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The estimation of $\int_a^b g(x)e^{ith(x)}dx$ using saddle point method

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1 Abstarct

It is known [E or D] that $\int_a^b g(x)e^{ith(x)}dx = \sqrt{2\pi}\alpha\frac{1}{\sqrt{t|h''(c)|}}g(c)e^{ith(c)}\left(1+O\left(\frac{1}{t}\right)\right)$, where a is the complex number with modulus 1.

In this paper we have detailed results including the dependancy of O term. In [G], we apply the Theorem 1 and Theorem 2 (below) to estimate the order of $\sum_{n=1}^{N} e^{2\pi ih(\alpha n\log n+\beta n)}$.

2 Lemmas

The following Lemma 1, 2, and 3 runs the same lines as that of [D]. We treat carefully the dependency of O term.

Lemma 1. *Let α, β, a, and d be real numbers such that $d \neq 0$, $0 < \alpha + 1 < \beta$. Then we have, for* $b \to \infty$ or $t \to \infty$,

$$\int_0^b x^\alpha e^{it(a+dx^\beta)}dx = A\frac{e^{iat}}{\beta(|d|t)^{(\alpha+1)/\beta}} - \frac{1}{i}\cdot\frac{1}{d}b^{(\alpha+1)-\beta}\frac{1}{\beta}\cdot\frac{1}{t}e^{iN}e^{iat} + O\left(b^{(\alpha+1)-\beta}\frac{1}{\beta(dt)^2}\right),$$

where $N = db^\beta t$, the constant implied by the O is absolute and

$$A = \int_0^\infty x^{(\alpha+1)/\beta}e^{iu}du = \begin{cases} e^{\frac{1}{2}\frac{\alpha+1}{\beta}\pi i}\Gamma\left(\dfrac{\alpha+1}{\beta}\right) & \text{if } d>0 \\ e^{-\frac{1}{2}\frac{\alpha+1}{\beta}\pi i}\Gamma\left(\dfrac{\alpha+1}{\beta}\right) & \text{if } d<0 \end{cases}.$$

Proof. We set $d > 0$ and $u = dtx^\beta$. We have

$$\int_0^b x^\alpha e^{it(a+dx^\beta)}dx = \frac{e^{iat}}{\beta(dt)^{(\alpha+1)/\beta}}\int_0^{db^\beta t} u^{(\alpha+1)/\beta-1}e^{iu}du.$$

We have to prove that the integral $\int_0^\infty u^{\lambda-1}e^{iu}du$ converges for $0<\lambda=(\alpha+1)/\beta<1$. Integrating by parts, we have

$$\int_N^\infty u^{\lambda-1}e^{iu}du = -\frac{1}{i}N^{\lambda-1}e^{iN} + \frac{\lambda-1}{i}\int_N^\infty u^{\lambda-2}e^{iu}du.$$

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The function $u^{\lambda-2}$ is monotone decreasing. Applying the second mean value theorem to the real and imaginary parts, we obtain

$$\int_N^\infty u^{\lambda-1}e^{iu}du = -\frac{1}{i}N^{\lambda-1}e^{iN} + O(N^{\lambda-2})$$

as N $\to\infty$, where the constant implied by the O absolute. Then we obtain

$$\frac{e^{iat}}{\beta(dt)^{(\alpha+1)/\beta}}\int_N^\infty u^{(\alpha+1)/\beta-1}e^{iu}du = \frac{e^{iat}}{\beta(dt)^{(\alpha+1)/\beta}}\frac{-1}{i}(db^\beta t)^{\lambda-1}e^{iN} + O\left(\frac{b^{\alpha+1-\beta}}{\beta(dt)^2}\right)$$

$$= \frac{-1}{id}b^{(\alpha+1)-\beta}\frac{1}{\beta t}e^{iat}e^{iN} + O\left(\frac{b^{\alpha+1-\beta}}{\beta(dt)^2}\right),$$

the constants implied by the O's are absolute.

