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著者 Author(s)	Hashimoto, Takashi
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(\mathfrak{g}, K) -module of $O(p, q)$ associated with the finite-dimensional representation of \mathfrak{sl}_2

Takashi Hashimoto

*University Education Center, Tottori University,
4-101, Koyama-Minami, Tottori, 680-8550, Japan
thashi@tottori-u.ac.jp*

The main aim of this paper is to show that one can construct (\mathfrak{g}, K) -modules of $O(p, q)$ associated with the finite-dimensional representation of \mathfrak{sl}_2 by quantizing the moment map on the symplectic vector space $(\mathbb{C}^{p+q})_{\mathbb{R}}$ and using the fact that $(O(p, q), \mathrm{SL}_2(\mathbb{R}))$ is a dual pair. Then one obtains the K -type formula, the Gelfand-Kirillov dimension and the Bernstein degree of them for all non-negative integers m satisfying $m + 3 \leq (p + q)/2$ when $p, q \geq 2$ and $p + q$ is even. In fact, one finds that the Gelfand-Kirillov dimension is equal to $p + q - 3$ and the Bernstein degree is equal to $4(m + 1)(p + q - 4)! / ((p - 2)!(q - 2)!)$.

Keywords: indefinite orthogonal group, moment map on symplectic vector space, canonical quantization, irreducible (\mathfrak{g}, K) -module, K -type formula

Mathematics Subject Classification 2010: 22E46, 17B20, 17B10

1. Introduction

Let G be a Lie group with \mathfrak{g}_0 its Lie algebra and \mathfrak{g} the complexification of \mathfrak{g}_0 . An action of G on a symplectic manifold (M, ω) is called symplectic if $g^*\omega = \omega$ for all $g \in G$, and a symplectic action is called Hamiltonian if there exists a smooth G -equivariant map $\mu : M \rightarrow \mathfrak{g}_0^*$ satisfying the condition (2.3) below, which is called a moment map, where \mathfrak{g}_0^* is the dual vector space of \mathfrak{g}_0 . We are concerned with the case where the symplectic manifold is a real symplectic vector space (W, ω) . It was shown in [3] that when $G = \mathrm{Sp}(n, \mathbb{R}), \mathrm{U}(p, q)$ and $O^*(2n)$, the canonical quantization of the moment map on $W = \mathbb{R}^{2n}, (\mathbb{C}^{p+q})_{\mathbb{R}}$ and $(\mathbb{C}^{2n})_{\mathbb{R}}$, with a choice of a Lagrangian subspace in each case, yields a representation of \mathfrak{g} that is the differentiation of the oscillator (or Segal-Shale-Weil) representation of $\mathrm{Mp}(n, \mathbb{R}), \mathrm{U}(p, q)$ and $O^*(2n)$ respectively, where $\mathrm{Mp}(n, \mathbb{R})$ is the metaplectic group, i.e., the double cover of $\mathrm{Sp}(n, \mathbb{R})$.

In this paper, we consider the case where $W = (\mathbb{C}^{p+q})_{\mathbb{R}}$, the real vector space underlying \mathbb{C}^{p+q} :

$$W = \{z = x + iy \mid x, y \in \mathbb{R}^{p+q}\},$$

which we regard as a symplectic vector space equipped with a symplectic form ω given by

$$\omega(z, w) = \mathrm{Im}(z^* I_{p,q} w) \quad (z, w \in W),$$

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and $G = \mathrm{O}(p, q)$, the indefinite orthogonal group defined by

$$\mathrm{O}(p, q) = \{g \in \mathrm{GL}_{p+q}(\mathbb{R}) \mid {}^t g I_{p,q} g = I_{p,q}\}$$

with $I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$. The action of $G = \mathrm{O}(p, q)$ on W defined by matrix multiplication is symplectic and Hamiltonian. The $\mathrm{O}(p, q)$ -case we consider here is closely related to the $\mathrm{U}(p, q)$ -case mentioned above. In fact, the symplectic vector space (W, ω) for $\mathrm{O}(p, q)$ is identical to the one for $\mathrm{U}(p, q)$, and the action of $\mathrm{O}(p, q)$ on W is the restriction of the action of $\mathrm{U}(p, q)$ induced from the canonical embedding of $\mathrm{O}(p, q)$ into $\mathrm{U}(p, q)$. Furthermore, the moment map for the $\mathrm{O}(p, q)$ -case is the real part of the one for the $\mathrm{U}(p, q)$ -case.

The canonical quantization of the moment map $\mu : W = (\mathbb{C}^{p+q})_{\mathbb{R}} \rightarrow \mathfrak{g}_0^*$ for $G = \mathrm{O}(p, q)$, with a choice of a Lagrangian subspace V of W , provides a representation π of \mathfrak{g} as in the cases mentioned above, which is shown to be a partial Fourier transformation of the representation π^\sharp of \mathfrak{g} obtained by differentiating the left regular representation of G on $C^\infty(V)$. Note that if we restrict the operator $\pi^\sharp(X)$, $X \in \mathfrak{g}$, to a subspace consisting of homogeneous functions on V with respect to the multiplicative group $\mathbb{R}_{>0}$, then the restricted representation is the degenerate principal series of G obtained by inducing up a one-dimensional representation of a parabolic subgroup of G (see [6]).

In the influential paper [5], Howe showed that one can treat the classical invariant theory from a unified viewpoint — the dual pair. In this paper, we focus our attention on the dual pair $(\mathrm{O}(p, q), \mathrm{SL}_2(\mathbb{R}))$, both components of which are noncompact, and apply the representation theory of \mathfrak{sl}_2 to cut out (\mathfrak{g}, K) -modules, which we denote by $M^+(m)$ and $M^-(m)$, $m = 0, 1, 2, \dots$, where $M^+(m)$ (resp. $M^-(m)$) consists of all highest (resp. lowest) weight vectors with respect to the \mathfrak{sl}_2 -action (see Definition 4.1 below for details). Note that $M^+(0) = M^-(0)$ by definition. We will see that such weight vectors are given in terms of harmonic polynomials and the Bessel functions of the first kind. Both $M^\pm(m)$ should correspond to the $(m+1)$ -dimensional irreducible representation of \mathfrak{sl}_2 under the Howe duality, and in fact are isomorphic to each other. They were originally considered in [12] without the condition of finite-dimensionality.

In the cases of the oscillator representations mentioned above, i.e., when $G = \mathrm{Sp}(n, \mathbb{R})$, $\mathrm{U}(p, q)$ and $\mathrm{O}^*(2n)$, we note that the counterpart G' of G for the dual pair (G, G') is compact, hence, all its irreducible representations are finite-dimensional. Furthermore, the oscillator representations give examples of the minimal representations (we refer to [7] and the references therein for the definition of the minimal representation). When $G = \mathrm{O}(p, q)$, its minimal representation is discussed e.g. in [1, 7–11, 16].

The main result of this paper is the K -type formula of $M^\pm(m)$ for nonnegative integers m satisfying

$$m + 3 \leq \frac{p + q}{2} \tag{1.1}$$

when $p + q$ is even, from which one can show that $M^\pm(m)$ are irreducible (\mathfrak{g}, K) -modules if $p, q \geq 2$ (Theorem 4.1). The fact that the elements of $M^\pm(m)$ are described in terms of the Bessel function plays a role in the proof of our main result. The K -type formula of $M^+(0) = M^-(0)$, which is associated with the one-dimensional trivial representation of \mathfrak{sl}_2 ,

shows that it corresponds to the (\mathfrak{g}, K) -module of the minimal representation of $O(p, q)$. We will see that the Gelfand-Kirillov dimension and the Bernstein degree of the irreducible $M^\pm(m)$, which we denote by $\text{Dim } M^\pm(m)$ and $\text{Deg } M^\pm(m)$ respectively, are given by

$$\begin{aligned}\text{Dim } M^\pm(m) &= p + q - 3, \\ \text{Deg } M^\pm(m) &= \frac{4(m+1)(p+q-4)!}{(p-2)!(q-2)!}\end{aligned}$$

(Corollary 4.1).

The rest of this paper is organized as follows. In §2, we calculate the moment map μ on W for $G = O(p, q)$, and construct the representation π of \mathfrak{g} via canonical quantization of μ . Then we show that π is a partial Fourier transform of the differential representation of the left regular representation of G on $C^\infty(V)$. In §3, we give an \mathfrak{sl}_2 -action that commutes with π , and find both highest weight vectors and lowest weight vectors with respect to the \mathfrak{sl}_2 -action. We remark that such weight vectors are given in terms of the Bessel functions of the first kind. In §4, we introduce (\mathfrak{g}, K) -modules $M^\pm(m)$ and prove that $M^+(m)$ and $M^-(m)$ are isomorphic to each other for any nonnegative integer m . When $p+q$ is even, we find the K -type formula of $M^\pm(m)$ for m satisfying (1.1) and show that they are irreducible if p, q are ≥ 2 . As a corollary, we obtain the Gelfand-Kirillov dimension and the Bernstein degree of $M^\pm(m)$.

Notation

Let \mathbb{N} denote the set of nonnegative integers $\{0, 1, 2, \dots\}$, and $[p]$ the set $\{1, 2, \dots, p\}$ for a positive integer p . We write $\bar{i} := p + i$ for $i \in [q]$ for the sake of simplicity. Finally, for $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote the rising and the falling factorials by

$$(\alpha)_n := \prod_{i=1}^n (\alpha + i - 1) \quad \text{and} \quad (\alpha)_n^- := \prod_{i=1}^n (\alpha - i + 1),$$

respectively.

