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A novice's guide to a calculation of the Bockstein spectral sequences

Katsumi SHIMOMURA*

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§ 1. Introduction

In the stable homotopy theory, the computation of homotopy groups $\pi_*(X)$ plays a central role, in a sense that it will generate a lot of theories by stimulating thinkers of homotopy theory. But unfortunately, the problem to determine the groups is very tough to attack. Ravenel introduced an L_n-localization (*cf.* [5]), and $\pi_*(L_nX)$ is much easier to access than $\pi_*(X)$. In fact, some of the computations have been done (*cf.* [6]). Once we know the groups, then it will help to understand the homotopy category of L_n-local spectra. This is one of main problems in the homotopy theory. Of course, we get some information on the category of spectra from them. Readers who have some interest in the stable homotopy theory, I recommend to consult Ravenel's books [4] and [5].

One way to compute the homotopy groups $\pi_*(L_nX)$ of a spectrum X with $v_j^{-1}BP_*(X)=0$ $(0 \le j < n)$ is to use the Adams-Novikov spectral sequence. The E_2-term of it is an algebraic object $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, v_n^{-1}BP_*(X))$, which we denote $H^*BP_*(X)$. One of the basic tools for computing those E_2-terms is the Bockstein spectral sequence introduced by Miller, Ravenel and Wilson in their celebrated paper [3]. The E_1-terms of the Bockstein spectral sequences are determined by Ravenel (*cf.* [4]) in one of his 1977 papers. Since then, many computations have been done (*cf.* [6]). One of good points of the Bockstein spectral sequence is, say, that it is easy to access for a novice to compute it. This is why, I have decided to write down this article for undergraduate students who know only definition of some elementary algebraic terms. For this sake, the readers will not be required any knowledge of homotopy theory. Besides, we will not use a word 'Hopf algebroid', though the idea of these computations based on the theory of Hopf algebroids. Here I prove theorems elementary. So some of the proofs are more complicated than those used the facts on the Brown-Peterson spectrum, and results on Hopf algebroids, spectra and the ring spectrum. For an undergraduate student, 'colimit' also seems hard to understand, and so I will not use colimits of comodules either here. Moreover some elementary explanations are added where it seems to be necessary.

As you have read above the homotopy groups are computed by the generalized Adams spectral sequence. If we give a usual generalized Adams resolution over a spectrum X based on a ring spectrum E, then we can show its E_2-term being $H^*E_*(X)$ by using the result of § 3. In this sense, this contains necessary facts to

*Faculty of Education, Tottori University, Tottori, 680, Japan

understand the homotopy theory. Moreover, the way how to compute it is included, by which many computations have been done (*cf.* [6]).

This article is organized as follows:

In the next section, we introduce a cohomology H^*M of a comodule M. Here H^*M means $\mathrm{Ext}_P^*(B, M)$ over the Hopf algebroid (B, P) associated to the Brown-Peterson spectrum BP, if Ext groups are defined. Here we did not define Ext groups. We just define a comodule M, and then define H^*M as a cohomology of a cobar complex over the comodule M. In section 3, we study the basic results of homological algebra, without using words 'injective' nor 'projective'. For example, H^*M is shown to be equivalent to another cohomology of a complex obtained from a(n injective) resolution over M. To state the Bockstein spectral sequence, we define comodules N_n^s and M_n^s in section 4. In section 5, we set up the Bockstein spectral sequence. The title of the next section indicate what is written in it. In the last section, I present how to obtain elements by which we can describe generators of the desired module $H^*M_n^s$.

§ 2. Cohomology of comodules

We begin with preparing some notation. First we define elementary terms:

DEFINITION 2.1. *Let K be a commutative ring. We call M a (left) K-**module** if*
1. *M is an additive group, and*
2. *There exists an **action** $\varphi: K \times M \to M$ such that*

$$(k + k')x = kx + k'x,\ (kk')x = k(k'x) \quad \text{and} \quad k(x + x') = kx + kx'$$

$$\text{for } k,\ k' \in K \quad \text{and} \quad x,\ x' \in M,$$

where we write $kx = \varphi(k, x)$.

If the second condition is replaced by

$$x(k + k') = xk + xk',\ x(kk') = (xk)k' \quad \text{and} \quad (x + x')k = xk + x'k$$

*for $xk = \varphi'(x, k)$, where $\varphi': M \times K \to M$, then we call M a right K-module. Besides, it is a K-**bimodule** if it is a right and left K-module.*

DEFINITION 2.2. *Let K be a commutative ring. M is a K-**algebra** if*

1. *M is a module,*
2. *M is a ring, and*
3. *there is a relation*

$$k(xx') = (kx)x'$$

for $k \in K$ *and* $x, x' \in M$. *Furthermore, a K-algebra is called a* ***K*-bialgebra** *if it further satisfies*

$$(xx')k = x(x'k),$$

for $k \in K$ *and* $x, x' \in M$.

Definition 2.3. *A subset N of a K-module M is called a* ***K*-submodule** *of M if it is a K-module by the addition and the action of M. A* **subalgebra** *of an algebra is similarly defined.*

Definition 2.4. *Let K be a commutative ring, M a right K-module and N a left K-module. Then a* **tensor product** $M \otimes_K N$ *of M and N is defined to be:*

$$M \otimes_K N = M \times N/R,$$

where R is a submodule of $M \times N$ *generated by the elements*

$$(x + x', y) - (x, y) - (x', y), \quad (x, y + y') - (x, y) - (x, y') \quad \textit{and} \quad (xk, y) - (x, ky),$$

where $k \in K$, $x, x' \in M$ *and* $y, y' \in N$. *We further denote*

$$x \otimes y$$

for an equivalent class of (x, y).

Definition 2.5. *Let K be a commutative ring, and M and N are K-modules. Then a map* $f: M \to N$ *of K-modules is called a* **module map** *if*

$$f(x + y) = f(x) + f(y), \quad \textit{and} \quad f(kx) = kf(x),$$

for $x, y \in M$ *and* $k \in K$. *If M and N are K-bimodules, then we require*

$$f(xk) = f(x)k$$

for a ***K*-bimodule map** *f. Furthermore, let M and N be (bi-)algebras. A map* $f: M \to N$ *is called an* **(bi-)algebra map** *if*
1. *f is a (bi-)module map, and*
2. $f(xy) = f(x)f(y)$ *for* $x, y \in M$.

Now consider a polynomial algebra $K[x_1, x_2, \ldots]$ over the generators x_n's. Let $E = (e_1, e_2, \ldots)$ denotes a sequence of non-negative integers e_n with $e_n = 0$ except for a finite number of n. Then we put

$$x^E = x_1^{e_1} x_2^{e_2} \ldots,$$

and so any element $\xi \in K[x_1, x_2, \dots]$ is expressed by

$$\xi = \sum_E k_E x^E \in K[x_1, x_2, \dots]$$

for $k_E \in K$. Then the addition and the multiplication are given by

$$\xi + \xi' = \sum_E (k_E + k'_E) x^E \quad \text{and} \quad \xi\xi' = \sum_{E,F} k_E k'_F x^{E+F},$$

where $\xi' = \sum_E k'_E x^E$.