Lemma 2. *Suppose that the real functions $g(x)$ and $h(x)$ satisfy the following conditions:* For $x \in (0, \mathrm{b}]$ *and* $\alpha = 0$, $\beta = 2$,

1. $g(x) = Dx^\alpha(1+\theta(x))$, $h(x) = a + dx^\beta(1+f(x))$, *as* $x \to 0$;

where θ and f are continuous in $(0, b)$ with $\theta(0) = f(0) = 0$;

2. *$g(x)$ is continuous, $h(x)$ twice continuously differentiable, and $h'(x) \neq 0$, where α, β, a, d, and D are real constants such that $d \neq 0$, $D \neq 0$.*

3. $\dfrac{d}{dx}\left(\dfrac{g(x)}{h'(x)}\right)$ *has a constant sign as* $x \to 0$ *and* $x \to b$.

Under these conditions, as $t \to \infty$, we have, for sufficiently small $\delta > 0$,

$$\int_0^b g(x)e^{ith(x)}dx$$

$$= AD\frac{e^{iat}}{\beta(|d|t)^{(\alpha+1)/\beta}} + o\left(\frac{d^{-(\alpha+1)/\beta}D}{\beta}\frac{1}{t^{(\alpha+1)/\beta}}\right) + O\left(\frac{1}{t\beta}\right) + O\left(\left(\left|\frac{g(b)}{h'(b)}\right| + \left|\frac{g(\delta)}{h'(\delta)}\right|\right)\frac{1}{t}\right),$$

where the constants implied by the O's and o are absolute.

Proof. Without loss of generality, we may suppose $d > 0$. Consider the function

$$\varphi(x) = x(d + df(x))^{1/\beta},$$

which is continuous and n-times continuously differentiable in $[0, \delta] \subset [0, b]$, and satisfies $\varphi(0) = 0$, $\varphi'(0) = \mathrm{d}^{1/\beta} > 0$.

It can be assumed that δ has been chosen sufficiently small such that φ is strictly increasing in $[0, \delta]$. Let $\Psi(u)$ be its inverse function, continuous, n-times continuously differentiable, and strictly increasing in the interval $[0, \varphi(\delta)]$ satisfying $\Psi(0) = 0$, $\Psi'(0) = \mathrm{d}^{-1/\beta}$. Then we have

$$g(\Psi(u)) = D(\Psi(u))^\alpha(1+\theta(\Psi(u))) = Dd^{-\alpha/\beta}u^\alpha + O(u^{\alpha+1}), \tag{1}$$

where the constant implied by the O is independent of α and β.

Devide the interval $[0, b]$ into $[0, \delta]$ and $[\delta, b]$. In the first interval, changing the variable by $u = \varphi(x)$, we have

$$\int_0^{\delta} g(x)e^{ith(x)}dx = \int_0^{\varphi(\delta)} g(\Psi(u))e^{ith(\Psi(u))}\Psi'(u)du$$
$$= Dd^{-(\alpha+1)/\beta}\int_0^{\varphi(\delta)} u^{\alpha}e^{it(a+u^{\beta})}du + \int_0^{\varphi(\delta)} g_1 e^{ith_1(u)}du,$$

where, by virtue of (1),

$$g_1(u) = g(\Psi(u))\Psi'(u) - Dd^{-(\alpha+1)/\beta}u^{\alpha} = O(u^{\alpha+1}),$$

and the constant implied by the O is independent of α and β. We have

$$\frac{g_1(u)}{h'_1(u)} = O(u^{\alpha+1})/\beta u^{\beta-1} = \frac{1}{\beta}O(u^{\alpha-\beta+2}) as \quad u \to 0,$$

where $h_1(u) = a + u^{\beta}$, and the constants implied by the O's are independent of α and β. Thus

$$\int_0^b g(x)e^{ith(x)}dx = \int_0^{\delta} g(x)e^{ith(x)}dx + \int_{\delta}^b g(x)e^{ith(x)}dx$$
$$= Dd^{-(\alpha+1)/\beta}\int_0^{\varphi(\delta)} u^{\alpha}e^{it(\alpha+u^{\beta})}du + \int_0^{\varphi(\delta)} g_1(u)e^{ith_1(u)}du + \int_{\delta}^b g(x)e^{ith(x)}dx.$$

