x)+a_2g_2(x)]=a_1\alpha_1+a_2\alpha_2.$$

Let $M_0(1)$ be the submodule defined as above. The following two theorems are obvious.

THEOREM 2.10. *$NL(f, M_0, b)g(x)=\alpha$, if and only if*

$$\lim_{x\to b} g(x)/f(x)=\alpha.$$

THEOREM 2.11. *Let $NL(f, M_0, b)g(x)=\alpha$, then there exists a neighbourhood U of b in S and a nonnegative real number B such that $|g(x)|/|f(x)|\le B$ for all $x\in U\cap E$.*

THEOREM 2.12. *Let*

$$NL(f_i,\ M_0,\ b)g_i(x)=\alpha_i \qquad (i=1,\ 2),$$

then

$$NL(f_1f_2,\ M_0,\ b)g_1(x)g_2(x)=\alpha_1\alpha_2.$$

PROOF. Since

$$\begin{aligned}&|g_1(x)g_2(x)/f_1(x)f_2(x)-\alpha_1\alpha_2|\\ &\le |g_1(x)|\,|g_2(x)-\alpha_2f_2(x)|/|f_1(x)|\,|f_2(x)|\\ &\quad+|\alpha_2|\,|g_1(x)-\alpha_1f_1(x)|/|f_1(x)|\\ &\to 0 \qquad \text{as } x\to b\end{aligned}$$

we have

$$NL(f_1f_2,\ M_0,\ b)g_1(x)g_2(x)=\alpha_1\alpha_2.$$

We would now like to consider $K=\boldsymbol{R}$, $E=(0,\infty)$ and $b=\infty$. We define sets $M_0^*(1)$ and $M_r(1)$ as follows:

$M_0^*(1)$ is the set of all linear sums of the functions

$$x^\lambda \ln^{r-1} x,\ \ln^r x \qquad \text{for} \quad \lambda>0 \quad \text{and} \quad r=1,2,\ldots,$$

and the functions belonging to the set $M_0(1)$, where the set $M_0(1)$ is defined as above.

$M_r(1)$ is the set of all functions $g(x)$ for which $\lim_{x\to\infty} p(x)g(x)=0$ for every polynomial $p(x)$ in $\boldsymbol{R}^{(0,\infty)}$.

The set $M_0^*(1)$ and $M_r(1)$ are submodules of the $\boldsymbol{R}$-module $\boldsymbol{R}^{(0,\infty)}$ and satisfy the condition (2.1), provided $f(x)=1$ for all x in $(0, \infty)$.

The following lemma is immediate.

LEMMA 2.13. *Let $M_0^*(1)\times M_r(1)$ be the set defined by*

$$M_0^*(1)\times M_r(1)=\{g(x)h(x)\colon g(x)\in M_0^*(1),\ h(x)\in M_r(1)\}.$$

Then

$$M_0^*(1)\times M_1(1)=M_r(1).$$

THEOREM 2.14. *Let*

$$NL(f_1, M_0^*, \infty)g_1(x)=\alpha_1, \quad NL(f_2, M_1, \infty)g_2(x)=\alpha_2.$$

Then

$$NL(f_1f_2, M_r, \infty)[g_1(x)g_2(x)-\alpha_2g_1(x)f_2(x)-\alpha_1g_2(x)f_1(x)]=-\alpha_1\alpha_2.$$

Thus if $f_1(x)=f_2(x)=1$ for all x in $(0, \infty)$ then

$$NL(1, M_r, \infty)[g_1(x)g_2(x)-\alpha_2g_1(x)-\alpha_1g_2(x)]=-\alpha_1\alpha_2.$$