2. Moment Map and its Quantization

Let G be the indefinite orthogonal group $O(p, q)$, which we realize by

$$O(p, q) = \{g \in \text{GL}_{p+q}(\mathbb{R}) \mid {}^t g I_{p,q} g = I_{p,q}\}$$

with $I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$. Let K be a maximal compact subgroup of G given by

$$K = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in G \mid a \in O(p), d \in O(q) \right\} \simeq O(p) \times O(q).$$

We denote the Lie algebra of K and its complexification by \mathfrak{k}_0 and \mathfrak{k} respectively.

Let $\{X_{i,j}^\pm\}$ be a basis for $\mathfrak{g}_0 = \mathfrak{o}(p, q)$ given by

$$\begin{aligned}X_{i,j}^+ &= E_{i,j} - E_{j,i} & (i, j \in [p]) \\ X_{i,j}^+ &= E_{i,\bar{j}} - E_{\bar{j},i} & (i, j \in [q]) \\ X_{i,j}^- &= E_{i,\bar{j}} + E_{\bar{j},i} & (i \in [p], j \in [q]),\end{aligned}\tag{2.1}$$

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which also forms a basis for $\mathfrak{g} = \mathfrak{o}_{p+q}$, the complexification of $\mathfrak{g}_0 = \mathfrak{o}(p, q)$. We often identify \mathfrak{g}^* with \mathfrak{g} via the invariant bilinear form B given by

$$B(X, Y) = \frac{1}{2} \operatorname{tr}(XY) \quad (X, Y \in \mathfrak{g}),$$

where \mathfrak{g}^* denotes the dual space of \mathfrak{g} . Finally, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition of \mathfrak{g} with

$$\mathfrak{k} = \sum_{i,j \in [p]} \mathbb{C}X_{i,j}^+ \oplus \sum_{i,j \in [q]} \mathbb{C}X_{i,j}^-, \quad \mathfrak{p} = \sum_{i \in [p], j \in [q]} \mathbb{C}X_{i,j}^-.$$

Let W be the real vector space $(\mathbb{C}^{p+q})_{\mathbb{R}}$ underlying the complex vector space \mathbb{C}^{p+q} :

$$W = \{z = x + iy \mid x = {}^t(x_1, \dots, x_{p+q}), y = {}^t(y_1, \dots, y_{p+q}) \in \mathbb{R}^{p+q}\},$$

which is equipped with a symplectic form ω given by

$$\omega(z, w) = \operatorname{Im}(z^* I_{p,q} w) \quad (z, w \in W) \quad (2.2)$$

Then G acts on (W, ω) symplectically via $z \mapsto gz$ (matrix multiplication) for $z \in W$ and $g \in G$. Furthermore, the action of G on (W, ω) is Hamiltonian, i.e., there exists a moment map $\mu : W \rightarrow \mathfrak{g}_0^*$, whose definition we briefly recall: if, in general, a Lie group G acts on a symplectic manifold (M, ω) symplectically, a smooth G -equivariant map $\mu : M \rightarrow \mathfrak{g}_0^*$ that satisfies

$$d\langle \mu, X \rangle = \iota(X_M)\omega \quad \text{for all } X \in \mathfrak{g}_0, \quad (2.3)$$

is called a moment map, where ι stands for the contraction and X_M denotes the vector field on M given by

$$X_M(p) = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX) \cdot p \quad (p \in M).$$

Under the identification that $e_i := {}^t(0, \dots, \overset{i}{1}, \dots, 0) \leftrightarrow \partial_{x_i}$ and $i e_i \leftrightarrow \partial_{y_i}$ for $i = 1, 2, \dots, p+q$, the symplectic form ω given in (2.2) can be rewritten as

$$\omega = \sum_{i=1}^{p+q} \epsilon_i dx_i \wedge dy_i$$

with $\epsilon_i = 1$ for $i \in [p]$ and $\epsilon_{p+i} = -1$ for $i \in [q]$.

Proposition 2.1. *The action of $G = \mathrm{O}(p, q)$ on (W, ω) is Hamiltonian, and the moment map $\mu : W \rightarrow \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by*

$$\begin{aligned} \mu(z) &= -\frac{i}{2} (zz^* - {}^t(zz^*)) I_{p,q} \\ &= (-x {}^t y + y {}^t x) I_{p,q} \\ &= \begin{bmatrix} -x' {}^t y' + y' {}^t x' & x' {}^t y'' - y' {}^t x'' \\ -x'' {}^t y' + y'' {}^t x' & x'' {}^t y'' - y'' {}^t x'' \end{bmatrix} \end{aligned} \quad (2.4)$$

for $z = x + iy \in W$ with $x = {}^t(x', x'')$, $y = {}^t(y', y'') \in \mathbb{R}^{p+q}$ and $x', y' \in \mathbb{R}^p$, $x'', y'' \in \mathbb{R}^q$.

Proof. Using the formula

$$\langle \mu(z), X \rangle = \frac{1}{2} \omega(z, Xz)$$

(see e.g. [2, Proposition 1.4.6]), which, in our case, can be written as

$$B(\mu(z), X) = \frac{1}{2} B((zz^* - \bar{z}{}^t z) I_{p,q}, X)$$

for all $X \in \mathfrak{g}_0$, we obtain (2.4) immediately. \square

Remark 2.1. Recall that the moment map $\mu_U : W \rightarrow \mathfrak{u}(p, q)^* \simeq \mathfrak{u}(p, q)$ for the action of $U(p, q)$ on our symplectic vector space (W, ω) is given by

$$\mu_U(z) = -i z z^* I_{p,q} \quad (z = x + iy \in W)$$

where we identify $\mathfrak{u}(p, q)^*$ with $\mathfrak{u}(p, q)$ via the invariant bilinear form B given by $B(X, Y) = (1/2) \operatorname{tr}(XY)$. Therefore, the moment map μ in the proposition is related to μ_U by

$$\mu(z) = \frac{\mu_U(z) + \overline{\mu_U(z)}}{2}.$$

Namely, one has $\mu = \operatorname{Re} \mu_U$.

We define a Poisson bracket by

$$\{f, g\} = \omega(\xi_g, \xi_f),$$

where ξ_f denotes the Hamiltonian vector field on W corresponding to $f \in C^\infty(W)$, i.e. the vector field that satisfies $\iota(\xi_f)\omega = df$. Then the Poisson bracket among the coordinate functions are given by

$$\{x_i, y_j\} = -\delta_{i,j} \epsilon_i, \quad \{x_i, x_j\} = \{y_i, y_j\} = 0$$

for $i, j = 1, 2, \dots, p+q$. Dirac's quantization conditions requires that

$$\{f_1, f_2\} = f_3 \quad \text{implies} \quad [\hat{f}_1, \hat{f}_2] = -i \hbar \hat{f}_3$$

for $f_i \in C^\infty(W)$ (see e.g. [15]). Thus, we quantize the coordinate functions as follows:

$$\begin{aligned} \hat{x}_i &= x_i, & \hat{y}_i &= -i \hbar \partial_{x_i}, & (i &= 1, \dots, p), \\ \hat{x}_j &= -i \hbar \partial_{y_j}, & \hat{y}_j &= y_j, & (j &= 1, \dots, q), \end{aligned} \quad (2.5)$$

where ∂_{x_i} and ∂_{y_j} denote $\partial/\partial x_i$ and $\partial/\partial y_j$ respectively. In what follows, we set $\hbar = 1$ for brevity.

The quantization (2.5) corresponds to a Lagrangian subspace V of W given by

$$V = \langle e_1, \dots, e_p, i e_{\bar{1}}, \dots, i e_{\bar{q}} \rangle_{\mathbb{R}} \quad (2.6)$$

^aMore precisely, one should denote $x = {}^t(x', {}^t x'')$ etc.; we will use this abbreviated notation in what follows.

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in the sense that the quantized operators are realized in $\mathcal{PD}(V)$, the ring of polynomial coefficient differential operators on V . Therefore, the quantized moment map $\hat{\mu}$ is given by

$$\hat{\mu} = (-\hat{x} \text{ }^t \hat{y} + \hat{y} \text{ }^t \hat{x}) I_{p,q} = \left[\begin{array}{cc} i(x' \text{ }^t \partial_{x'} - \partial_{x'} \text{ }^t x') & x' \text{ }^t y'' + \partial_{x'} \text{ }^t \partial_{y''} \\ \partial_{y''} \text{ }^t \partial_{x'} + y'' \text{ }^t x' & i(y'' \text{ }^t \partial_{y''} - \partial_{y''} \text{ }^t y'') \end{array} \right],$$

where

$$\begin{aligned} \hat{x} &= \text{ }^t(\hat{x}_1, \dots, \hat{x}_{p+q}) = \text{ }^t(x', -i \partial_{y''}), \\ \hat{y} &= \text{ }^t(\hat{y}_1, \dots, \hat{y}_{p+q}) = \text{ }^t(-i \partial_{x'}, y''), \end{aligned}$$

and

$$\begin{aligned} x' &= \text{ }^t(x_1, \dots, x_p), & \partial_{x'} &= \text{ }^t(\partial_{x_1}, \dots, \partial_{x_p}), \\ y'' &= \text{ }^t(y_{\bar{1}}, \dots, y_{\bar{q}}), & \partial_{y''} &= \text{ }^t(\partial_{y_{\bar{1}}}, \dots, \partial_{y_{\bar{q}}}). \end{aligned}$$

Note that $x_1, \dots, x_p, y_{\bar{1}}, \dots, y_{\bar{q}}$ are considered to be the coordinate functions on V with respect to the basis $e_1, \dots, e_p, i e_{\bar{1}}, \dots, i e_{\bar{q}}$.