Now we consider the three integrals, respectively. By Lemma 1, we have, for $N = t\varphi(\delta)^{\beta}$

$$\int_0^{\varphi(\delta)} u^{\alpha}e^{it(a+u^{\beta})}du = \frac{e^{iat}}{\beta t^{(\alpha+1)/\beta}}A - \frac{e^{iat}}{\beta t^{(\alpha+1)/\beta}}\int_N^{\infty} u^{(\alpha+1)/\beta-1}e^{iu}du.$$

Since $h'(x) \neq 0$ on $[\delta, b]$ and $\frac{d}{dx}\left(\frac{g(x)}{h'(x)}\right)$ has a constant sign, we have

$$\int_{\delta}^b g(x)e^{ith(x)}dx = \frac{1}{it}\int_{\delta}^b \frac{g(x)}{h'(x)}\frac{d}{dx}e^{ith(x)}dx$$
$$= \frac{1}{it}\left[\frac{g(b)}{h'(b)}e^{ith(b)} - \frac{g(\delta)}{h'(\delta)}e^{ith(\delta)}\right] - \frac{1}{it}\int_{\delta}^b e^{ith(x)}\frac{d}{dx}\left(\frac{g(x)}{h'(x)}\right)dx = O\left(\left(\frac{g(b)}{h'(b)} + \frac{g(\delta)}{h'(\delta)}\right)\frac{1}{t}\right),$$

as $t \to \infty$, where the constants implied by the O is absolute.

Since $\frac{d}{dx}\left(\frac{g_1(x)}{h'_1(x)}\right)$ has a constant sign, we have

$$\int_0^{\varphi(\delta)} g_1(u)e^{ith_1(u)}du = \frac{1}{it}\int_0^{\varphi(\delta)} \frac{g_1(u)}{h'_1(u)}\frac{d}{du}e^{ith_1(u)}du$$
$$= \frac{1}{it}\left[\frac{g_1(\varphi(\delta))}{h'_1(\varphi(\delta))}e^{ith_1(\varphi(\delta))} - \frac{g_1(0)}{h'_1(0)}e^{ith_1(0)}\right] - \frac{1}{it}\int_0^{\varphi(\delta)} e^{ith_1(x)}\frac{d}{dx}\left(\frac{g_1(x)}{h'_1(x)}\right)dx.$$

Thus

$$\left|\int_0^{\varphi(\delta)} g_1(u)e^{ith_1(u)}du\right| \leq \left|\frac{2}{t}\frac{g_1(\varphi(\delta))}{h'_1(\varphi(\delta))}\right| = \frac{1}{t}\cdot\frac{1}{\beta}O\left(\varphi(\delta)^{\alpha-\beta+2}\right).$$

Since $\alpha = 0$, $\beta = 2$, we have, as $t \to \infty$,

$$\int_0^{\varphi(\delta)} g_1(u) e^{ith_1(u)} du = O\left(\frac{2}{t\beta}\right),$$

where the constants implied by the O is independent of α, β, δ, and t.

Thus, for sufficiently small fixed $\delta > 0$, we have, as $t \to \infty$,

$$\int_0^b g(x) e^{ith(x)} dx$$

$$= Dd^{-(\alpha+1)/\beta} \int_0^{\varphi(\delta)} u^{\alpha} e^{it(a+u^{\beta})} du + O\left(\frac{2}{t\beta}\right) + O\left(\left(\left|\frac{g(b)}{h'(b)}\right| + \left|\frac{g(\delta)}{h'(\delta)}\right|\right)\frac{1}{t}\right)$$

$$= Dd^{-(\alpha+1)/\beta}\left\{\frac{e^{iat}}{\beta t^{(\alpha+1)/\beta}} A - \frac{e^{iat}}{\beta t^{(\alpha+1)/\beta}} \int_N^{\infty} u^{(\alpha+1)/\beta-1} e^{iu} du\right\} + O\left(\frac{2}{\beta t}\right)$$