PROOF. Since

$$g_1(x)/f_1(x)-\alpha_1\in M_0^*(1), \quad g_2(x)/f_2(x)-\alpha_2\in M_r(1)$$

we have

$$\begin{aligned}&[g_1(x)/f_1(x)-\alpha_1][g_2(x)/f_2(x)-\alpha_2]\\&\qquad=[g_1(x)g_2(x)-\alpha_2g_1(x)f_2(x)-\alpha_1g_2(x)f_1(x)]/f_1(x)f_2(x)+\alpha_1\alpha_2\end{aligned}$$

is in $M_0^*(1)\times M_r(1)$ and so

$$NL(f_1f_2, M_r, \infty)[g_1(x)g_2(x)-\alpha_2g_1(x)f_2(x)-\alpha_1g_2(x)f_1(x)]=-\alpha_1\alpha_2.$$

Now suppose that $A(x)$ is in $M_0^*(1)$. Then we can express $A(x)$ uniquely in the form

$$A(x)=s(x)+m(x),$$

where $s(x)$ is a linear sum of the functions

$$x^\lambda \ln^{r-1} x,\ \ln^r x \qquad \text{for} \quad \lambda>0 \quad \text{and} \quad r=1, 2,\ldots$$

and $m(x)$ is in $M_0(1)$. Letting

$$NL(1, M_0^*, \infty)g_1(x)=\alpha_1, \quad NL(1, M_0^*, \infty)g_2(x)=\alpha_2,$$

we have

$$g_i(x)=\alpha_i+s_i(x)+m_i(x),$$

where $s_i(x)$ is a linear sum of the functions $x^\lambda \ln^{r-1} x$, $\ln^r x$ for $\lambda>0$ and $r=1, 2,\ldots$ and $m_i(x)$ is in $M_0(1)$ for $i=1, 2$.

Since

$$g_1(x)g_2(x)=\alpha_1\alpha_2+\alpha_1 s_2(x)+\alpha_1 m_2(x)+\alpha_2 s_1(x)+s_1(x)s_2(x)+$$
$$+s_1(x)m_2(x)+\alpha_2 m_1(x)+s_2(x)m_1(x)+m_1(x)m_2(x)$$

and $\alpha_1 s_2(x)$, $\alpha_1 m_2(x)$, $\alpha_2 s_1(x)$, $\alpha_2 m_1(x)$, $s_1(x)s_2(x)$, $m_1(x)m_2(x)$ are in $M_0^*(1)$, we have the following theorem.

THEOREM 2.15. *Let* $NL(1, M_0^*, \infty)g_i(x)=\alpha_i$ *for* $i=1, 2$. *Then we have*

(2.3) $$NL(1, M_0^*, \infty)g_1(x)g_2(x)=\alpha_1\alpha_2$$

if and only if $s_1(x)m_2(x)+s_2(x)m_1(x)$ *is in* $M_0^*(1)$, *and*

(2.4) *if* $m_1(x)$, $m_2(x)$ *are in* $M_r(1)$, *then*

$$NL(1, M_0^*, \infty)g_1(x)g_2(x)=\alpha_1\alpha_2.$$

Let $M_r^*(1)$ be the set of all linear sums of the functions

$$x^\lambda \ln^{r-1} x,\ \ln^r x \qquad \text{for} \quad \lambda>0 \quad \text{and} \quad r=1, 2,\ldots$$

and functions belonging to the set $M_r(1)$.
The set $M_r^*(1)$ is a submodule of the R-module $R^{(0,\infty)}$ and atisfies the condition (2.1), provided $f(x)=1$ for all x in $(0, \infty)$.

The following theorem is immediate.

THEOREM 2.16. *Let* $NL(1, M_r^*, \infty)g_i(x)=\alpha_i$ *for* $i=1, 2$. *Then*

$$NL(1, M_r^*, \infty)g_1(x)g_2(x)=\alpha_1\alpha_2.$$

THEOREM 2.17. *Let* $NL(1, M_0^*, \infty)g_i(x)=\alpha_i$ *for* $i=1, 2$. *Then for any* A *in* $\boldsymbol{R}$, *there exist functions*

$$h_i(x)\in\alpha_i+M_0^*(1)$$

for $i=1, 2$ *such that*

$$NL(1, M_0^*, \infty)h_1(x)h_2(x)=A.$$

PROOF. Let $h_1(x)=\alpha_1+(A-\alpha_1\alpha_2)x$, $h_2(x)=\alpha_2+1/x$. Then $h_i(x)$ is in $\alpha_i+M_0^*(1)$ for $i=1, 2$ and