Theorem 2.1. *For $X \in \mathfrak{g}$, set $\pi(X) := i \langle \hat{\mu}, X \rangle$. Then $\pi : \mathfrak{g} \rightarrow \mathcal{PD}(V)$ is a Lie algebra homomorphism. In terms of the basis (2.1), it is given by*

$$\pi(X) = \begin{cases} -x_j \partial_{x_i} + x_i \partial_{x_j} & \text{if } X = X_{i,j}^+; \\ -y_j \partial_{y_i} + y_i \partial_{y_j} & \text{if } X = X_{i,j}^-; \\ i(x_i y_j + \partial_{x_i} \partial_{y_j}) & \text{if } X = X_{i,j}^-. \end{cases} \quad (2.7)$$

Proof. This is proved in the same manner as [3, Theorem 2.3] (or, one can verify the commutation relations by direct calculation). \square

There is another canonical quantization that corresponds to the same Lagrangian subspace V of W as given in (2.6). Namely, if we quantize the coordinate functions as

$$\begin{aligned} \hat{x}_i &= x_i, & \hat{y}_i &= -i \partial_{x_i}, & (i = 1, \dots, p), \\ \hat{x}_j &= y_j, & \hat{y}_j &= i \partial_{y_j}, & (j = 1, \dots, q), \end{aligned} \quad (2.8)$$

then the quantized moment map, which we denote by $\hat{\mu}^\sharp$, is given by

$$\hat{\mu}^\sharp = (-\hat{x} \text{ }^t \hat{y} + \hat{y} \text{ }^t \hat{x}) I_{p,q} = i \left[\begin{array}{cc} x' \text{ }^t \partial_{x'} - \partial_{x'} \text{ }^t x' & x' \text{ }^t y'' + \partial_{x'} \text{ }^t y'' \\ y'' \text{ }^t \partial_{x'} + \partial_{y''} \text{ }^t x' & y'' \text{ }^t \partial_{y''} - \partial_{y''} \text{ }^t y'' \end{array} \right],$$

where

$$\begin{aligned} \hat{x} &= \text{ }^t(\hat{x}_1, \dots, \hat{x}_{p+q}) = \text{ }^t(x', y''), \\ \hat{y} &= \text{ }^t(\hat{y}_1, \dots, \hat{y}_{p+q}) = \text{ }^t(-i \partial_{x'}, i \partial_{y''}). \end{aligned}$$

Hence one obtains a representation $\pi^\sharp : \mathfrak{g} \rightarrow \mathcal{PD}(V)$ if one sets $\pi^\sharp(X) := i \langle \hat{\mu}^\sharp, X \rangle$ for $X \in \mathfrak{g}$. It is given in terms of the basis (2.1) by

$$\pi^\sharp(X) = \begin{cases} -x_j \partial_{x_i} + x_i \partial_{x_j} & \text{if } X = X_{i,j}^+; \\ -y_j \partial_{y_i} + y_i \partial_{y_j} & \text{if } X = X_{i,j}^-; \\ -(x_i \partial_{y_j} + y_j \partial_{x_i}) & \text{if } X = X_{i,j}^-. \end{cases} \quad (2.9)$$

Remark 2.2. (i) Comparing (2.8) with (2.5), one sees that π^\sharp is related to π through the partial Fourier transform on \mathbb{R}^{p+q} with respect to the variables $y_{\bar{1}}, \dots, y_{\bar{q}}$. In fact, if we denote the dual variable of $y_{\bar{j}}$ by $\eta_{\bar{j}}$, $j = 1, 2, \dots, q$, then π and π^\sharp interchange with each other under the correspondence

$$-i \partial_{y_{\bar{j}}} \longleftrightarrow \eta_{\bar{j}}, \quad y_{\bar{j}} \longleftrightarrow i \partial_{\eta_{\bar{j}}} \quad (j = 1, \dots, q);$$

the former operators $-i \partial_{y_{\bar{j}}}$ and $\eta_{\bar{j}}$ are the realizations of $\hat{x}_{\bar{j}}$, while the latter operators $y_{\bar{j}}$ and $i \partial_{\eta_{\bar{j}}}$ are the realizations of $\hat{y}_{\bar{j}}$.

(ii) Recall that one can obtain π^\sharp by differentiating the left regular representation of $G = O(p, q)$ on $C^\infty(V)$, the space of complex-valued smooth functions on V , where G acts on V by matrix multiplication under the identification of V with \mathbb{R}^{p+q} given by ${}^t(x', i y'') \leftrightarrow {}^t(x', y'')$ (see e.g. [6, 12]). As one can see from (2.7) and (2.9), $\pi^\sharp(X)$ coincides with $\pi(X)$ for all $X \in \mathfrak{k}$. Thus, the action π restricted to \mathfrak{k}_0 lifts to the action of K on $C^\infty(V)$.

3. Dual Pair $(O(p, q), \mathfrak{sl}_2(\mathbb{R}))$

Henceforth, let us denote $x' = {}^t(x_1, \dots, x_p)$ and $y'' = {}^t(y_{\bar{1}}, \dots, y_{\bar{q}})$ by

$$x = {}^t(x_1, \dots, x_p) \quad \text{and} \quad y = {}^t(y_1, \dots, y_q)$$

respectively for the sake of simplicity if there exists no risk of confusion. Namely, we regard (x_1, \dots, x_p) and (y_1, \dots, y_q) as the canonical coordinate functions on \mathbb{R}^p and on \mathbb{R}^q respectively.

If we denote the Casimir element of \mathfrak{g} by $\Omega_{\mathfrak{g}}$, then the corresponding Casimir operator is given by

$$\begin{aligned} \pi(\Omega_{\mathfrak{g}}) &= (E_x - E_y)^2 + (p - q)(E_x - E_y) - 2(E_x + E_y) \\ &\quad - \left(r_x^2 r_y^2 + r_x^2 \Delta_x + r_y^2 \Delta_y + \Delta_x \Delta_y \right) - pq, \end{aligned} \quad (3.1)$$

where

$$\begin{aligned} E_x &= \sum_{i \in [p]} x_i \partial_{x_i}, & r_x^2 &= \sum_{i \in [p]} x_i^2, & \Delta_x &= \sum_{i \in [p]} \partial_{x_i}^2, \\ E_y &= \sum_{j \in [q]} y_j \partial_{y_j}, & r_y^2 &= \sum_{j \in [q]} y_j^2, & \Delta_y &= \sum_{j \in [q]} \partial_{y_j}^2. \end{aligned} \quad (3.2)$$

Now, taking account of the fact that our realization of the representation operators of \mathfrak{g} given in (2.7) is a partial Fourier transform of the ones given in [6, 12] as we mentioned in Remark 2.2 (i) above, we define elements H, X^+, X^- of $\mathcal{PD}(V)$ by

$$H = -E_x - \frac{p}{2} + E_y + \frac{q}{2}, \quad X^+ = -\frac{1}{2}(\Delta_x + r_y^2), \quad X^- = \frac{1}{2}(r_x^2 + \Delta_y). \quad (3.3)$$

Then, it is immediate to see that the commutation relations among them are given by

$$[H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = H.$$

Proposition 3.1. *Let $\mathfrak{g}' := \mathbb{C}\text{-span}\{H, X^+, X^-\}$. Then \mathfrak{g}' is a Lie subalgebra of $\mathcal{PD}(V)^\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2 (= \mathfrak{sl}_2(\mathbb{C}))$, where $\mathcal{PD}(V)^\mathfrak{g}$ denotes the commutant of \mathfrak{g} in $\mathcal{PD}(V)$.*

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Proof. Note that $\pi(X_{i,j}^+)$, $i, j \in [p]$, span the Lie subalgebra isomorphic to \mathfrak{o}_p commuting with E_x, Δ_x and r_x^2 , and that $\pi(X_{i,j}^+)$, $i, j \in [q]$, span the Lie subalgebra isomorphic to \mathfrak{o}_q commuting with E_y, Δ_y and r_y^2 . Hence, it remains to show that each $\pi(X_{i,j}^-)$ commutes with H, X^+ and X^- given in (3.3)

We will only show here that $[\pi(X_{i,j}^-), X^+] = 0$. The other cases can be shown similarly. Now, one sees

$$\begin{aligned} -2i [\pi(X_{i,j}^-), X^+] &= [x_i y_j + \partial_{x_i} \partial_{y_j}, -\Delta_x - r_y^2] \\ &= \sum_{k=1}^p [\partial_{x_k}^2, x_i] y_j - \sum_{l=1}^q \partial_{x_i} [\partial_{y_j}, y_l^2] \\ &= \sum_{k=1}^p 2\delta_{k,i} \partial_{x_k} y_j - \sum_{l=1}^q 2\partial_{x_i} \delta_{j,l} y_l \\ &= 2\partial_{x_i} y_j - 2\partial_{x_i} y_j = 0. \end{aligned}$$

This completes the proof. \square

If one denotes the Casimir element of \mathfrak{g}' by $\Omega_{\mathfrak{g}'}$, then the corresponding Casimir operator that is defined by

$$\begin{aligned} \pi(\Omega_{\mathfrak{g}'}) &= H^2 + 2(X^+ X^- + X^- X^+) \\ &= H^2 - 2H + 4X^+ X^- \\ &= H^2 + 2H + 4X^- X^+ \end{aligned}$$

is concretely written in terms of the operators given by (3.2) as follows:

$$\begin{aligned} \pi(\Omega_{\mathfrak{g}'}) &= (E_x - E_y)^2 + (p - q)(E_x - E_y) - 2(E_x + E_y) \\ &\quad - \left(r_x^2 r_y^2 + r_x^2 \Delta_x + r_y^2 \Delta_y + \Delta_x \Delta_y \right) + \frac{1}{4}(p - q)^2 - (p + q). \end{aligned} \quad (3.4)$$

It follows from (3.1) and (3.4) that

$$\pi(\Omega_{\mathfrak{g}}) = \pi(\Omega_{\mathfrak{g}'}) - \frac{1}{4}(p + q)^2 + (p + q)$$

(see [4, 12]).