Let $\boldsymbol{Z}_{(p)}$ denote the ring of integers localized away from the prime p. That is, $\boldsymbol{Z}_{(p)} = \left\{ \frac{r}{s} \in \boldsymbol{Q} \mid r \in \boldsymbol{Z}, s \in \boldsymbol{Z} - p\boldsymbol{Z} \right\}$. We consider the $\boldsymbol{Z}_{(p)}$-algebras

$$B = \boldsymbol{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad P = B[t_1, t_2, \dots],$$

the polynomial algebras over the generators v_i and t_i with dimension $|v_i| = 2p^i - 2 = |t_i|$.

We also consider the $\boldsymbol{Q}$-algebras

$$BQ = \boldsymbol{Q}[m_1, m_2, \dots] \quad \text{and} \quad PQ = BQ[t_1, t_2, \dots],$$

where each generator m_i has dimension $2p^i - 2$. The algebra B is supposed to be embedded in BQ by the formula

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}. \tag{2.6}$$

In the same way, P is embedded in PQ. Consider further K-algebra maps η_L, η_R: $BQ \to PQ$ defined by

$$\begin{aligned} \eta_L(m_n) &= m_n \\ \eta_R(m_n) &= \sum_{i+j=n} m_i t_j^{p^i} \end{aligned} \tag{2.7}$$

for $n \geq 0$. Then PQ is a BQ-(bi)algebra by

$$bx = \eta_L(b)x \quad \text{and} \quad xb = \eta_R(b)x$$

for $b \in BQ$ and $x \in PQ$, where the right hand sides are obtained by the multiplication of PQ.

We now define a coproduct Δ: $PQ \to PQ \otimes_{BQ} PQ$ to be a BQ-algebra map given by the formula

$$\sum_{i+j=n} m_i \Delta(t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}. \tag{2.8}$$

Hereafter, we set that

$$m_0 = 1 = t_0, \qquad v_0 = p.$$

We also define a counit $\varepsilon: PQ \to BQ$ to be a BQ-algebra map satisfying

$$\varepsilon(t_n) = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \end{cases}$$

Then we see easily the following

PROPOSITION 2.9.

$$\varepsilon\eta_L = 1_{BQ} = \varepsilon\eta_R \quad \text{and} \quad (\varepsilon \otimes 1_{PQ})\Delta = 1_{PQ} = (1_{PQ} \otimes \varepsilon)\Delta.$$

Moreover we have

PROPOSITION 2.10. *The coproduct Δ satisfies the coassociative law. That is,* $(\Delta \otimes 1_{PQ})\Delta = (1_{PQ} \otimes \Delta)\Delta$.

PROOF. If suffices to show it on each generator t_n. In fact, for any element $\xi \in PQ$, $\xi = \sum_E k_E t^E$ and so

$$(\Delta \otimes 1_{PQ})\Delta(\xi) = \sum_E k_E(\Delta \otimes 1_{PQ})\Delta(t)^E = \sum_E k_E(1_{PQ} \otimes \Delta)\Delta(t)^E = (1_{PQ} \otimes \Delta)\Delta(\xi),$$

where $(\Delta \otimes 1_{PQ})\Delta(t)^E = (\Delta \otimes 1_{PQ})\Delta(t_1)^{e_1}(\Delta \otimes 1_{PQ})\Delta(t_2)^{e_2}\dots$ as above.

Since Δ is a BQ-algebra map and $PQ \otimes_{BQ} PQ$ is an integral domain, we see that $\Delta(1) = 1 \otimes 1$. By (2.8), we see that $\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1$, and so

$$(\Delta \otimes 1_{PQ})\Delta(t_1) = t_1 \otimes 1 \otimes 1 + 1 \otimes t_1 \otimes 1 + 1 \otimes 1 \otimes t_1 = (1_{PQ} \otimes \Delta)\Delta(t_1).$$

Now suppose $(\Delta \otimes 1_{PQ})\Delta(t_i) = (1_{PQ} \otimes \Delta)\Delta(t_i)$ for $i < n$. Apply $(\Delta \otimes 1_{PQ})$ on (2.8), and we compute

$$\begin{aligned}
\sum_{i+j=n} m_i(\Delta \otimes 1_{PQ})\Delta(t_j)^{p^i} &= \sum_{i+j+k=n} m_i\Delta(t_j)^{p^i} \otimes t_k^{p^{i+j}} \\
&= \sum_{k=0}^{n} \sum_{i+j=n-k} m_i\Delta(t_j)^{p^i} \otimes t_k^{p^{n-k}} \\
&= \sum_{k=0}^{n} \sum_{i+j+l=n-k} m_i t_j^{p^i} \otimes t_l^{p^{i+j}} \otimes t_k^{p^{n-k}} \\
&= \sum_{i+j+l+k=n} m_i t_j^{p^i} \otimes t_l^{p^{i+j}} \otimes t_k^{p^{n-k}}
\end{aligned}$$

On the other hand, apply $(1_{PQ} \otimes \Delta)$ on (2.8),

$$\begin{aligned}
\sum_{i+j=n} m_i(1_{PQ} \otimes \Delta)\Delta(t_j)^{p^i} &= \sum_{i+j+k=n} m_i t_j^{p^i} \otimes \Delta(t_k)^{p^{i+j}} \\
&= \sum_{k=0}^{n} \left(\sum_{i+j=n-k} m_i t_j^{p^i} \right) \otimes \Delta(t_k)^{p^{n-k}} \\
&= \sum_{k=0}^{n} \eta_R(m_{n-k}) \otimes \Delta(t_k)^{p^{n-k}}
\end{aligned}$$

$$= \sum_{k=0}^{n} 1 \otimes m_{n-k} \Delta(t_k)^{p^{n-k}}$$
$$= \sum_{i+j+k=n} 1 \otimes m_i t_j^{p^i} \otimes t_k^{p^{i+j}}$$
$$= \sum_{i+j+k=n} \eta_R(m_i) \otimes t_j^{p^i} \otimes t_k^{p^{i+j}}$$
$$= \sum_{i_2+i_2+j+k=n} m_{i_1} t_{i_2}^{p^{i_1}} \otimes t_j^{p^{i_1+i_2}} \otimes t_k^{p^{i+j}}$$

by (2.7). These show

$$\sum_{i+j=n} m_i(\Delta \otimes 1_{PQ})\Delta(t_j)^{p^i} = \sum_{i+j=n} m_i(1_{PQ} \otimes \Delta)\Delta(t_j)^{p^i},$$

which implies $(\Delta \otimes 1_{PQ})\Delta(t_n) = (1_{PQ} \otimes \Delta)\Delta(t_n)$ by the inductive hypothesis. q.e.d.