$$h_1(x)h_2(x)=A+\alpha_2(A-\alpha_1\alpha_2)x+\alpha_1/x.$$

Thus

$$NL(1, M_0^*, \infty)h_1(x)h_2(x)=A.$$

EXAMPLE 2.18. We define two functions $g_1(x)$ and $g_2(x)$ by

$$g_1(x)=\sum_{k=-m}^{m} a_k x^k,\quad g_2(x)=\sum_{k=-m}^{m} b_k x^k$$

where a_k, b_k are in $\boldsymbol{R}$ for $k=-m, -m+1,\dots, m$.
It follows immediately that

$$NL(x^k, M_0^*, \infty)g_1(x)=a_k \qquad (-m\le k\le m)$$

and since

$$g_1(x)g_2(x)=\sum_{k=1}^{2m}\sum_{i=-m}^{m-k} a_i b_{-k-i}x^{-k}+\sum_{k=0}^{2m}\sum_{i=k-m}^{m} a_i b_{k-i}x^k$$

we have

$$NL(x^{-k}, M_0^*, \infty)g_1(x)g_2(x)=\sum_{i=-m}^{m-k} a_i b_{-k-i} \qquad (1\le k\le 2m),$$

$$NL(x^k, M_0^*, \infty)g_1(x)g_2(x)=\sum_{i=k-m}^{m} a_i b_{k-i} \qquad (0\le k\le 2m).$$

Now let $K=R$, $E=(0, \infty)$ and let $b=0$. Suppose that $M_0^{**}(1)$ is the set of all linear sums of the functions

$$x^{-\lambda}\ln^{r-1}x,\ \ln^r x \qquad \text{for}\quad \lambda>0 \quad\text{and}\quad r=1, 2,\dots$$

and all functions $g(x)$ for which $\lim_{x\to 0} g(x)=0$.
The set $M_0^{**}(1)$ is a submodule of the R-module $R^{(0,\infty)}$ and satisfies the condition (2.1), provided that $f(x)=1$ for all x in $(0, \infty)$.

The following results follow similarly

$$NL(x^k, M_0^{**}, 0)g_1(x)=a_k \qquad (-m\le k\le m),$$

$$NL(x^{-k}, M_0^{**}, 0)g_1(x)g_2(x)=\sum_{i=-m}^{m-k} a_i b_{-k-i} \qquad (1\le k\le 2m),$$

$$NL(x^k, M_0^{**}, 0)g_1(x)g_2(x)=\sum_{i=k-m}^{m} a_i b_{k-i} \qquad (0\le k\le 2m).$$

3. A Neutrix Product of Distributions

Now let $K=\boldsymbol{R}$, $E=\{1, 2,\dots\}$, let $b=\infty$ and let $M_0^*(1)$ be the set of all linear sums of the functions

$$n^{\lambda}\ln^{r-1}n,\ \ln^r n \qquad \text{for}\quad \lambda>0 \quad\text{and}\quad r=1, 2,\dots$$

and all functions $g(n)$ for which $\lim_{n\to\infty} g(n)=0$. The set $M_0^*(1)$ is a submodule of the R-module R^N and satisfies the condition (2.1), provided that $f(n)=1$ for $n=1, 2,\dots$.

Let ρ be a fixed infinitely differentiable function having the properties:

(3.1) $$\rho(x)=0 \quad \text{for} \quad |x|\geq 1,$$

(3.2) $$\rho(x)\geq 0$$

(3.3) $$\rho(x)=\rho(-x),$$

(3.4) $$\int_{-1}^{1}\rho(x)dx=1.$$

The function δ_n is defined by

$$\delta_n(x)=n\rho(nx)$$

for $n=1, 2,\dots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta function δ. For an arbitrary distribution f the function f_n is defined by

$$f_n(x)=f*\delta_n(x)=\int_{-1/n}^{1/n}f(x-t)\delta_n(t)dt.$$

It follows that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to f.