In what follows, we denote by $\mathcal{H}^k(\mathbb{R}^n)$ the space of homogeneous harmonic polynomials on \mathbb{R}^n of degree k and set $\mathcal{H}(\mathbb{R}^n) := \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{R}^n)$. It is well known that $\mathcal{H}^k(\mathbb{R}^n)$ is an irreducible $O(n)$ -module and its dimension is given by

$$\begin{aligned} \dim \mathcal{H}^k(\mathbb{R}^n) &= \binom{k+n-1}{n-1} - \binom{k+n-3}{n-1} \\ &= \frac{(k+n-3)!}{k!(n-2)!} (2k+n-2) \end{aligned}$$

if $n \geq 2$ and $k \in \mathbb{N}$, where $\binom{\nu}{i}$ denotes the binomial coefficient. Note that it can be further rewritten as

$$\dim \mathcal{H}^k(\mathbb{R}^n) = \frac{2(k+n/2-1)}{(n-2)!} (k+1)(k+2) \cdots (k+n-3). \quad (3.5)$$

Now, we will find a highest weight vector with respect to the \mathfrak{g}' -action (3.3), i.e. a function f on V which satisfies

$$Hf = \lambda f \quad \text{and} \quad X^+ f = 0 \quad (3.6)$$

for some $\lambda \in \mathbb{C}$. Taking account of the fact that the algebra of polynomial functions on V , which we denote by $\mathcal{P}(V)$, can be written as

$$\begin{aligned} \mathcal{P}(V) &\simeq \mathbb{C}[x_1, \dots, x_p] \otimes \mathbb{C}[y_1, \dots, y_q] \\ &\simeq \bigoplus_{k=0}^{\infty} \left(\mathbb{C}[r_x^2] \otimes \mathcal{H}^k(\mathbb{R}^p) \right) \otimes \bigoplus_{l=0}^{\infty} \left(\mathbb{C}[r_y^2] \otimes \mathcal{H}^l(\mathbb{R}^q) \right) \\ &\simeq \bigoplus_{k,l=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \otimes \mathbb{C}[r_x^2, r_y^2], \end{aligned}$$

we will seek for a function that satisfies (3.6) of the form

$$f(x, y) = h_1(x)h_2(y)\phi(r_x^2, r_y^2), \quad (3.7)$$

where $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ are harmonic polynomials, and $\phi(s, t) \in \mathbb{C}[[s, t]]$ is a formal power series (**Caution:** we do *not* assume that ϕ is a *polynomial*). Namely, our function f on V lives in the space $\tilde{\mathcal{E}}$ defined by

$$\tilde{\mathcal{E}} := \bigoplus_{k,l=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \otimes \mathbb{C}[[r_x^2, r_y^2]] \quad (\text{algebraic direct sum}). \quad (3.8)$$

Recall that the action π of \mathfrak{k}_0 lifts to the action of K on $\tilde{\mathcal{E}}$ as we mentioned in Remark 2.2 (ii), which we denote by the same letter π .

Lemma 3.1. *Let $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ and $r^2 = \sum_{i=1}^n x_i^2$. For a homogeneous harmonic polynomial $h = h(x_1, \dots, x_n)$ on \mathbb{R}^n of degree d and for a smooth function $\varphi(u)$ in a single variable u , we have*

$$\Delta(h\varphi(r^2)) = (4d + 2n)h\varphi'(r^2) + 4r^2 h\varphi''(r^2). \quad (3.9)$$

Proof. Since $\partial_{x_i} \varphi(r^2) = 2x_i \varphi'(r^2)$ and $\partial_{x_i}^2 \varphi(r^2) = 2\varphi'(r^2) + 4x_i^2 \varphi''(r^2)$, one obtains

$$\Delta\varphi(r^2) = 2n\varphi'(r^2) + 4r^2 \varphi''(r^2).$$

Thus,

$$\begin{aligned} \Delta(h\varphi(r^2)) &= \sum_{i=1}^n \left(\partial_{x_i}^2 h \cdot \varphi(r^2) + 2\partial_{x_i} h \cdot \partial_{x_i} \varphi(r^2) + h \cdot \partial_{x_i}^2 \varphi(r^2) \right) \\ &= 4dh\varphi'(r^2) + h\Delta\varphi(r^2) \\ &= 4dh\varphi'(r^2) + h \left(2n\varphi'(r^2) + 4r^2 \varphi''(r^2) \right) \\ &= (4d + 2n)h\varphi'(r^2) + 4r^2 h\varphi''(r^2). \end{aligned}$$

This completes the proof. \square

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For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ (resp. $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$), we define its shifted degree by $\kappa_+(h_1) := k + p/2$ (resp. $\kappa_-(h_2) := l + q/2$), which we denote just by κ_+ (resp. κ_-) if there is no risk of confusion.

It follows from Lemma 3.1 that if f is of the form in (3.7) then

$$\begin{aligned} X^+ f &= -\frac{1}{2}(\Delta_x(h_1 h_2 \phi) + r_y^2 h_1 h_2 \phi) \\ &= -2h_1 h_2 \left(r_x^2 (\partial_s^2 \phi)(r_x^2, r_y^2) + \kappa_+ (\partial_s \phi)(r_x^2, r_y^2) + \frac{r_y^2}{4} \phi(r_x^2, r_y^2) \right), \end{aligned}$$

which shows that $f = h_1(x)h_2(y)\phi(r_x^2, r_y^2)$ satisfies $X^+ f = 0$ if and only if ϕ is a solution to a differential equation

$$s\partial_s^2 \phi + \kappa_+ \partial_s \phi + \frac{t}{4} \phi = 0 \quad (3.10)$$

with $\kappa_+ = \kappa_+(h_1) = k + p/2$. Solving the differential equation (3.10) by power series, one obtains that

$$\phi(s, t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (\kappa_+)_n} \left(\frac{st}{4} \right)^n, \quad (3.11)$$

where a_0 is an arbitrary formal power series in t . Note that if one defines a power series Ψ_α by

$$\Psi_\alpha(u) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (\alpha)_n} u^n = 1 - \frac{u}{\alpha} + \frac{u^2}{2! \alpha(\alpha+1)} - \frac{u^3}{3! \alpha(\alpha+1)(\alpha+2)} + \cdots \quad (3.12)$$

for $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$, then it converges on the whole \mathbb{C} and is a unique solution to a differential equation

$$u\Psi_\alpha''(u) + \alpha\Psi_\alpha'(u) + \Psi_\alpha(u) = 0 \quad (3.13)$$

that satisfies the initial condition $\Psi_\alpha(0) = 1$.

In the sequel, we set

$$\psi_\alpha^{(n)} := \Psi_\alpha^{(n)}(r_x^2 r_y^2 / 4) \quad (n \in \mathbb{N}) \quad (3.14)$$

for brevity, where $\Psi_\alpha^{(n)}(u)$ denotes the n -th derivative of $\Psi_\alpha(u)$ in u .

If, in addition, f satisfies that $Hf = \lambda f$ for some $\lambda \in \mathbb{C}$, then the factor a_0 in (3.11) is equal to t^{μ_-} up to a constant multiple, with $\mu_- = (1/2)(\lambda + \kappa_+ - \kappa_-) \in \mathbb{N}$, $\kappa_+ = \kappa_+(h_1)$ and $\kappa_- = \kappa_-(h_2)$.

Thus, for given $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, if $f = f(x, y)$ of the form (3.7) is a highest weight vector with respect to the \mathfrak{g}' -action, i.e. it satisfies (3.6), then $\phi(r_x^2, r_y^2)$ is a constant multiple of $r_y^{2\mu_-} \psi_{\kappa_+}$, and the weight of f is given by

$$\lambda = -\kappa_+ + \kappa_- + 2\mu_- \quad (\mu_- \in \mathbb{N}). \quad (3.15)$$

Similarly, for given $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, one can show that if $f = f(x, y)$ of the form (3.7) is a lowest weight vector with respect to the \mathfrak{g}' -action, i.e. it satisfies

$$Hf = \lambda f \quad \text{and} \quad X^- f = 0 \quad (3.16)$$

for some $\lambda \in \mathbb{C}$, then $\phi(r_x^2, r_y^2)$ is a constant multiple of $r_x^{2\mu_+} \psi_{\kappa_-}$, and the weight of f is given by

$$\lambda = -\kappa_+ + \kappa_- - 2\mu_+ \quad (\mu_+ \in \mathbb{N}). \quad (3.17)$$

Let us summarize the above argument in the following.