DEFINITION 2.11. *The (right) BQ-module M is a* **(right)** *PQ*-**comodule** *if there is a BQ-module map* $\psi: M \to M \otimes_{BQ} PQ$ *such that*

$$(\psi \otimes 1_{PQ})\psi = (1_M \otimes \Delta)\psi.$$

We call ψ *a* **structure map** *of the PQ-comodule M.*

As an example of PQ-comodules, we have

PROPOSITION 2.12. *The map* η_R *makes BQ a PQ-comodule. That is,* η_R *is a structure map of the comodule BQ.*

PROOF. By the same reason as above, it suffices to show that

$$(\eta_R \otimes 1_{PQ})\eta_R(m_n) = (1_{BQ} \otimes \Delta)\eta_R(m_n)$$

for each $n \geq 0$. Here η_R is considered to be the map $\eta_R: BQ \to PQ = BQ \otimes_{BQ} PQ$ given by $\eta_R(x) = 1 \otimes \eta_R(x)$. This is shown by a direct computation

$$(\eta_R \otimes 1_{PQ})\eta_R(m_n) = (\eta_R \otimes 1_{PQ})\left(\sum_{i+j=n} m_i \otimes t_j^{p^i}\right)$$
$$= \sum_{i+j=n} \left(\sum_{k+l=i} m_k \otimes t_l^{p^k}\right) \otimes t_j^{p^i}$$
$$= \sum_{k+l+j=n} 1 \otimes m_k t_l^{p^k} \otimes t_j^{p^{k+l}}$$
$$= \sum_{i+j=n} 1 \otimes m_i \Delta(t_j)^{p^i}$$
$$= (1_{BQ} \otimes \Delta)\left(\sum_{i+j=n} 1 \otimes m_i t_j^{p^i}\right)$$
$$= (1_{BQ} \otimes \Delta)(\eta_R(m_n))$$

as desired. q.e.d.

The following is obtained by Hazewinkel [1].

THEOREM 2.13.† *The maps η_L, η_R, Δ, ε on BQ and PQ yield the maps on B and P.*

PROOF. To prove this, we introduce an algebra CT for an algebra C: CT is $C[T]$ as a C-module, and the multiplication is defined by $(aT^i)(bT^j) = ab^{p^i}T^{i+j}$. We note that if $f\colon C \to C'$ is an algebra map, then it is extended to be another one $f\colon CT \to C'T$ by setting

$$f(\sum c_i T^i) = \sum f(c_i)T^i$$

for $c_i \in C$.

The formulae (2.6), (2.7) and (2.8) can be rewritten to be:

$$pm + mp = mv \in BQT, \; \eta_R(m) = mt \in PQT \quad \text{and}$$

$$m\Delta(t) = m(t \otimes 1)(1 \otimes t) \in (PQ \otimes_{BQ} PQ)T.$$

Here $v = \sum_{k\geq 0} v_k T^k \in BT$, $m = \sum_{k\geq 0} m_k T^k \in BQT$, $t = \sum_{k\geq 0} t_k T^k \in PT$, $t \otimes 1 = \sum_{k\geq 0} (t_k \otimes 1)T^k \in (P \otimes_B P)T$ and $1 \otimes t = \sum_{k\geq 0} (1 \otimes t_k)T^k \in (P \otimes_B P)T$. Since $m = 1 + \cdots$, we have $b = \sum_{k\geq 0} b_k T^k \in BQT$ such that $mb = 1$. We then see that $bm = 1$. In fact, $(bm)(bm) = b(mb)m = bm$ and so $(bm)(bm - 1) = 0$. Now that BQT is an integral domain and $bm \neq 0$, we see that $bm = 1$.

Turn now to prove the theorem. For Δ, multiplying b on the left of $m\Delta(t) = m(t \otimes 1)(1 \otimes t)$ shows the relation $\Delta(t) = (t \otimes 1)(1 \otimes t)$. Applying η_R on $pm + mp = mv$ shows

$$\eta_R(pm + mp) = \eta_R(mv)$$

$$p\eta_R(m) + \eta_R(m)p = \eta_R(m)\eta_R(v)$$

$$pmt + mtp = mt\eta_R(v)$$

Multiply b from the left, and we have $bpmt + tp = t\eta_R(v)$. Substitute $pm = mv - mp$ to it, and we get $b(mv - mp)t + tp = t\eta_R(v)$, and so $vt - pt + tp = t\eta_R(v)$ by the formula $bm = 1$. The elements p, t and v are all in PT, and so is $\eta_R(v)$ as desired. q.e.d.

We use the same notation η_L, η_R, Δ, ε for (B, P), that is, η_L, $\eta_R\colon B \to P$, $\Delta\colon P \to P \otimes_B P$ and $\varepsilon\colon P \to B$. Then, we immediately have the following

COROLLARY 2.14.

$$\varepsilon\eta_R = 1_B = \varepsilon\eta_L, \; (\varepsilon \otimes 1_P)\Delta = 1_P = (1_P \otimes \varepsilon)\Delta,$$

$$(1_P \otimes \Delta)\Delta = (\Delta \otimes 1_P)\Delta \quad \text{and} \quad (\eta_R \otimes 1_P)\eta_R = (1_P \otimes \eta_R)\eta_R.$$

DEFINITION 2.15. *A submodule I of an algebra A is called an* **ideal** *of A if $IA \subset I$.*

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†This is trivial, since (B, P) is obtained by the Brown-Peterson spectrum BP, though Hazewinkel showed it without using the spectrum.

As an example, we have

LEMMA 2.16. *Let* $a_1, \dots, a_n$ *be elements of* A. *Then* $I = \{\sum_{i=1}^n a_i x_i | x_i \in A\}$ *is an ideal.*

The proof is trivial by definition.

DEFINITION 2.17. *The ideal* I *of Lemma* 2.16 *is denoted by*

$$I = (a_1, a_2, \dots, a_n)$$

and said to be an **ideal generated by the elements** $a_1, a_2, \dots, a_n$.

Let I_n denote an ideal of B generated by $p, v_1, \dots, v_{n-1}$. For η_R, we have the following Landweber's formula:

LEMMA 2.18. $\eta_R(v_n) \equiv v_n + v_{n-1} t_1^{p^{n-1}} - v_{n-1}^p t_1 \bmod I_{n-1}$.