DEFINITION 3.1. Let f and g be arbitrary distributions and let $g_n=g*\delta_n$. We say that the neutrix product of order n^k of f and g exists and is equal to h on the open interval (a, b) if

$$NL(n^k, M_0^*, \infty)(fg_n, \phi)=(h, \phi)$$

for all test function ϕ with compact support contained in the interval (a, b). We then write $f\overset{k}{\circ}g=h$. If no misundrestanding is possible, we will simply say the neutrix product instead of neutrix product of order n^k. Further if $k=0$, then we will omit 0 and write $f\circ g$.

DEFINITION 3.2. We define the ordinary summable function x_+^λ for $\lambda>-1$ by

$$x_+^\lambda=\begin{cases} x^\lambda & \text{for} \quad x>0, \\ 0 & \text{for} \quad x<0 \end{cases}$$

and we define the distribution x_+^λ for $\lambda<-1$ and $\lambda\neq -2, -3,\dots$ inductively by

$$x_+^\lambda=(\lambda+1)^{-1}(x_+^{\lambda+1})'.$$

We define the distribution x_-^λ by

$$x_-^\lambda=(-x)_+^\lambda$$

for $\lambda\neq -1, -2,\dots$.

The following two theorems were proved in [3].

THEOREM 3.3. *The neutrix products $x_+^\lambda \circ x_-^{-\lambda-r}$ and $x_-^{-\lambda-r} \circ x_+^\lambda$ exist and*

$$x_+^\lambda \circ x_-^{-\lambda-r} = x_-^{-\lambda-r} \circ x_+^\lambda = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{(r-1)}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $r=1, 2, \ldots$.

THEOREM 3.4. *The neutrix product $x_+^r \circ \delta^{(r+p)}$ and $\delta^{(r+p)} \circ x_+^r$ exist and*

$$x_+^r \circ \delta^{(r+p)} = \delta^{(r+p)} \circ x_+^r = \frac{(-1)^r (r+p)!}{2p!}\delta^{(p)}$$

for $r, p=0, 1, 2, \ldots$.

We now prove the following theorems.

THEOREM 3.5. *Let f and g be distributions and suppose that the neutrix product $f \overset{k}{\circ} g$ and $f \overset{k}{\circ} g'$ exist on the open interval (a, b). Then the neutrix product $f' \overset{k}{\circ} g$ exists and*

$$(f \overset{k}{\circ} g)' = f' \overset{k}{\circ} g + f \overset{k}{\circ} g'$$

on the interval (a, b).

PROOF. Let ϕ be an arbitrary test function with compact support contained in the interval (a, b) and let $g_n = g * \delta_n$. Then

$$(f \overset{k}{\circ} g', \phi) = NL(n^k, M_0^*, \infty)(f, g_n'\phi)$$

and

$$\begin{aligned} -((f \overset{k}{\circ} g)', \phi) &= (f \overset{k}{\circ} g, \phi') \\ &= NL(n^k, M_0^*, \infty)(f, g_n\phi') \\ &= NL(n^k, M_0^*, \infty)(f, (g_n\phi)' - g_n'\phi) \end{aligned}$$

and so

$$(f, g_n'\phi)n^{-k} - (f \overset{k}{\circ} g', \phi)$$

and

$$[(f, (g_n\phi)') - (f, g_n'\phi)]n^{-k} + ((f \overset{k}{\circ} g)', \phi)$$

are in $M_0^*(1)$. Thus

$$(f, (g_n\phi)')n^{-k} - (f \overset{k}{\circ} g', \phi) + ((f \overset{k}{\circ} g)', \phi)$$

is in $M_0^*(1)$ and so

$$NL(n^k, M_0^*, \infty)(f, (g_n\phi)') = (f \overset{k}{\circ} g', \phi) - ((f \overset{k}{\circ} g)', \phi).$$

The result of theorem follows.