Proposition 3.2. *Given $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, let $f = f(x, y)$ be a function of the form*

$$f(x, y) = h_1(x)h_2(y)\phi(r_x^2, r_y^2) \quad (\phi(r_x^2, r_y^2) \in \mathbb{C}[[r_x^2, r_y^2]]). \quad (3.18)$$

If f given in (3.18) is a highest (resp. lowest) weight vector with respect to the \mathfrak{g}' -action, i.e. it satisfies (3.6) (resp. (3.16)), then $\phi(r_x^2, r_y^2)$ is a constant multiple of $r_y^{2\mu_-} \psi_{\kappa_+}$ (resp. $r_x^{2\mu_+} \psi_{\kappa_-}$) and the weight λ of f is equal to $-\kappa_+ + \kappa_- + 2\mu_-$ (resp. $-\kappa_+ + \kappa_- - 2\mu_+$).

Here $\kappa_+ = \kappa_+(h_1) = k + p/2$, $\kappa_- = \kappa_-(h_2) = l + q/2$, $\mu_+, \mu_- \in \mathbb{N}$, and $\psi_{\kappa_{\pm}}$ is an element of $\mathbb{C}[[r_x^2, r_y^2]]$ given by (3.14) with $\alpha = \kappa_{\pm}$.

Taking account of the discussion so far, let us introduce the subspace \mathcal{E} of $\tilde{\mathcal{E}}$ by

$$\mathcal{E} := \mathbb{C}\text{-span} \left\{ h_1(x)h_2(y)r_x^{2a}r_y^{2b}\psi_{\alpha} \in \tilde{\mathcal{E}} \mid \begin{array}{l} h_1 \in \mathcal{H}(\mathbb{R}^p), h_2 \in \mathcal{H}(\mathbb{R}^q), \\ a, b \in \mathbb{N}, \alpha \in \mathbb{C} \setminus (-\mathbb{N}) \end{array} \right\}.$$

Then one will find that \mathcal{E} is stable under the action of (\mathfrak{g}, K) as well as that of \mathfrak{g}' (see Propositions 3.3 and 3.4 below).

Remark 3.1. (i) The function Ψ_{α} given in (3.12) can be written in terms of the Bessel function J_{ν} of the first kind of order ν

$$J_{\nu}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left(\frac{t}{2}\right)^{\nu+2n}$$

that solves the Bessel's differential equation

$$\frac{d^2 w}{dt^2} + \frac{1}{t} \frac{dw}{dt} + \left(1 - \frac{\nu^2}{t^2}\right) w = 0 \quad (3.19)$$

(see e.g. [14]). Namely, one has

$$\Psi_{\alpha}(u) = \Gamma(\alpha) u^{-(\alpha-1)/2} J_{\alpha-1}(2u^{1/2}). \quad (3.20)$$

Therefore,

$$\psi_{\alpha} = \Gamma(\alpha) \left(\frac{r_x r_y}{2}\right)^{-(\alpha-1)} J_{\alpha-1}(r_x r_y).$$

Note that (3.13) corresponds to (3.19) under (3.20).

(ii) Recall that our representation π is related to π^{\sharp} via the partial Fourier transform with respect to y_1, \dots, y_q , as we mentioned in Remark 2.2 (i). Namely, one can obtain π^{\sharp} by replacing $-i \partial_{y_j}$ and y_j in π by η_j and $i \partial_{\eta_j}$, $j = 1, \dots, q$, respectively. Under this correspondence, one finds that $H = -E_x - p/2 + E_y + q/2$ and $X^+ = -\frac{1}{2}(\Delta_x + r_y^2)$ transform,

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up to constant multiples, into the shifted degree operator $\tilde{E}_{p,q}$ and the d'Alembertian $\square_{p,q}$ on \mathbb{R}^{p+q} that are given by

$$\begin{aligned}\tilde{E}_{p,q} &= \sum_{i=1}^p x_i \partial_{x_i} + \sum_{j=1}^q \eta_j \partial_{\eta_j} + \frac{p+q}{2}, \\ \square_{p,q} &= \sum_{i=1}^p \partial_{x_i}^2 - \sum_{j=1}^q \partial_{\eta_j}^2,\end{aligned}$$

respectively. Therefore, the highest weight vector f satisfying $Hf = \lambda f$ for some $\lambda \in \mathbb{C}$ and $X^+ f = 0$ corresponds to a homogeneous solution \tilde{f} to the equation $\square_{p,q} \tilde{f} = 0$.

Note that $\Psi_\alpha^{(n)}$ is equal to $\Psi_{\alpha+n}$ up to a constant multiple. In fact, differentiating both sides of (3.13) n times, one obtains

$$u\Psi_\alpha^{(n+2)}(u) + (\alpha+n)\Psi_\alpha^{(n+1)}(u) + \Psi_\alpha^{(n)}(u) = 0. \quad (3.21)$$

Since $\Psi_{\alpha+n}$ is a unique solution to (3.13) with α replaced by $\alpha+n$ that satisfies $\Psi_{\alpha+n}(0) = 1$, it follows that $\Psi_\alpha^{(n)} = (-1)^n / (\alpha)_n \Psi_{\alpha+n}$. Thus, one obtains

$$\psi_\alpha^{(n)} = \frac{(-1)^n}{(\alpha)_n} \psi_{\alpha+n} \quad (n \in \mathbb{N}). \quad (3.22)$$

In what follows, we set $\rho_x := r_x^2/2$ and $\rho_y := r_y^2/2$ for economy of space. Then, it follows from (3.21) and (3.22) that

$$\rho_x \rho_y \psi_{\alpha+2} = \alpha(\alpha+1)(\psi_{\alpha+1} - \psi_\alpha) \quad (3.23)$$

for $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$. Furthermore, setting $\rho := r^2/2$, one can rewrite (3.9) as

$$\frac{1}{2} \Delta(h\varphi(\rho)) = \left(d + \frac{n}{2}\right) h\varphi'(\rho) + h\rho\varphi''(\rho), \quad (3.24)$$

where h, Δ, r^2 and φ are as in Lemma 3.1.

Proposition 3.3. *For $f = h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha \in \mathcal{E}$, one has*

$$H(h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha) = (-\kappa_+ + \kappa_- - 2a + 2b) h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha, \quad (3.25)$$

$$X^+(h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha) = h_1 h_2 \left(-a(\kappa_+ + a - 1) \rho_x^{a-1} \rho_y^b \psi_\alpha + \frac{\kappa_+ + 2a - \alpha}{\alpha} \rho_x^a \rho_y^{b+1} \psi_{\alpha+1} \right), \quad (3.26)$$

$$X^-(h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha) = h_1 h_2 \left(b(\kappa_- + b - 1) \rho_x^a \rho_y^{b-1} \psi_\alpha - \frac{\kappa_- + 2b - \alpha}{\alpha} \rho_x^{a+1} \rho_y^b \psi_{\alpha+1} \right). \quad (3.27)$$

In particular, the \mathfrak{g}' -action preserves the K -type of each element of \mathcal{E} .

Proof. It is immediate to show (3.25), and we will only show (3.26) here; the other case (3.27) can be shown similarly.

Setting $\varphi(u) := u^a \Psi_\alpha(\rho_y u)$, one sees

$$\begin{aligned}\varphi'(u) &= a u^{a-1} \Psi_\alpha(\rho_y u) + u^a \rho_y \Psi'_\alpha(\rho_y u), \\ \varphi''(u) &= a(a-1) u^{a-2} \Psi_\alpha(\rho_y u) + 2a u^{a-1} \rho_y \Psi'_\alpha(\rho_y u) + u^a \rho_y^2 \Psi''_\alpha(\rho_y u).\end{aligned}$$

Hence it follows from (3.24) that

$$\begin{aligned} \frac{1}{2}\Delta_x(h_1\rho_x^a\psi_\alpha) &= a(\kappa_+ + a - 1)h_1\rho_x^{a-1}\psi_\alpha + (\kappa_+ + 2a)h_1\rho_x^a\rho_y\psi'_\alpha + h_1\rho_x^{a+1}\rho_y^2\psi''_\alpha \\ &= h_1\left(a(\kappa_+ + a - 1)\rho_x^{a-1}\psi_\alpha + (\kappa_+ + 2a - \alpha)\rho_x^a\rho_y\psi'_\alpha - \rho_x^a\rho_y\psi_\alpha\right) \end{aligned}$$

since $\rho_x\rho_y\psi''_\alpha = -\alpha\psi'_\alpha - \psi_\alpha$. Therefore, one obtains that

$$\begin{aligned} X^+(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) &= -\frac{1}{2}(\Delta_x + 2\rho_y)(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) \\ &= -a(\kappa_+ + a - 1)h_1h_2\rho_x^{a-1}\rho_y^b\psi_\alpha - (\kappa_+ + 2a - \alpha)h_1h_2\rho_x^a\rho_y^{b+1}\psi'_\alpha, \end{aligned}$$

which, by (3.22), equals the right-hand side of (3.26). This completes the proof. \square

We conclude this section by calculating the action of \mathfrak{p} on \mathcal{E} , i.e. $\pi(X_{i,j}^-)f$ for $X_{i,j}^- \in \mathfrak{p}$ and $f \in \mathcal{E}$.