PROOF. We notice that $(m_1, \dots, m_{n-2}) = (v_1, \dots, v_{n-2})$ in PQ. By (2.6) and (2.7), we see that

$$\begin{aligned}\eta_R(v_n) &= p\eta_R(m_n) - \sum_{i=1}^{n-1} \eta_R(m_i)\eta_R(v_{n-i})^{p^i} \\ &= p\sum_{i=0}^{n} m_i t_{n-i}^{p^i} - \sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1} m_j t_{i-j}^{p^j}\right)\eta_R(v_{n-i})^{p^i} \\ &\equiv p(t_n + m_{n-1}t_1^{p^{n-1}} + m_n) - \sum_{i=2}^{n-1} t_i\eta_R(v_{n-i}) - t_1\eta_R(v_{n-1})^{p^i} \bmod (m_1, \dots, m_{n-2}).\end{aligned}$$

The right hand side is found to be an element of $P/(m_1, \dots, m_{n-2})$ and so we have

$$\eta_R(v_n) \equiv pt_n + v_{n-1}t_1^{p^{n-1}} + v_n + t_1\eta_R(v_{n-1})^{p^i} \bmod (v_1, \dots, v_{n-2})$$

by the inductive hypothesis, which shows the desired formula in the lemma. q.e.d.

PROPOSITION 2.19. $PI_n = I_nP$, *where* $PI_n = \{x\eta_R(w) | x \in P, w \in I_n\}$ *and* $I_nP = \{\eta_L(w)x | x \in P, w \in I_n\}$.

PROOF. By Lemma 2.18, $\eta_R(v_k) \equiv \eta_L(v_k) \bmod I_k$ for each k. Therefore the proposition follows. q.e.d.

COROLLARY 2.20. *The maps* η_L, η_R, Δ *and* ε *on* (B, P) *yield the maps on* $(B/I_n, P/I_n)$.

PROOF. By Proposition 2.19,

$$\begin{aligned}I_n \otimes_B P \otimes_B P &= B \otimes_B I_nP \otimes_B P = B \otimes_B PI_n \otimes_B P \\ &= B \otimes_B P \otimes_B I_nP.\end{aligned}$$

Therefore, we see that $I_n \otimes_B P \otimes_B P = I_n \otimes_B P \otimes_B P + B \otimes_B I_nP \otimes_B P + B \otimes_B P \otimes_B I_nP$. Put $I = I_n \otimes_B P \otimes_B P$ and $J = I_n \otimes_B P \otimes_B P + B \otimes_B I_nP \otimes_B P + B \otimes_B P \otimes_B I_nP$ and we obtain

$$B/I_n \otimes_B P \otimes_B P = (B \otimes_B P \otimes_B P)/I = (B \otimes_B P \otimes_B P)/J.$$

Therefore, $P/I_n \otimes_B P/I_n = (B \otimes_B P \otimes_B P)/J$. Furthermore, we have $f(vx) = vf(x)$ for $v \in B$, if f is a B-module map. Therefore, if f is one of the above maps, then $f(I_nM) \subset I_nf(M)$ for $M = B$, P. Hence it induces a map on $(B/I_n, P/I_n)$. q.e.d.

DEFINITION 2.21. *A* **comodule** *M is a B-module with* **coaction** *$\psi: M \to M \otimes_B P$ such that ψ is a B-module map and satisfies $(1 \otimes \varepsilon)\psi = 1_M$ and $(1_M \otimes \Delta)\psi = (\psi \otimes 1_P)\psi$.*

PROPOSITION 2.22. *B is a comodule with the coaction $\eta_R: B \to P$. Furthermore, B/I_n is also a comodule with a coaction induced from η_R.*

PROOF. The first statement follows from Proposition 2.12 and Theorem 2.13. The other also follows from Corollary 2.20 and Propositions 2.9 and 2.10. q.e.d.

DEFINITION 2.23. *A* **cobar complex** *Ω^*M consists of B-modules Ω^sM and B-module maps called* **differential** *$d_s: \Omega^sM \to \Omega^{s+1}M$ defined by*

$$\Omega^sM = M \otimes_B P \otimes_B \cdots \otimes_B P \qquad (s \text{ copies of } P)$$

and

$$d_s(m \otimes p_1 \otimes \cdots \otimes p_s) = \psi(m) \otimes p_1 \otimes \cdots \otimes p_s + \sum_{k=1}^{s} m \otimes p_1 \otimes \cdots \otimes \Delta(p_k) \otimes \cdots \otimes p_s$$
$$- (-1)^s m \otimes p_1 \otimes \cdots \otimes p_s \otimes 1$$

Then we see easily that

LEMMA 2.24. *For each integer $s \geq 0$, $d_{s+1}d_s = 0$.*

PROOF. Let $\Delta_i^{(s)}: \Omega^sM \to \Omega^{s+1}M$ for $0 < i \leq s$ be a map defined by

$$\Delta_i^{(s)}(m \otimes p_1 \otimes \cdots \otimes p_s) = m \otimes p_1 \otimes \cdots \otimes \Delta(p_i) \otimes \cdots \otimes p_s.$$

Then,

$$d_s(m \otimes x) = \psi(m) \otimes x + \sum_{i=1}^{s} (-1)^i m \otimes \Delta_i^{(s)}(x) - (-1)^s m \otimes x \otimes 1, \tag{2.25}$$

for $x = p_1 \otimes \cdots \otimes p_s \in P \otimes_B \cdots \otimes_B P$ (s factors of P). Therefore,

$$d_{s+1}d_s(m \otimes x) = d_{s+1}\left(\psi(m) \otimes x + \sum_{i=1}^{s} (-1)^i m \otimes \Delta_i^{(s)}(x) - (-1)^s m \otimes x \otimes 1\right)$$
$$= (\psi \otimes 1 - 1 \otimes \Delta)(\psi(m)) \otimes x - \sum_{i=1}^{s} (-1)^i \psi(m) \otimes \Delta_i^{(s)}(x)$$
$$+ (-1)^s \psi(m) \otimes x \otimes 1 + \sum_{i=1}^{s} (-1)^i \psi(m) \otimes \Delta_i^{(s)}(x)$$
$$+ \sum_{i=1}^{s} \sum_{j=1}^{s+1} (-1)^{i+j} m \otimes \Delta_j^{(s+1)}\Delta_i^{(s)}(x) + \sum_{i=1}^{s} (-1)^{i+s} m \otimes \Delta_i^{(s)}(x) \otimes 1$$

$$- (-1)^s\psi(m) \otimes x \otimes 1 - (-1)^s \sum_{i=1}^{s+1} (-1)^i m \otimes \Delta_i^{(s+1)}(x \otimes 1)$$

$$- (-1)^s m \otimes x \otimes 1 \otimes 1.$$

The first term is zero by the definition of ψ. The second and fourth terms offset each other. Similar is the terms of the third and the seventh. By definition, we see that $\Delta_i^{(s+1)}(x \otimes 1) = \Delta_i^{(s)}(x) \otimes 1$ if $i \leq s$, and $\Delta_{s+1}^{(s+1)}(x \otimes 1) = x \otimes 1 \otimes 1$. This shows that the sixth term cancels out the eighth an the ninth terms. Thus if we put $x_{i,j} = (-1)^{i+j} m \otimes \Delta_j^{(s+1)}\Delta_i^{(s)}(x)$, then

$$\begin{aligned} d_{s+1}d_s(m \otimes x) &= \sum_{i=1}^{s} \sum_{j=1}^{s+1} x_{i,j} \\ &= \sum_{i=1}^{s} \left(\sum_{j=1}^{i-1} x_{i,j} + x_{i,i} + x_{i,i+1} + \sum_{j=i+2}^{s+1} x_{i,j} \right) \end{aligned}$$