$$+ O\left(\left(\left|\frac{g(b)}{h'(b)}\right| + \left|\frac{g(\delta)}{h'(\delta)}\right|\right)\frac{1}{t}\right),$$

where the constants implied by the O's are independent of α, β, δ, b, N, and t. The second term of above equation is, as $t \to \infty$,

$$- Dd^{-(\alpha+1)/\beta} \frac{e^{iat}}{\beta t^{(\alpha+1)/\beta}} \int_N^{\infty} u^{(\alpha+1)/\beta-1} e^{iu} du$$

$$= Dd^{-(\alpha+1)/\beta} \frac{e^{iat}}{\beta t^{(\alpha+1)/\beta}} \frac{1}{i}\left(t\varphi(\delta)^{\beta}\right)^{\lambda-1} e^{iN} + O\left(Dd^{-(\alpha+1)/\beta} \frac{1}{\beta t^{(\alpha+1)/\beta}}\left(t\varphi(\delta)^{\beta}\right)^{\lambda-2}\right)$$

$$= Dd^{-(\alpha+1)/\beta} e^{iat+iN} \frac{1}{i\beta t}(\varphi(\delta))^{\alpha+1-\beta} + O\left(Dd^{-(\alpha+1)/\beta} \frac{1}{\beta t^{(\alpha+1)/\beta}}\left(t\varphi(\delta)^{\beta}\right)^{\lambda-2}\right)$$

$$= Dd^{-(\alpha+1)/\beta} e^{iat+iN} \frac{1}{i\beta} \frac{N^{(\alpha+1-\beta)/\beta}}{t^{(\alpha+1)/\beta}} + O\left(Dd^{-(\alpha+1)/\beta} \frac{N^{(\alpha+1-2\beta)/\beta}}{\beta t^{(\alpha+1)/\beta}}\right)$$

$$= o\left(\frac{Dd^{-(\alpha+1)/\beta}}{\beta t^{(\alpha+1)/\beta}}\right),$$

where $N = t\varphi(\delta)^{\beta}$ sufficiently large and the constants implied by the O and o are independent of α, β, δ, N, and t.

Thus, as $t \to \infty$, we have

$$\int_0^b g(x) e^{ith(x)} dx$$

$$= DA \frac{e^{iat}}{\beta (dt)^{(\alpha+1)/\beta}} + o\left(\frac{d^{-(\alpha+1)/\beta} D}{\beta t^{(\alpha+1)/\beta}}\right) + O\left(\frac{1}{\beta t}\right) + O\left(\left(\frac{g(b)}{h'(b)} + \frac{g(\delta)}{h'(\delta)}\right)\frac{1}{t}\right),$$

where the constants implied by the O's are absolute. This completes the proof.

Lemma 3. *Suppose that the real functions* $g(x)$ and $h(x)$ *satisfy the following conditions*: For $x \in (c, b]$,

1. $g(x)= D(1 + \theta(x))$, $h(x) = a + d(x-c)^2(1+f(x))$, as $x \to 0$;

2. *$g(x)$ is continuous and $h(x)$ is twice continuously differentiable and $h'(x) \neq 0$, where a, d,* and *D are real constants such that $d \neq 0$, $D \neq 0$. The function θ and f are continuous in* (c, b) with $\theta(c) = f(c) = 0$;

3. $\frac{d}{dx}\left(\frac{g(x)}{h'(x)}\right)$ *has a constant sign as $x \to c$ and $x \to b$.*

Under these conditions, as $t \to \infty$, we have, for sufficiently small $\delta > 0$,

$$\int_c^b g(x)e^{ith(x)}dx$$
$$= AD\frac{e^{iat}}{2\sqrt{|d|t}} + o\left(\frac{D}{\sqrt{|d|t}}\right) + O\left(\frac{1}{t}\right) + O\left(\left(\left|\frac{g(b)}{h'(b)}\right| + \left|\frac{g(c+\delta)}{h'(c+\delta)}\right|\right)\frac{1}{t}\right),$$

where the constants implied by the O's are absolute.