For a homogeneous polynomial $P = P(x_1, \dots, x_n)$ on \mathbb{R}^n of degree d , set

$$P^\dagger := P - \frac{r^2}{4(d + n/2 - 2)}\Delta P,$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ and $r^2 = \sum_{i=1}^n x_i^2$. Note that if $\Delta^2 P = 0$ then P^\dagger is harmonic by Lemma 3.1, and that if $h = h(x_1, \dots, x_n)$ is harmonic then $\Delta(x_i h) = 2\partial_{x_i} h$ and $\Delta^2(x_i h) = 0$.

Proposition 3.4. *For $f = h_1h_2\rho_x^a\rho_y^b\psi_\alpha \in \mathcal{E}$, one has*

$$\begin{aligned} & -i\pi(X_{i,j}^-)(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) \\ &= (\partial_{x_i}h_1)(\partial_{y_j}h_2)\rho_x^a\rho_y^b\left(\frac{(\kappa_+ + a - \alpha)(\kappa_- + b - \alpha)}{(\kappa_+ - 1)(\kappa_- - 1)}\psi_\alpha + \frac{(\alpha - 1)(\kappa_+ + \kappa_- + a + b - \alpha - 1)}{(\kappa_+ - 1)(\kappa_- - 1)}\psi_{\alpha-1}\right) \\ & \quad + (\partial_{x_i}h_1)(y_jh_2)^\dagger\left(-\frac{\kappa_+ + a + b - \alpha}{\alpha(\kappa_+ - 1)}\rho_x^{a+1}\rho_y^b\psi_{\alpha+1} + \frac{b(\kappa_+ + a - 1)}{\kappa_+ - 1}\rho_x^a\rho_y^{b-1}\psi_\alpha\right) \\ & \quad + (x_ih_1)^\dagger(\partial_{y_j}h_2)\left(-\frac{\kappa_- + a + b - \alpha}{\alpha(\kappa_- - 1)}\rho_x^a\rho_y^{b+1}\psi_{\alpha+1} + \frac{a(\kappa_- + b - 1)}{\kappa_- - 1}\rho_x^{a-1}\rho_y^b\psi_\alpha\right) \\ & \quad + (x_ih_1)^\dagger(y_jh_2)^\dagger\left(-\frac{a+b+1-\alpha}{\alpha}\rho_x^a\rho_y^b\psi_{\alpha+1} + ab\rho_x^{a-1}\rho_y^{b-1}\psi_\alpha\right), \end{aligned} \quad (3.28)$$

where one regards $\partial_{x_i}h_1/(\kappa_+ - 1)$ (resp. $\partial_{y_j}h_2/(\kappa_- - 1)$) as zero when $\kappa_+ = 1$ (resp. $\kappa_- = 1$).

Proof. Since $\partial_{x_i}\psi_\alpha = \rho_y x_i\psi'_\alpha$ and $\partial_{y_j}\psi_\alpha = \rho_x y_j\psi'_\alpha$, one obtains

$$\begin{aligned} & -i\pi(X_{i,j}^-)f = (\partial_{x_i}\partial_{y_j} + x_i y_j)(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) \\ &= (\partial_{x_i}h_1)(\partial_{y_j}h_2)\rho_x^a\rho_y^b\psi_\alpha + (x_ih_1)(\partial_{y_j}h_2)\left(a\rho_x^{a-1}\rho_y^b\psi_\alpha + \rho_x^a\rho_y^{b+1}\psi'_\alpha\right) \\ & \quad + (\partial_{x_i}h_1)(y_jh_2)\left(b\rho_x^a\rho_y^{b-1}\psi_\alpha + \rho_x^{a+1}\rho_y^b\psi'_\alpha\right) \\ & \quad + (x_ih_1)(y_jh_2)\left(\rho_x^a\rho_y^b\psi_\alpha + ab\rho_x^{a-1}\rho_y^{b-1}\psi_\alpha + (a+b+1)\rho_x^a\rho_y^b\psi'_\alpha + \rho_x^{a+1}\rho_y^{b+1}\psi''_\alpha\right). \end{aligned} \quad (3.29)$$

Now, by definition, one has

$$x_ih_1 = (x_ih_1)^\dagger + \frac{\rho_x}{\kappa_+ - 1}\partial_{x_i}h_1 \quad \text{and} \quad y_jh_2 = (y_jh_2)^\dagger + \frac{\rho_y}{\kappa_- - 1}\partial_{y_j}h_2. \quad (3.30)$$

Substituting (3.30) into (3.29), and using the relations (3.22) and (3.23), one sees that the coefficient of $(\partial_{x_i} h_1)(\partial_{y_j} h_2)$ in (3.29) equals the one of $(\partial_{x_i} h_1)(\partial_{y_j} h_2)$ in the right-hand side of (3.28). One can verify that each coefficient of $(\partial_{x_i} h_1)(y_j h_2)^\dagger, (x_i h_1)^\dagger(\partial_{y_j} h_2)$ and $(x_i h_1)^\dagger(y_j h_2)^\dagger$ in (3.29) equals the one of the corresponding terms in (3.28) similarly. This completes the proof. \square

4. (\mathfrak{g}, K) -module associated with finite-dimensional \mathfrak{sl}_2 -module

If a nonzero $f \in \mathcal{E}$ satisfies $Hf = \lambda f$, $X^+ f = 0$ and $(X^-)^{m+1} f = 0$ (resp. $Hf = \lambda f$, $X^- f = 0$ and $(X^+)^{m+1} f = 0$) for some $m \in \mathbb{N}$, then it follows from the representation theory of $\mathfrak{g}' = \mathfrak{sl}_2$ that $\lambda = m$ (resp. $\lambda = -m$). Thus, we introduce (\mathfrak{g}, K) -modules associated with the finite-dimensional \mathfrak{sl}_2 -module F_m of dimension $m + 1$ as follows, which are the main objects of this paper.

Definition 4.1. Given $m \in \mathbb{N}$, we define (\mathfrak{g}, K) -modules $M^\pm(m)$ by

$$\begin{aligned} M^+(m) &:= \{f \in \mathcal{E} \mid Hf = mf, X^+ f = 0, (X^-)^{m+1} f = 0\}, \\ M^-(m) &:= \{f \in \mathcal{E} \mid Hf = -mf, X^- f = 0, (X^+)^{m+1} f = 0\}. \end{aligned}$$

The modules $M^\pm(m)$ were originally introduced in [12] without the condition of finite dimensionality. Note that $M^+(0)$ is identical to $M^-(0)$ by definition and that both $M^\pm(m)$ should correspond to the \mathfrak{sl}_2 -module F_m under the Howe duality (cf. [5]).

If $M^+(m) \neq \{0\}$ (resp. $M^-(m) \neq \{0\}$), then one sees that $p \equiv q \pmod{2}$; for, if one takes a nonzero $f = h_1 h_2 \rho_y^{\mu_+} \psi_{\kappa_+} \in M^+(m)$ (resp. $f = h_1 h_2 \rho_x^{\mu_-} \psi_{\kappa_-} \in M^-(m)$) with $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ and $\mu_\pm \in \mathbb{N}$, then

$$\pm m = -\kappa_+ + \kappa_- \pm 2\mu_\mp = -k + l - \frac{p-q}{2} \pm 2\mu_\mp \in \mathbb{Z} \quad (4.1)$$

by (3.15) (resp. (3.17)). Hence one obtains $(p-q)/2 \in \mathbb{Z}$. Therefore, we assume that $p \equiv q \pmod{2}$ in the rest of this paper.

Lemma 4.1. For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ and $m \in \mathbb{N}$, let

$$v^+ = h_1 h_2 \rho_y^{\mu_+} \psi_{\kappa_+} \in M^+(m) \quad \text{and} \quad v^- = h_1 h_2 \rho_x^{\mu_-} \psi_{\kappa_-} \in M^-(m), \quad (4.2)$$

where $\mu_+, \mu_- \in \mathbb{N}$ such that $\mu_+ + \mu_- = m$. Then the \mathfrak{g}' -module generated by v^+ coincides with the one generated by v^- :

$$\langle v^+ \rangle_{\mathfrak{g}'} = \langle v^- \rangle_{\mathfrak{g}'}$$

Proof. Both v^+ and $(X^+)^m v^-$ (resp. v^- and $(X^-)^m v^+$) are elements of $\mathcal{E} \subset \tilde{\mathcal{E}}$ that are highest (resp. lowest) weight vectors of weight m (resp. $-m$) under \mathfrak{g}' -action. Namely, they are solutions in $\tilde{\mathcal{E}}$ to the differential equation

$$Hf = \pm m f \quad \text{and} \quad X^\pm f = 0.$$

As we mentioned in Proposition 3.2, they are respectively equal to each other up to a constant multiple. This completes the proof. \square

Proposition 4.1. For $m \in \mathbb{N}$, $M^+(m)$ and $M^-(m)$ are isomorphic to each other.