Here, if $j > i + 1$, then $\Delta_j^{(s+1)}\Delta_i^{(s)} = \Delta_i^{(s+1)}\Delta_{j-1}^{(s)}$. Therefore we see that $\sum_{i=1}^{s} \sum_{j=i+2}^{s+1} x_{i,j} = -\sum_{i=1}^{s} \sum_{j=1}^{i-1} x_{i,j}$. Furthermore, $x_{i,i} = -x_{i,i+1}$ since $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. Hence $d_{s+1}d_s(m \otimes x) = 0$. q.e.d.

By this, we obtain $\operatorname{Im} d_{s-1} \subset \operatorname{Ker} d_s$ and define the homology of M by

(2.26) $$H^s M = \operatorname{Ker} d_s / \operatorname{Im} d_{s-1},$$

for $s \geq 0$, where $d_{-1} = 0\colon 0 \to \Omega^0 M$. In the following we will write an element $\xi \in H^s M$ by its representative $x \in \Omega^s M$ as follows:

$$\xi = [x].$$

In other words, $[x] = x + \operatorname{Im} d_{s-1}$, and so $[x] = [y]$ if and only if

$$\exists z \in \Omega^{s-1} M \qquad \text{such that } d_{s-1}(z) = x - y.$$

§ 3. Homological algebra

We begin with defining a comodule map between comodules:

DEFINITION 3.1. *Let (M, ψ_M) and (N, ψ_N) be comodules. Then a module map $f\colon M \to N$ is called a* **comodule map** *if the following diagram is commutative:*

(3.2) $$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \psi_M} & & \downarrow{\scriptstyle \psi_N} \\ M \otimes_B P & \xrightarrow{f \otimes 1} & N \otimes_B P. \end{array}$$

PROPOSITION 3.3. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence of comodules and comodule maps. Then we have a long exact sequence.*

$$0 \to H^0L \xrightarrow{f_*} H^0M \xrightarrow{g_*} H^0N \xrightarrow{\delta} H^1L \to \cdots \xrightarrow{\delta} H^nL \xrightarrow{f_*} H^nM \xrightarrow{g_*} H^nN \xrightarrow{\delta} \cdots.$$

Here $\delta\colon H^nN \to H^{n+1}L$ is defined by $\delta(x) = [f_{\#}^{-1} d_n g_{\#}^{-1}(x)]$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^n L & \xrightarrow{f_{\#}} & \Omega^n M & \xrightarrow{g_{\#}} & \Omega^n N & \longrightarrow & 0 \\ & & \downarrow{d_n} & & \downarrow{d_n} & & \downarrow{d_n} & & \\ 0 & \longrightarrow & \Omega^{n+1} L & \xrightarrow[f_{\#}]{} & \Omega^{n+1} M & \xrightarrow[g_{\#}]{} & \Omega^{n+1} N & \longrightarrow & 0 \end{array}$$

First check that δ is a well defined map, and then we can check the proposition by chasing the above diagram. These are done by an easy diagram chasing. q.e.d.

Consider a sequence of comodules (M_s, ψ_s) and comodule maps $f_s\colon M_s \to M_{s+1}$:

$$0 \longrightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{s-1}} M_s \xrightarrow{f_s} \cdots.$$

DEFINITION 3.4. *We call a sequence above a* **resolution** *of M_0 if it is an exact sequence and $H^tM_s = 0$ for each s, $t > 0$. A sequence is called a* **complex** *if $f_{s+1}f_s = 0$ for each $s \geq 0$.*

An an example of complex, we have the cobar complex Ω^*M introduced in the previous section, by virtue of Lemma 2.24.

Consider a submodule $\pi(M)$ of a comodule M defined by

$$\pi(M) = \{x \mid x \in M, \psi(x) = x \otimes 1\}.$$

Then we have the following lemma by definition:

LEMMA 3.5. *$H^0M = \pi(M)$ for a comodule M.*

We further obtain

LEMMA 3.6. *Let M and N be comodules and $f\colon M \to N$ a comodule map. Then f induces a module map*

$$\pi(f)\colon \pi(M) \to \pi(N),$$

which is a restriction of f.

PROOF. By the definition of a comodule map, $\psi_N(f(x)) = (f \otimes 1)\psi_M(x)$ for $x \in M$. Therefore, if $x \in \pi(M) \subset M$, then $\psi_N(f(x)) = f(x) \otimes 1$, and so $f(x) \in \pi(N)$ as desired. q.e.d.

By this lemma, we obtain a complex $\{\pi(M_s); \pi(f_s)\}$ from a resolution $\{M_s; f_s\}$.

PROPOSITION 3.7. *Let (M, ψ) be a comodule and consider a resolution of M. Then*

$$H^sM = \mathrm{Ker}\,(\pi(f_s)\colon \pi(M_s) \to \pi(M_{s+1}))/\mathrm{Im}\,(\pi(f_{s-1})\colon \pi(M_{s-1}) \to \pi(M_s)).$$

Here H^*M is defined in (2.26).

PROOF.[¶] First note that the long exact sequence

$$0 \longrightarrow M \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} \cdots$$

splits into short ones

$$0 \to K_n \xrightarrow{g_n} M_n \xrightarrow{h_n} K_{n+1} \to 0$$

for $n > 0$ with $K_1 = M$ and $K_n = \mathrm{Ker}\, f_n = \mathrm{Coker}\, f_{n-2}$. Besides,

(3.8) $$f_n = g_{n+1}h_n.$$

Proposition 3.3 produces a long exact sequence

$$\to H^sK_n \to H^sM_n \to H^sK_{n+1} \xrightarrow{\delta} H^{s+1}K_n \to .$$

Since $H^sM_n = 0$ for $s > 0$ by the definition of a resolution, we have exact sequences

(3.9) $$0 \to H^0K_n \xrightarrow{g_*} H^0M_n \xrightarrow{h_*} H^0K_{n+1} \to H^1K_n \to 0$$

and isomorphisms

(3.10) $$H^sK_{n+1} \cong H^{s+1}K_n.$$

Therefore, noticing that $H^0M = \pi(M)$ by Lemma 3.5, we compute

$$\begin{aligned} H^sM &= H^sK_1 = H^1K_s && \text{(by (3.10))} \\ &= H^0K_{s+1}/\mathrm{Im}\, h_* && \text{(by (3.9))} \\ &= \mathrm{Ker}\, h_*/\mathrm{Im}\, h_* && \text{(by (3.9))} \\ &= \mathrm{Ker}\, \pi(f_s)/\mathrm{Im}\, \pi(f_{s-1}) && \text{(by (3.8))} \end{aligned}$$

q.e.d.