Proof. Substituting x by $x-c$, and putting $\alpha = 0$, $\beta = 2$, in Lemma 2, we obtain Lemma 3.

Lemma 4. *Suppose that the real functions $g(x)$* and *$h(x)$ satisfy the following conditions*: For $x \in [a, c)$,

1. $g(x) = D(1 + \theta(x))$, $h(x) = a + d(x-c)^2(1+f(x))$, as $x \to 0$;

2. *$g(x)$ is continuous and $h(x)$ is twice continuously differentiable and $h'(x) \neq 0$, where a, d, and D are real constants such that $d \neq 0$, $D \neq 0$. The function θ and f are continuous in (a, c) with $\theta(c) = f(c) = 0$;*

3. $\frac{d}{dx}\left(\frac{d(x)}{h'(x)}\right)$ *has a constant sign as $x \to a$ and $x \to c$.*

Under these conditions, we have, for sufficiently small $\delta > 0$,

$$\int_a^c g(x)e^{ith(x)}dx$$
$$= AD\frac{e^{iat}}{2\sqrt{|d|t}} + o\left(\frac{D}{\sqrt{|d|t}}\right) + O\left(\frac{1}{t}\right) + O\left(\left(\left|\frac{g(a)}{h'(a)}\right| + \left|\frac{g(c-\delta)}{h'(c-\delta)}\right|\right)\frac{1}{t}\right),$$

as $t \to \infty$, where the constants implied by the O's and o are absolute.

Proof. Substituting x by $c-x$, in Lemma 2, we obtain Lemma 4.

3 Theorems

Theorem 1 (cf. [T: lemman 4.7, E]). *Suppose that the real function $g(x)$ is continuous and $h(x)$ is twice continuously differentiable on $[a, b]$, and $h'(c) = 0$ at just one point c with $a < c < b$, $g(c) \neq 0$,* and $h''(c) \neq 0$.

Moreover, $\frac{d}{dx}\left(\frac{g(x)}{h'(x)}\right)$ *has a constant sign as $x \to c + 0$* and *$x \to c - 0$. Then we have for all sufficiently small $\delta > 0$,*

$$\int_a^b g(x)e^{ith(x)}dx = 2\sqrt{\frac{\pi}{2t|h''(c)|}}g(c)\exp\left(ith(c)+\frac{1}{4}i\pi\,\mathrm{sgn}(h''(c))\right)$$

$$+o\left(\frac{g(c)}{\sqrt{|h''(c)|t}}\right)+O\left(\frac{1}{t}\right)+O\left(\left(\left|\frac{g(c-\delta)}{h'(c-\delta)}\right|+\left|\frac{g(c+\delta)}{h'(c+\delta)}\right|+\left|\frac{g(a)}{h'(a)}\right|+\left|\frac{g(b)}{h'(b)}\right|\right)\frac{1}{t}\right),$$

as $t \to \infty$ or $b \to \infty$, *where the constants implied by the O's and o are absolute.*

Proof. By the mean value theorem, we obtain

$$h(x) = h(c)+(x-c)h'(c)+\frac{1}{2}(x-c)^2h''(\xi),$$

$$= h(c)+\frac{1}{2}h''(c)(x-c)^2\frac{h''(\xi)}{h''(c)},$$

where ξ is the number between c and x.

Since $h(x)$ is twice continuously differentiable, we have

$$\left.\frac{h''(\xi)}{h''(c)}\right|_{x\to c} = \frac{h''(c)}{h''(c)} = 1, \quad \xi = h''^{-1}\left(\frac{2(h(x)-h(c))}{(x-c)^2}\right),$$

where ξ is continuous function of x. Thus we can set $\dfrac{h''(\xi)}{h''(c)} := 1+f(x)$, continuous function such that $f(c)=0$. Also we set $\theta(x)$ continuous with $\theta(c)=0$ by $g(x)$ being continuous. Applying Lemma 3 and 4 with $\alpha=0$, $\beta=2$, a $=h(c)$ and $d=h''(c)/2$, we obtain