Proof. For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, set

$$v^+ = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in M^+(m) \quad \text{and} \quad v^- = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-} \in M^-(m).$$

with $\mu_+ + \mu_- = m$ as in (4.2). Then, $(X^+)^m v^-$ (resp. $(X^-)^m v^+$) is equal to v^+ (resp. v^-) up to a constant multiple as we saw in Lemma 4.1, and thus, $(X^+)^m (X^-)^m v^+$ is equal to v^+ up to a constant multiple. In fact, $(X^+)^m (X^-)^m v^+ = (m!)^2 v^+$. Therefore,

$$(X^-)^m : M^+(m) \longrightarrow M^-(m)$$

provides an isomorphism of (g, K)-module. This completes the proof. \square

Now we prepare two lemmas to prove our main result. Note that Lemma 4.2 below is just a special case of Proposition 3.4. However, we state it separately to highlight the rôle of the relation (3.23).

Lemma 4.2. Let $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, and set $\kappa_+ = k + p/2$, $\kappa_- = l + q/2$.

(1) For a highest weight vector $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in \mathcal{E}$, $\pi(X_{i,j}^-)f$ is given by

$$\begin{aligned} -i \pi(X_{i,j}^-)(h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+}) &= \frac{\kappa_- + \mu_- - 1}{\kappa_- - 1} (\partial_{x_i} h_1) (\partial_{y_j} h_2) \rho_y^{\mu_-} \psi_{\kappa_+ - 1} \\ &\quad + \mu_- (\partial_{x_i} h_1) (y_j h_2)^\dagger \rho_y^{\mu_- - 1} \psi_{\kappa_+ - 1} \\ &\quad + \frac{\kappa_+ - \kappa_- - \mu_-}{\kappa_+ (\kappa_- - 1)} (x_i h_1)^\dagger (\partial_{y_j} h_2) \rho_y^{\mu_- + 1} \psi_{\kappa_+ + 1} \\ &\quad + \frac{\kappa_+ - \mu_- - 1}{\kappa_+} (x_i h_1)^\dagger (y_j h_2)^\dagger \rho_y^{\mu_-} \psi_{\kappa_+ + 1}. \end{aligned} \quad (4.3)$$

(2) For a lowest weight vector $f = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-}$, $\pi(X_{i,j}^-)f$ is given by

$$\begin{aligned} -i \pi(X_{i,j}^-)(h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-}) &= \frac{\kappa_+ + \mu_+ - 1}{\kappa_+ - 1} (\partial_{x_i} h_1) (\partial_{y_j} h_2) \rho_x^{\mu_+} \psi_{\kappa_- - 1} \\ &\quad + \frac{\kappa_- - \kappa_+ - \mu_+}{\kappa_- (\kappa_+ - 1)} (\partial_{x_i} h_1) (y_j h_2)^\dagger \rho_x^{\mu_+ + 1} \psi_{\kappa_- + 1} \\ &\quad + \mu_+ (x_i h_1)^\dagger (\partial_{y_j} h_2) \rho_x^{\mu_+ - 1} \psi_{\kappa_- - 1} \\ &\quad + \frac{\kappa_- - \mu_+ - 1}{\kappa_-} (x_i h_1)^\dagger (y_j h_2)^\dagger \rho_x^{\mu_+} \psi_{\kappa_- + 1}. \end{aligned} \quad (4.4)$$

Proof. We only show (4.3) here. The other formula (4.4) can be shown similarly.

Set $a = 0$, $b = \mu_-$ and $\alpha = \kappa_+$ in (3.28). Then, using the relation (3.23) with $\alpha = \kappa_+ - 1$, i.e.

$$\rho_x \rho_y \psi_{\kappa_+ + 1} = \kappa_+ (\kappa_+ - 1) (\psi_{\kappa_+} - \psi_{\kappa_+ - 1}),$$

one sees that the coefficient of $(\partial_{x_i} h_1) (y_j h_2)^\dagger$ equals

$$\begin{aligned} & - \frac{\mu_-}{\kappa_+ (\kappa_+ - 1)} \rho_x \rho_y^{\mu_-} \psi_{\kappa_+ + 1} + \mu_- \rho_y^{\mu_- - 1} \psi_{\kappa_+} \\ &= \mu_- \rho_y^{\mu_- - 1} \psi_{\kappa_+ - 1}. \end{aligned}$$

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To show for the other coefficients is trivial and omitted. \square

Lemma 4.3. Let $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, and set $\kappa_+ = k + p/2$, $\kappa_- = l + q/2$.

(1) For a highest weight vector $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+}$ of weight $\lambda = -\kappa_+ + \kappa_- + 2\mu_-$, one has

$$(X^-)^\nu f = h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(-\lambda + \nu - 1)_i^- (\mu_-)_{\nu-i}^- (\kappa_- + \mu_- - 1)_{\nu-i}^-}{(\kappa_+)_i} \rho_x^i \rho_y^{\mu_- - \nu + i} \psi_{\kappa_+ + i} \quad (4.5)$$

for $\nu = 0, 1, 2, \dots$

(2) For a lowest weight vector $f = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-}$ of weight $\lambda = -\kappa_+ + \kappa_- - 2\mu_+$, one has

$$(X^+)^\nu f = (-1)^\nu h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + \nu - 1)_i^- (\mu_+)_{\nu-i}^- (\kappa_+ + \mu_+ - 1)_{\nu-i}^-}{(\kappa_-)_i} \rho_x^{\mu_+ - \nu + i} \rho_y^i \psi_{\kappa_- + i} \quad (4.6)$$

for $\nu = 0, 1, 2, \dots$

Proof. We only show (4.6) by induction on ν here. The other case (4.5) can be shown similarly.

It is trivial if $\nu = 0$, and it is nothing but Propostion 3.3 if $\nu = 1$. Assume that it is true for $\nu \geq 1$, and apply X^+ to the both sides of (4.6). Then, one sees that the right-hand side equals

$$\begin{aligned} & (-1)^\nu h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + \nu - 1)_i^- (\mu_+)_{\nu-i}^- (\kappa_+ + \mu_+ - 1)_{\nu-i}^-}{(\kappa_-)_i} \\ & \times \left(-(\mu_+ - \nu + i)(\kappa_+ + \mu_+ - \nu + i - 1) \rho_x^{\mu_+ - \nu + i - 1} \rho_y^i \psi_{\kappa_- + i} + \frac{-\lambda + i - 2\nu}{\kappa_- + i} \rho_x^{\mu_+ - \nu + i} \rho_y^{i+1} \psi_{\kappa_- + i + 1} \right). \end{aligned} \quad (4.7)$$

The coefficient of $(-1)^\nu h_1 h_2 \rho_x^{\mu_+ - \nu + j - 1} \rho_y^j \psi_{\kappa_- + j}$ in (4.7), $j = 0, 1, \dots, \nu + 1$, equals

$$\begin{aligned} & \binom{\nu}{j} \frac{(\lambda + \nu - 1)_j^- (\mu_+)_{\nu-j}^- (\kappa_+ + \mu_+ - 1)_{\nu-j}^-}{(\kappa_-)_j} \cdot (-1)(\mu_+ - \nu + j)(\kappa_+ + \mu_+ - \nu + j - 1) \\ & + \binom{\nu}{j-1} \frac{(\lambda + \nu - 1)_{j-1}^- (\mu_+)_{\nu-j+1}^- (\kappa_+ + \mu_+ - 1)_{\nu-j+1}^-}{(\kappa_-)_{j-1}} \cdot \frac{-(\lambda + 2\nu - j + 1)}{\kappa_- + j - 1} \\ & = - \left\{ \binom{\nu}{j} (\lambda + \nu - j) + \binom{\nu}{j-1} (\lambda + 2\nu - j + 1) \right\} \\ & \quad \times \frac{(\mu_+)_{\nu-j+1}^- (\kappa_+ + \mu_+ - 1)_{\nu-j+1}^- (\lambda + \nu - 1)_{j-1}^-}{(\kappa_-)_j} \\ & = - \binom{\nu+1}{j} \frac{(\lambda + \nu)_j^- (\mu_+)_{\nu-j+1}^- (\kappa_+ + \mu_+ - 1)_{\nu-j+1}^-}{(\kappa_-)_j}. \end{aligned}$$

This completes the proof. \square

The following is our main result.

Theorem 4.1. Assume that $p \geq 1$, $q \geq 1$ and $p + q \in 2\mathbb{N}$. Let $m \in \mathbb{N}$ be a nonnegative integer satisfying $m + 3 \leq (p + q)/2$. Then one has the following.

The K -type formula of $M^\pm(m)$ is given by

$$M^\pm(m)|_K \simeq \bigoplus_{\substack{k, l \geq 0 \\ k - l + \frac{p-q}{2} \in \Lambda_m}} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q), \quad (4.8)$$

where $\Lambda_m = \{-m, -m + 2, -m + 4, \dots, m - 2, m\}$, the set of H -weights of F_m ;

(2) Suppose further that $p, q \geq 2$. Then $M^\pm(m)$ are irreducible (\mathfrak{g}, K)-modules.