LEMMA 3.11. *Consider a comodule $M \otimes_B P$ with coaction $1 \otimes \Delta\colon M \otimes_B P \to (M \otimes_B P) \otimes_B P$. Then $H^sM \otimes_B P = 0$ for $s > 0$.*

PROOF. It suffices to show that $d_s(x) = 0$ implies the existence of an element y such that $d_{s-1}(y) = x$ for each $s > 0$. Put first $x = (m \otimes x_0) \otimes x'$ for $x' = x_1 \otimes \cdots \otimes x_s$ and compute

$$\begin{aligned} d_s(x) &= d_s((m \otimes x_0) \otimes x') \\ &= (m \otimes \Delta(x_0)) \otimes x' + \sum_{i=1}^{s} (-1)^i (m \otimes x_0)\Delta_i^{(s)}(x') - (-1)^s (m \otimes x_0) \otimes x' \otimes 1 \end{aligned}$$

[¶] This is obvious if you use the spectral sequence associated to the double complex $M_s \otimes \Omega^t B$.

as in (2.25). Suppose that $d_s(x)=0$ and send this by $1\otimes\varepsilon\otimes 1$ to obtain

$$0=(m\otimes x_0)\otimes x'+\sum_{i=1}^{s}(-1)^i(m\varepsilon(x_0))\Delta_i^{(s)}(x')-(-1)^s(m\varepsilon(x_0))\otimes x'\otimes 1$$
$$=(m\otimes x_0)\otimes x'+d_{s-1}(m\varepsilon(x_0)\otimes x')$$

using $(\varepsilon\otimes 1)\Delta(x_0)=x_0$ of Corollary 2.14. Now put $y=-(m\varepsilon(x_0)\otimes x_1)\otimes x''=-m\varepsilon(x_0)\otimes x'$ for $x''=x_2\otimes\cdots\otimes x_s$. Then $d_s(y)=x$ by the above equality. q.e.d.

COROLLARY 3.12. *Let*

$$0\to M\to I_1\otimes_B P\xrightarrow{f_1} I_2\otimes_B P\xrightarrow{f_2}\cdots$$

be a resolution of M. *Then*

$$H^sM=\operatorname{Ker}\pi(f_s)/\operatorname{Im}\pi(f_{s-1}).$$

Furthermore, note that $\pi(f_s)\colon I_s\to I_{s+1}$.

PROOF. It is enough to show that $\pi(M\otimes_B P)=M$. Since $\psi(m\otimes x)=m\otimes\Delta(x)$ by the definition of the coaction of $M\otimes_B P$, $m\otimes\Delta(x)=m\otimes x\otimes 1$ if $m\otimes x\in\pi(M\otimes_B P)$. Therefore, $x\in \boldsymbol{Z}_{(p)}$ and we see $\pi(M\otimes_B P)\subset M$ by identifying $m\otimes 1=m$. q.e.d.

§ 4. The chromatic spectral sequence

Let M be a comodule and $v\in B$ acts on M, that is a multiplication by v induces a comodule map $v\colon M\to M$. In other words, the diagram

(4.1)
$$\begin{array}{ccc} M & \xrightarrow{v} & M\\ \downarrow{\scriptstyle\psi_M} & & \downarrow{\scriptstyle\psi_M}\\ M\otimes_B P & \xrightarrow[v\otimes 1]{} & M\otimes_B P\end{array}$$

commutes. Then define

(4.2) $$v^{-1}M=\{v^jm\mid m\in M, j\in \boldsymbol{Z}\},$$

satisfying

$$w(v^jm)=(wv^j)m,\ (u+w)(v^jm)=u(v^jm)+w(v^jm)\quad\text{and}\quad v^j(m+m')=v^jm+v^jm'$$

for $u, w\in B$ and $m, m'\in M$. By this, $v^{-1}M$ is a B-module.

LEMMA 4.3. (*cf.* [2]) *Suppose that* v *is a monomorphism. Then* $v^{-1}M$ *has a comodule structure so that the inclusion* $i\colon M\to v^{-1}M$ *is a comodule map.*

PROOF. We define a coaction $\psi\colon v^{-1}M\to v^{-1}M\otimes_B P$ by

(4.4) $$\psi(v^jm)=(v^j\otimes 1)\psi_M(m),$$

where ψ_M is a coaction of M. By (4.1), we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M\otimes_B P \\ \downarrow i & & \downarrow i \\ v^{-1}M & \xrightarrow[\psi]{} & v^{-1}M\otimes_B P, \end{array}$$

so that $i: M \to v^{-1}M$ is a module map. This shows that i is a comodule map. It is easy to be checked that ψ is an algebra map. In fact,

$$\begin{aligned}(1\otimes\varepsilon)\psi(v^j m) &= (1\otimes\varepsilon)(v^j\otimes 1)\psi_M(m)\\ &= (v^j\otimes 1)(1\otimes\varepsilon)\psi_M(m)\\ &= v^j m\end{aligned}$$

and

$$\begin{aligned}(\psi\otimes 1)\psi(v^j m) &= (v^j\otimes 1\otimes 1)(\psi_M\otimes 1)\psi_M(m)\\ &= (v^j\otimes 1\otimes 1)(1\otimes\Delta)\psi_M(m)\\ &= (1\otimes\Delta)(v^j\otimes 1)\psi_M(m)\\ &= (1\otimes\Delta)\psi(v^j m)\end{aligned}$$

for $v^j m \in v^{-1}M$, since they are B-module maps. q.e.d.

Then we denote $M/(v^\infty)$ the cokernel of the inclusion $i: M \to v^{-1}M$, that is, the sequence

$$0 \to M \hookrightarrow v^{-1}M \to M/(v^\infty) \to 0$$

is exact.

LEMMA 4.5. *Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence such that L and M are comodules and $f: L \to M$ is a comodule map. Then the cokernel N also has a comodule structure so that the projection $g: M \to N$ is a comodule map.*

PROOF. Define a coaction $\psi_N: N \to N\otimes_B P$ of N by

$$\psi_N(x) = (g\otimes 1)\psi_M(y) \qquad \text{for } y\in M \text{ with } g(y) = x.$$

A diagram chasing shows that this is well defined. In fact, $(g\otimes 1)\psi_M(w) = (g\otimes 1)\psi_M(f(u)) = (g\otimes 1)(f\otimes 1)\psi_L(u) = 0$ for $w\in M$ such that $g(w)=0$, where u with $f(u) = w$ exists by the exactness.