$$\int_a^b g(x)e^{ith(x)}dx = \int_a^b g(x)e^{ith(x)}dx + \int_c^b g(x)e^{ith(x)}dx$$

$$= 2AD\frac{e^{iat}}{2\sqrt{|d|t}}+o\left(\frac{D}{\sqrt{|d|t}}\right)+O\left(\frac{1}{t}\right)+O\left(\left(\left|\frac{g(b)}{h'(b)}\right|+\left|\frac{g(c+\delta)}{h'(c+\delta)}\right|\right)\frac{1}{t}\right)$$

$$+O\left(\left(\left|\frac{g(a)}{h'(a)}\right|+\left|\frac{g(c-\delta)}{h'(c-\delta)}\right|\right)\frac{1}{t}\right),$$

where the constants implied by the O's and o are absolute.

This completes the theorem.

Theorem 2 ([cf.[Gr-Ko] [T: lemma 2.4]) *Suppose $g'(x)$ is monotone with $g(x)>0$ and $g'(x)\le 0$. Suppose $f(x)$ has twice continuously differentiable on $[a, b]$ and also that $f'(x)$ is monotone. Let H_1 and H_2 be such that $H_1 < f'(x) \le H_2$ and $H = H_2 - H_1 + 2\varepsilon$ (≥ 2). Then for any $\varepsilon > 0$, we have*

$$\sum_{n\in[a,b]} g(n)e^{2\pi i f(n)} = \sum_{H_1-\varepsilon<m<H_2+\varepsilon}\int_a^b g(x)e^{2\pi i(f(x)-mx)}dx + O(g(a)\log H),$$

where the constant implied by the O is an absolute constant.

Proof. The proof runs along the same lines as [T].

$$\sum_{n=a}^{b} g(n)e^{2\pi i f(n)} = \int_a^b g(x)e^{2\pi i f(x)}dx - \int_a^b g(x)e^{2\pi i f(x)}d\Psi(x) + c_1, \tag{2}$$

where $c_1 = g(b)e^{2\pi i f(b)} - g(a)e^{2\pi i f(a)}$.
Integration by parts shows that

$$\begin{aligned} &-\int_a^b g(x)e^{2\pi i f(x)}d\Psi(x) \\ &= \int_a^b \left(g'(x) + 2\pi i f'(x)g(x)\right)e^{2\pi i f(x)}\Psi(x)dx + \left[g(x)e^{2\pi i f(x)}\Psi(x)\right]_a^b \\ &= \int_a^b \left(g'(x) + 2\pi i f'(x)g(x)\right)e^{2\pi i f(x)}\Psi(x)dx + \left(|g(b)| + |g(a)|\right)c_2, \end{aligned}$$

where $|c_2| \le 1$. Since $g'(x)$ is monotone,

$$\left|\int_a^b g'(x)e^{2\pi i f(x)}\Psi(x)dx\right| \le \frac{1}{2}\int_a^b |g'(x)|dx \le \frac{1}{2}\left(|g(b) - g(a)|\right) \le \frac{1}{2}\left(|g(b)| + |g(a)|\right).$$

And

$$\int_a^b 2\pi i f'(x)g(x)e^{2\pi i f(x)}\Psi(x)dx = \sum_{m\ne 0}\frac{1}{m}\int_a^b g(x)e^{2\pi i(f(x)-mx)}f'(x)dx. \tag{3}$$

If m $< H_1 - \varepsilon$ or $H_2 + \varepsilon < m$, then $F'(x) - m$ is monotonic and non-zero on $[a, b]$.