Proof. It suffices to show the theorem for $M^+(m)$. Let $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \neq 0$ be an element of $M^+(m)$, where $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$. Then by Lemma 4.3, one obtains

$$\begin{aligned} (X^-)^{m+1} f &= h_1 h_2 \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(0)_i^- (\mu_-)_{m+1-i}^- (\kappa_- + \mu_- - 1)_{m+1-i}^-}{(\kappa_+)_i} \rho_x^i \rho_y^{\mu_- - m - 1 + i} \psi_{\kappa_+ + i} \\ &= (\mu_-)_{m+1}^- (\kappa_- + \mu_- - 1)_{m+1}^- h_1 h_2 \rho_y^{\mu_- - m - 1} \psi_{\kappa_+}. \end{aligned}$$

Thus, $(X^-)^{m+1} f = 0$ implies that $(\mu_-)_{m+1}^- = 0$ or $(\kappa_- + \mu_- - 1)_{m+1}^- = 0$. Namely,

$$\mu_- = 0, 1, \dots, m, \quad \text{or} \quad (4.9)$$

$$\mu_- = -\kappa_- + 1, -\kappa_- + 2, \dots, -\kappa_- + m + 1. \quad (4.10)$$

The assumption that $m + 3 \leq (p + q)/2$, however, implies that (4.10) is impossible; if it holded true, then by (4.1), one would obtain

$$-m + 2 \leq \kappa_+ + \kappa_- \leq m + 2,$$

which contradicts $\kappa_+ + \kappa_- \geq (p + q)/2 \geq m + 3$. Therefore, it follows from (4.9) that

$$k - l + \frac{p - q}{2} = \kappa_+ - \kappa_- = -m + 2\mu_- \in \Lambda_m,$$

which proves (1).

Let us consider a closed subset $D_m \subset \mathbb{R}^2$ (with respect to the Euclidean topology of \mathbb{R}^2) given by

$$D_m = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1 \geq p/2, t_2 \geq q/2, |t_1 - t_2| \leq m\}, \quad (4.11)$$

and the set of integral points of D_m given by

$$\Sigma_m = \left\{ (t_1, t_2) \in D_m \mid \begin{array}{l} t_1 - p/2 \in \mathbb{N}, t_2 - q/2 \in \mathbb{N}, \\ t_1 - t_2 \in \Lambda_m \end{array} \right\}. \quad (4.12)$$

Note that the sum in the right-hand side of (4.8) can be written as the one with (κ_+, κ_-) running over the set Σ_m .

Now, applying (4.3) to $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in M^+(m)$, we denote the coefficient of

$$(\partial_{x_i} h_1)(\partial_{y_j} h_2), \quad (x_i h_1)^\dagger (\partial_{y_j} h_2), \quad (\partial_{x_i} h_1)(y_j h_2)^\dagger \quad \text{and} \quad (x_i h_1)^\dagger (y_j h_2)^\dagger$$

in the right-hand side of (4.3) by C_{--} , C_{+-} , C_{-+} and C_{++} respectively, where $\mu_- = 0, 1, \dots, m$. Namely,

$$\begin{aligned} C_{--} &= \frac{\kappa_- + \mu_- - 1}{\kappa_- - 1} \rho_y^{\mu_-} \psi_{\kappa_+ - 1}, & C_{-+} &= \mu_- \rho_y^{\mu_- - 1} \psi_{\kappa_+ - 1}, \\ C_{+-} &= \frac{\kappa_+ - \mu_- - \kappa_-}{\kappa_+ (\kappa_- - 1)} \rho_y^{\mu_- + 1} \psi_{\kappa_+ + 1}, & C_{++} &= \frac{\kappa_+ - \mu_- - 1}{\kappa_+} \rho_y^{\mu_-} \psi_{\kappa_+ + 1}. \end{aligned}$$

(i) First, let us consider the case where $(\kappa_+, \kappa_-) \in \Sigma_m$ is an interior point of D_m . Note then that κ_+ and κ_- are > 1 by the assumption that $p, q \geq 2$. Then, one obtains $\mu_- = 1, 2, \dots, m-1$ by (4.9). In particular, $C_{-+} \neq 0$. Now, $C_{--} = 0$ would imply $\mu_- = -\kappa_- + 1$, which contradicts $m+3 \leq (p+q)/2$ as we saw above. It also follows from (4.1) that $\kappa_- - \kappa_+ + \mu_- = m - \mu_-$, and $C_{+-} \neq 0$. Finally, $C_{++} = 0$ would imply that $\kappa_+ + \kappa_- = m+2$, which is absurd. Thus, all the coefficients in (4.3) never vanish.

(ii) Next, let us consider the case where $(\kappa_+, \kappa_-) \in \Sigma_m$ is in the boundary of D_m . Then there are three sub-cases:

- (ii-a) $\mu_- = 0$,
- (ii-b) $\mu_- = m$,
- (ii-c) $0 < \mu_- < m$ and $k = 0$ or $l = 0$.

In Case (ii-a), $C_{-+} = 0$, and, C_{--} , C_{+-} and C_{++} are nonzero by the same reason as Case (i). In Case (ii-b), $C_{+-} = 0$ since $\kappa_+ - \kappa_- = m$, and all the other coefficients are nonzero. In Case (ii-c), all the coefficients are nonzero, but $\partial_{x_i} h_1 = 0$ or $\partial_{y_j} h_2 = 0$.

Therefore, by applying $\pi(X)$, $X \in \mathfrak{p}$, one can move from any K -type in $M^+(m)$ to any other K -type in $M^+(m)$, while $\pi(X)$, $X \in \mathfrak{k}$, preserves the K -type of each element of $M^+(m)$. This completes the proof of (2), and of the theorem. \square

Example 4.1. Figure 1 below illustrates D_m in (4.11) and Σ_m in (4.12) in the case where $p = 14$, $q = 12$ and $m = 4$. The colored area and the dots sitting in the area indicates D_m and Σ_m respectively. Each K -type of $M^\pm(m)$ corresponds to a dot by the correspondence

$$\mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \longleftrightarrow (k + p/2, l + q/2).$$

Let us apply $\pi(X_{i,j}^-)$ to an element f of $M^\pm(m)$. Then, if the K -type of f corresponds to a white dot \circ in Fig. 1, one can move to any adjacent dots in the north-east, north-west, south-east, and south-west direction; if it corresponds to a black dot \bullet in Fig. 1, one can move only to adjacent dots in the interior or in the boundary of D_m .

Now, let us briefly recall the definitions of the Gelfand-Kirillov dimension and the Bernstein degree of a finitely generated $U(\mathfrak{g})$ -module M , where $U(\mathfrak{g})$ denotes the universal enveloping algebra of \mathfrak{g} . Namely, we choose a finite-dimensional subspace M_0 so that $M = U(\mathfrak{g})M_0$, and for each nonnegative integer n , we set $M_n := U_n(\mathfrak{g})M_0$, with $U_n(\mathfrak{g})$ denoting the subspace of $U(\mathfrak{g})$ spanned by products of at most n elements of \mathfrak{g} . Then there exists a polynomial $\psi_M(t) \in \mathbb{Q}[t]$ of degree $d-1$ such that

$$\psi_M(n) = \dim(M_n/M_{n-1}) \quad \text{for all sufficiently large } n.$$

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where, in each summand, μ_- is determined by $\mu_- = (1/2)(\kappa_+ - \kappa_- + m)$. If one sets $M_n := U_n(\mathfrak{g})M_0$ ($M_{-1} := 0$), then it follows from (4.8) and (3.5) that

$$\begin{aligned}
& \dim(M_n/M_{n-1}) \\
&= \sum_{j=0}^m \dim(\mathcal{H}^{n+j}(\mathbb{R}^p) \otimes \mathcal{H}^{n+m-j+\frac{p-q}{2}}(\mathbb{R}^q)) \\
&= 4 \sum_{j=0}^m \frac{n+j+\frac{p}{2}-1}{(p-2)!} (n+j+1)(n+j+2) \cdots (n+j+p-3) \\
&\quad \times \frac{n+m-j+\frac{p-q}{2}+\frac{q}{2}-1}{(q-2)!} (n+m-j+\frac{p-q}{2}+1)(n+m-j+\frac{p-q}{2}+2) \\
&\quad \quad \quad \cdots (n+m-j+\frac{p-q}{2}+q-3) \\
&= \frac{4(m+1)}{(p-2)!(q-2)!} n^{p+q-4} + (\text{lower order terms in } n) \tag{4.17}
\end{aligned}$$

for all $n \in \mathbb{N}$, which implies (4.13). Furthermore, since the leading term of (4.17) can be rewritten as

$$\frac{4(m+1)}{(p-2)!(q-2)!} n^{p+q-4} = \frac{4(m+1)(p+q-4)!}{(p-2)!(q-2)!} \frac{n^{p+q-4}}{(p+q-4)!},$$

one obtains (4.14). This completes the proof. \square

Remark 4.1. One can show that the nonnegative integer c in (4.15) is in fact equal to $\max\{m+p, m+q\}$.

It is well known that the Gelfand-Kirillov dimension of the minimal representation of $O(p, q)$ is equal to $p+q-3$ (cf. [8–10, 16]). The K -type formula (4.8) of $M^+(0) = M^-(0)$ in Theorem 4.1 shows that it corresponds to the (\mathfrak{g}, K) -module of the minimal representation of $O(p, q)$. However, as we have seen in Corollary 4.1, the Gelfand-Kirillov dimension of $M^\pm(m)$ is equal to $p+q-3$ for any $m \in \mathbb{N}$ satisfying $m+3 \leq (p+q)/2$. The Bernstein degree distinguishes the minimal representation from the others.

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