THEOREM 3.6. *The neutrix product* $x_+^\lambda \overset{k}{\circ} x_-^{-\lambda-r}$ *exists and*

$$x_+^\lambda \overset{k}{\circ} x_-^{-\lambda-r} = (-1)^{r-k-1}\alpha(r, k)\,\frac{\Gamma(\lambda+1)B(\lambda+r-k, -\lambda)}{\Gamma(\lambda+r)\,(r-k-1)!}\,\delta^{(r-k-1)}$$

for $-1<\lambda<0$, $r=1, 2,\dots$ *and* $k\le r-1$,
where

$$\alpha(r, k)=\int_0^1 s^{r-k-1}\rho^{(r-1)}(s)ds.$$

PROOF. Since $-1<\lambda<0$, we have

$$x_-^{-\lambda-r}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\frac{d^{r-1}}{dx^{r-1}}\,x_-^{-\lambda-1}$$

and

$$\begin{aligned}(x_-^{-\lambda-r})_n &= x_-^{-\lambda-r}*\delta_n\\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\,x_-^{-\lambda-1}*\delta_n^{(r-1)}\\ &= \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}\int_x^{1/n}(t-x)^{-\lambda-1}\delta_n^{(r-1)}(t)dt.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{\Gamma(\lambda+r)}{\Gamma(\lambda+1)}\int_{-\infty}^{\infty} x_+^\lambda\,(x_-^{-\lambda-r})_n x^m dx\\ &=\int_0^{1/n} x^{\lambda+m}\int_x^{1/n}(t-x)^{-\lambda-1}\delta_n^{(r-1)}(t)dtdx\\ &=\int_0^{1/n}\delta_n^{(r-1)}(t)\int_0^t x^{\lambda+m}(t-x)^{-\lambda-1}dxdt\\ &=\int_0^{1/n} t^m\delta_n^{(r-1)}(t)\int_0^1 v^{\lambda+m}(1-v)^{-\lambda-1}dvdt\\ &=B(\lambda+m+1, -\lambda)\int_0^{1/n} t^m\delta_n^{(r-1)}(t)dt\end{aligned}$$

where the substitution $x=tv$ has been made and B denotes the beta function.
Making the substitution $nt=s$ we have

$$\int_0^{1/n} t^m\delta_n^{(r-1)}(t)dt=n^{r-m-1}\int_0^1 s^m\rho^{(r-1)}(s)ds$$

for $m=0, 1, 2,\dots$.

Now let ϕ be an arbitrary test function. Then

$$\phi(x)=\sum_{m=0}^{j-1}\frac{x^m}{m!}\phi^{(m)}(0)+\frac{x^j}{j!}\phi^{(j)}(\xi x)$$

where $0\leq\xi\leq 1$ and so

$$(x_+^\lambda, (x_-^{-\lambda-r})_n\phi(x))$$
$$=\sum_{m=0}^{j-1}\frac{\phi^{(m)}(0)}{m!}\int_{-\infty}^{\infty}x_+^\lambda(x_-^{-\lambda-r})_n x^m dx+\frac{1}{j!}\int_{-\infty}^{\infty}x_+^\lambda(x_-^{-\lambda-r})_n x^j\phi^{(j)}(\xi x)\,dx.$$

We have

$$\left|\int_{-\infty}^{\infty}x_+^\lambda(x_-^{-\lambda-r})_n x^j\phi^{(j)}(\xi x)dx\right|$$
$$\leq\sup_x\{|\phi^{(j)}(x)|\}\int_{-\infty}^{\infty}|x_+^\lambda(x_-^{-\lambda-r})_n x^j|dx=O(n^{r-j-1})$$

for $r-1<j$ and

$$\frac{\phi^{(m)}(0)}{m!}\int_{-\infty}^{\infty}x_+^\lambda(x_-^{-\lambda-r})_n x^m dx$$
$$=\frac{\phi^{(m)}(0)}{m!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}B(\lambda+m+1,\ -\lambda)\int_0^{1/n}t^m\delta_n^{(r-1)}(t)dt$$
$$=\frac{\phi^{(m)}(0)}{m!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}B(\lambda+m+1,\ -\lambda)n^{r-m-1}\int_0^1 s^m\rho^{(r-1)}(s)ds$$
$$=\frac{\phi^{(r-k-1)}(0)}{(r-k-1)!}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r)}B(\lambda+r-k,\ -\lambda)n^k\alpha(r,\ k)$$