Moreover $f'(x)/(f'(x) - m)$ is monotonic and $g(x)$ is monotonely decreasing. By applying the second mean value theorem to the real and imaginary parts, we have

$$\begin{aligned} \int_a^b g(x)e^{2\pi i(f(x)-mx)}f'(x)dx &= \frac{1}{2\pi i}\int_a^b g(x)\frac{f'(x)}{f'(x)-m}d\left(e^{2\pi i(f(x)-mx)}\right) \\ &\le 2g(a)\left(\frac{|f'(b)|}{|f'(b)-m|} + \frac{|f'(a)|}{|f'(a)-m|}\right) \le 4g(a)\log(H_2+\varepsilon). \end{aligned}$$

Thus if $m \ge H_2 + \varepsilon$, the above is $2\dfrac{H_2}{m - H_2}$. Therefore we have

$$\begin{aligned} \left|\sum_{m\ge H_2+\varepsilon}\frac{1}{m}\int_a^b g(x)e^{2\pi i(f(x)-mx)}f'(x)dx\right| &\le 4g(a)\sum_{m\ge H_2+\varepsilon}\frac{H_2}{m(m-H_2)} \\ = 4g(a)\sum_{m\ge H_2+\varepsilon}\left\{\frac{1}{m-H_2} - \frac{1}{m}\right\} &< 4g(a)\log(H_2+\varepsilon) \le 4g(a)\log H. \end{aligned}$$

We have the similar upper bound for the terms with m $\le H_1 - \varepsilon$:

$$\begin{aligned} \left|\sum_{m\le H_1-\varepsilon}\frac{1}{m}\int_a^b g(x)e^{2\pi i(f(x)-mx)}f'(x)dx\right| &\le -g(a)\sum_{m\le H_1-\varepsilon}\frac{H_1}{m(H_2-m)} \\ &\le 4g(a)\log(H_1-\varepsilon) \le 4g(a)\log H. \end{aligned}$$

Therefore

$$\left| \sum_{m\le H_1-\varepsilon, m\ge H_2+\varepsilon} \frac{1}{m}\int_a^b g(x)e^{2\pi i(f(x)-mx)} f'(x)dx \right| \le c_3 g(a)\log H,$$

where $|c_3| \le 8$.

We integrate the remaning terms by parts to get

$$I = \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{m}\int_a^b g(x)f'(x)e^{2\pi i(f(x)-mx)}dx = \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{2\pi im}\int_a^b g(x)e^{-2\pi imx}de^{2\pi if(x)}$$

$$= \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \left[\frac{g(x)e^{2\pi i(f(x)-mx)}}{2\pi im} \right]_a^b$$

$$- \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{2\pi im}\int_a^b \left(g'(x)e^{-2\pi imx} - g(x)2\pi ime^{-2\pi imx}\right)e^{2\pi if(x)}dx$$

Since

$$\left| \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \left[\frac{g(x)e^{2\pi i(f(x)-mx)}}{2\pi im} \right]_a^b \right| \le \frac{1}{2}\left(|g(b)|+|g(a)|\right) \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{m\pi},$$

and

$$\left| \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{2\pi im}\int_a^b g'(x)e^{-2\pi imx}e^{2\pi if(x)}dx \right|$$
$$\le \frac{1}{2}\left(|g(b)|+|g(a)|\right) \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{m\pi}.$$

Thus

$$I = \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \int_a^b g(x)e^{2\pi i(f(x)-mx)}dx + \left(|g(b)|+|g(a)|\right)c_4 \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{\pi m},$$

where $|c_4| \le 1$.

Therefore

$$\sum_{m\ne 0}\frac{1}{m}\int_a^b g(x)e^{2\pi i(f(x)-mx)}f'(x)dx \tag{4}$$

$$= c_3 g(a)\log H + \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \int_a^b g(x)e^{2\pi i(f(x)-mx)}dx + \left(|g(b)|+|g(a)|\right)c_4 \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\ne 0} \frac{1}{\pi m}.$$

Therefore by (2), (3), and (4)

$$\sum_{n=a}^{b} g(n)e^{2\pi if(n)} = \int_a^b g(x)e^{2\pi if(x)}dx + \sum_{H_1-\varepsilon<m<H_2+\varepsilon, m\neq 0} \int_a^b g(x)e^{2\pi i(f(x)-mx)}dx$$
$$+c_3 g(a)\log H + c_5\left(|g(a)|+|g(b)|\right)\log H + c_6\left(|g(a)|+|g(b)|\right),$$

where $|c_5| \leq 1, |c_6| \leq \frac{5}{2}$.
This completes the proof.

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