Then the relations $(1\otimes\varepsilon)\psi_N = 1_N$ and $(\psi_N\otimes 1)\psi_N = (1\otimes\Delta)\psi_N$ follow from those on ψ_M, and g is a comodule map by definition. q.e.d.

COROLLARY 4.6. *Suppose that M is a comodule and a comodule map $v: M \to M$*

is a monomorphism. Then $M/(v^\infty)$ *is also a comodule such that the projection* $v^{-1}M \to M/(v^\infty)$ *is a comodule map.*

NOTATION 4.7. *For* $x \in M/(v^\infty)$, $x = m/v^j$ *for* $m \in M$ *and* $j > 0$, *and* $v_1^j x = 0$.

That is, the notation of fraction '/' means that the fraction would be zero if the denominator is reduced away. Now we define the chromatic spectral sequence introduced in [3].

DEFINITION 4.8. *Put* $N_n^0 = B/I_n$. *Suppose inductively that a comodule* N_n^s *is defined. Then put* $M_n^s = v_{n+s}^{-1}N_n^s$ *and define* N_n^{s+1} *to fit into the exact sequence*

$$0 \to N_n^s \hookrightarrow M_n^s \to N_n^{s+1} \to 0.$$

Here $M_n^s = v_{n+s}^{-1}N_n^s$ *is defined in* (4.13) *later.*

Note that $N_n^0 = B/I_n$ is a comodule with the coaction induced from the right unit $\eta_R: B \to P$ by Proposition 2.22.

Here the following four lemmas show the fact that B-modules N_n^s and M_n^s are all comodules with coaction induced from η_R. The proofs of them are inductively given after a comodule M_n^s is defined in (4.13) below so that $N_n^s \hookrightarrow M_n^s$ is a comodule map.

LEMMA 4.9. *For any element* x *of* N_n^s, *there exists an integer* $k > 0$ *such that* $I_{n+s}^k x = 0$. *Here* $I_{n+s}^k x = \{a_1 \dots a_k x \mid a_i \in I_{n+s}, 0 < i \leq k\}$.

By virtue of this lemma, consider a B-submodules $N_n^s(k)$ defined by

$$N_n^s(k) = \{x \mid x \in N_n^s, I_{n+s}^k x = 0\}.$$

LEMMA 4.10. $N_n^s(k)$ *is a comodule such that the inclusion* $N_n^s(k) \subset N_n^s$ *is a comodule map.*

LEMMA 4.11. $N_n^s = \cup_{k>0} N_n^s(k)$.

LEMMA 4.12. *There exists an integer* $l(k)$ *such that* $v_{n+s}^{p^{l(k)}}: N_n^s \to N_n^s$ *is an injective comodule map.*

Now we define

$$M_n^s = \bigcup_{k>0} v(k)^{-1}N_n^s(k), \tag{4.13}$$

where $v(k)$ is an injective map of Lemma 4.12.

LEMMA 4.14. M_n^s *is a comodule such that* $N_n^s \to M_n^s$ *is a comodule map.*

Now the short exact sequences $0 \to N_n^s \to M_n^s \to N_n^{s+1} \to 0$ induce long ones

$$\to H^t N_n^s \to H^t M_n^s \to H^t N_n^{s+1} \xrightarrow{\delta} H^{t+1} N_n^s \to.$$

Therefore, if we know modules $H^t M_n^s$ and the maps δ for each s, t, we will obtain a module $H^{s+t}N_n^0$. In fact, by the exact sequence $0 \to H^0 N_n^{s+t} \to H^0 M_n^{s+t}$ we get $H^0 N_n^{s+t}$ from $H^0 M_n^{s+t}$. Then suppose that we have $H^t N_n^s$, and consider the exact sequence $H^t N_n^s \xrightarrow{\delta} H^{t+1} N_n^{s-1} \to H^{t+1} M_n^{s-1}$, and we obtain $H^{t+1} N_n^{s-1}$. Proceed these steps, and we obtain $H^{s+t} N_n^0$. We will denote these process by

$$H^t M_n^s \Rightarrow H^{s+t} N_n^0,$$

and call it a *chromatic spectral sequence*‡.

Now we will prove the above lemmas. First we assume that we have these lemmas for N_n^i and M_n^i for $0 \le i < s$. In fact, if $i = 0$, then $I_n x = 0$ for any $x \in N_n^0$. Therefore, Lemmas 4.9 follows. Since $N_n^0 = N_n^0(1)$, Lemmas 4.10 and 4.11 is trivial. For Lemma 4.12, take $l(1) = 0$. Then Lemma 2.18 shows the lemma for $i = 1$. Lemma 4.14 follows from Lemma 4.3 by taking $v = v_n$. Therefore, the first step of the induction is shown.

PROOF OF LEMMA 4.9. Take an element x of N_n^s. Then we have an element y in M_n^{s-1} which is sent to x by the map $M_n^{s-1} \to N_n^s$. By definition, we have $y = v_{n+s-1}^a z$ for some integer a and an element z of N_n^{s-1}. Here since $x = 0$ if $a \ge 0$, we may assume that $a < 0$. By the inductive hypothesis, we have an integer $k' > 0$ such that $I_{n+s-1}^{k'} z = 0$, and so $I_{n+s-1}^{k'} y = 0$. Now put $k = -a + k'$. Then $I_{n+s}^k x = 0$. In fact, consider $w = a_1 \dots a_i a_{i+1} \dots a_k z$ for $a_j \in I_{n+s-1}$ if $1 \le j \le i$, and $a_j = v_{n+s-1}^{e_j}$ if $i < j < n+s$. Then if $i \ge k'$, then $w = 0$ since $I_{n+s-1}^{k'} z = 0$. If $i < k'$, then v_{n+s-1}^{-a} divides $a_1 \cdots a_k$, and so $w = w' v_{n+s-1}^{-a} x = 0$, since $v_{n+s-1}^{-a} x = z = 0 \in N_n^s$. Thus we have proved the lemma. q.e.d.

PROOF OF LEMMA 4.10. Consider the diagram

$$\begin{array}{ccc} N_n^s(k) & \xrightarrow{inc} & N_n^s \\ \vdots & & \downarrow \psi \\ N_n^s(k) \otimes_B P & \longrightarrow & N_n^s \otimes_B P. \end{array}$$

It suffices to show that $\mathrm{Im}\,(\psi \circ inc) \subset N_n^s(k) \otimes_B P$. In fact, $\psi \circ inc$ is pulled back to the dotted arrow. Since any element a of I_{n+s} acts on the maps f as $af(x) = f(ax)$, we see that $a(\psi \circ inc)(x) = (\psi \circ inc)(ax) = 0$ for $a \in I_{n+s}^k$. q.e.d.

PROOF OF LEMMA 4.11. This follows immediately from Lemma 4.9. q.e.d.