where the substitution $k=r-m-1$ has been made. Since

$$\phi^{(r-k-1)}(0)=(-1)^{r-k-1}(\delta^{(r-k-1)},\ \phi)$$

we have

$$NL(n^k,\ M_0^*,\ \infty)(x_+^\lambda(x_-^{-\lambda-r})_n,\ \phi)$$
$$=(-1)^{r-k-1}\alpha(r,\ k)\frac{\Gamma(\lambda+1)\ B(\lambda+r-k,\ -\lambda)}{\Gamma(\lambda+r)(r-k-1)!}(\delta^{(r-k-1)},\ \phi).$$

The result of the theorem follows.

THEOREM 3.7. *The neutrix product* $x_+^r \overset{k}{\circ}\delta^{(r+p)}$ *exists and*

$$x_+^r\overset{k}{\circ}\delta^{(r+p)}=(-1)^{p-k}\frac{a(r,\ p,\ k)}{(p-k)!}\delta^{(p-k)}$$

where

$$a(r, p, k) = \int_0^1 s^{p+r-k} \rho^{(r+p)}(s)\, ds$$

for $r, p = 1, 2, \ldots$ *and* $k \leq p$.

PROOF. We have

$$\int_{-\infty}^{\infty} x_+^r \delta_n^{(r+p)}(x) x^m dx = \int_0^{1/n} x^{m+r} \delta_n^{(r+p)}(x) dx$$
$$= n^{p-m} \int_0^1 s^{m+r} \rho^{(r+p)}(s) ds$$

where the substitution $nx = s$ has been made and so

$$\int_{-\infty}^{\infty} |x_+^r \delta_n^{(r+p)}(x)\ x^{p+j}| dx = n^{-j} \int_0^1 |s^{r+p+j} \rho^{(r+p)}(s)| ds$$
$$= 0(n^{-j})$$

for $j \geq 1$.

Now let ϕ be an arbitrary test function. Then

$$(x_+^r, \delta_n^{(r+p)}(x)\phi(x))$$
$$= \sum_{m=0}^{p+j} \frac{\phi^{(m)}(0)}{m!} \int_{-\infty}^{\infty} x_+^r \delta_n^{(r+p)}(x)\ x^m dx$$
$$+ \frac{1}{(p+j+1)!} \int_{-\infty}^{\infty} x_+^r \delta_n^{(r+p)}(x)\ x^{p+j+1} \phi^{(p+j+1)}\ (\xi x) dx.$$

Since

$$\frac{1}{(p+j+1)!} \int_{-\infty}^{\infty} |x_+^r \delta_n^{(r+p)}\ (x) x^{p+j+1} \phi^{(p+j+1)}(\xi x)| dx$$
$$\leq \sup_x \{|\phi^{(p+j+1)}(x)|\} \frac{1}{(p+j+1)!} \int_0^{\infty} |x_+^r \delta_n^{(r+p)}(x) x^{p+j+1}| dx$$
$$= \sup_x \{|\phi^{(p+j+1)}(x)|\} \frac{1}{(p+j+1)!} O(n^{-j-1})$$

we have

$$\frac{\phi^{(m)}(0)}{m!} \int_{-\infty}^{\infty} x_+^r \delta_n^{(r+p)}(x) x^m dx$$
$$= \frac{\phi^{(p-k)}(0)}{(p-k)!} n^k \int_0^1 s^{p+r-k} \rho^{(r+p)}(s) ds$$
$$= \frac{a(r, p, k)}{(p-k)!} n^k (-1)^{p-k} (\delta^{(p-k)}, \phi)$$

where the substitutions $nx = s$ and $m = p - k$ have been made. The result of the theorem follows.

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