PROOF OF LEMMA 4.12. We can show by induction that if $x \equiv y \bmod (p, a)$, then $x^{p^{i+j-1}} \equiv y^{p^{i+j-1}} \bmod (p^i, a^{p^j})$. Therefore, we obtain the lemma by setting $l(k) = p^k$. q.e.d.

PROOF OF LEMMA 4.14. This is easily checked by (4.13), Lemma 4.11 and Lemma 4.3. q.e.d.

‡Usually, a spectral sequence is constructed more systematic, but we need not here such a general spectral sequence.

§ 5. The Bockstein spectral sequence

By using Notation 4.7, an element x of M_n^s is written by

$$x/v_n^{e(n)}\dots v_{n+s-1}^{e(n+s-1)}$$

for $x \in B$ and an integer $e(i) > 0$ for $n \leq i < n+s$, and besides it is zero if the denominator is reduced away. Therefore we obtain an exact sequence

$$0 \to M_{n+1}^{s-1} \xrightarrow{\varphi} M_n^s \xrightarrow{v_n} M_n^s \to 0, \qquad (\varphi(x) = x/v_n)$$

for each integer $n \geq 0$, and it yields an long exact one

$$\to H^k M_{n+1}^{s-1} \xrightarrow{\varphi_*} H^k M_n^s \xrightarrow{v_n} H^k M_n^s \xrightarrow{\delta} H^{k+1} M_{n+1}^{s-1} \to,$$

By this exact sequence, we can compute $H^* M_n^s$ from $H^* M_{n+1}^{s-1}$. We call this the *Bockstein spectral sequence*††.

Lemma 5.1. (*cf.* [3, Remark 3.11]) *If we have a commutative diagram*

$$\begin{array}{ccccccc}
H^k M_{n+1}^{s-1} & \xrightarrow{\varphi_*} & B^k & \xrightarrow{v_n} & B^k & \xrightarrow{\delta} & H^{k+1} M_{n+1}^{s-1} \\
\| & & \downarrow g & & \downarrow g & & \| \\
H^k M_{n+1}^{s-1} & \xrightarrow{\varphi_*} & H^k M_n^s & \xrightarrow{v_n} & H^k M_n^s & \xrightarrow{\delta} & H^{k+1} M_{n+1}^{s-1},
\end{array}$$

and every element $x \in B^k$ has an integer j such that $v_n^j x = 0$, then $H^k M_n^s = B^k$.

This is proved by an induction on j using a diagram chasing. But here I will give a proof using the Five Lemma.

Proof. For brevity, put $A = H^k M_{n+1}^{s-1}$, $A' = H^{k+1} M_{n+1}^{s-1}$, $A(j) = \{x \mid x \in H^k M_n^s,\ v_n^j x = 0\}$ and $B(j) = \{x \mid x \in B^k,\ v_n^j x = 0\}$. Then we have a commutative diagrams

$$\begin{array}{ccccccc}
A & \xrightarrow{\varphi} & A(j) & \xrightarrow{v_n} & A(j) & \xrightarrow{\delta} & A' \\
\| & & \downarrow & & \downarrow & & \| \\
A & \xrightarrow{\varphi} & H^k M_n^s & \xrightarrow{v_n} & H^k M_n^s & \xrightarrow{\delta} & A',
\end{array}$$

and

$$\begin{array}{ccccccc}
A & \xrightarrow{\varphi} & B(j) & \xrightarrow{v_n} & B(j) & \xrightarrow{\delta} & A' \\
\| & & \downarrow & & \downarrow & & \| \\
A & \xrightarrow{\varphi} & B^k & \xrightarrow{v_n} & B^k & \xrightarrow{\delta} & A'.
\end{array}$$

††Originally, this is considered for the mod p spectral sequence. In this sequence with $n = 0$, it contains all differentials of the original Bockstein spectral sequence, where $v_0 = p$.

Comparing these diagrams with the diagram of Lemma 5.1 produces another commutative one:

$$\begin{array}{ccccccc} A & \xrightarrow{\varphi} & B(j) & \xrightarrow{v_n} & B(j) & \xrightarrow{\delta} & A' \\ \| & & \downarrow & & \downarrow & & \| \\ A & \xrightarrow{\varphi} & A(j) & \xrightarrow{v_n} & A(j) & \xrightarrow{\delta} & A'. \end{array}$$

This splits into two commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im}\varphi_* & \xrightarrow{i} & B(j) & \xrightarrow{v_n} & v_nB(j) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle g_j} & & \downarrow{\scriptstyle v_ng_j} & & \\ 0 & \longrightarrow & \operatorname{Im}\varphi_* & \xrightarrow{i} & A(j) & \xrightarrow{v_n} & v_nA(j) & \longrightarrow & 0, \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & v_nB(j) & \xrightarrow{i'} & B(j-1) & \longrightarrow & A' \\ & & \downarrow{\scriptstyle v_ng_j} & & \downarrow{\scriptstyle g_{j-1}} & & \| \\ 0 & \longrightarrow & v_nA(j) & \xrightarrow{i'} & A(j-1) & \longrightarrow & A'. \end{array}$$

Here g_j is defined by $g_j(x) = g(x)$ for $g: B^k \to H^kM_n^s$ of Lemma 5.1, i's are the inclusions and i''s are also found to be inclusions since $v_n^{j-1}(v_nx) = 0$ for $v_nx \in v_nB(j)$.

Since $B(1) = A$, we see that g_1 is an isomorphism. Now suppose that g_{j-1} is an isomorphism. Then by the Five Lemma in the last diagram, we see that v_ng_j is an isomorphism. Again the Five Lemma in the diagram before the last shows that g_j is an isomorphism. Thus we obtain isomorphisms g_j inductively. Note that $H^kM_n^s = \bigcup_j A(j)$ and $B^k = \bigcup_j B(j)$ since any element x of $H^kM_n^s$ and B^k satisfies a condition that there exists an integer k such that $v_n^{kx} = 0$. Furthermore g is obtained from g_j's, and we see the desired lemma. q.e.d.

§ 6. How to compute the Bockstein spectral sequences

Now we will show how to compute the Bockstein spectral sequence. In order to explain it, we prepare the following comodules. First consider an ideal $I(j)_n$ generated by $p, v_1, \ldots, v_{n-2}, v_{n-1}^j$. That is,

$$I(j)_n = I_{n-1} + (v_{n-1}^j) \quad \text{for} \quad n > 0 \quad \text{and} \quad j > 0$$

and $I(1)_n = I_n$. Define $N(j)_n^s$ and $M(j)_n^s$, in the same manner as defining N_n^s and M_n^s, inductively by

$$N(j)_n^0 = B/I(j)_n, \; M(j)_n^s = v_{n+s}^{-1}N(j)_n^s \quad \text{and} \quad 0 \to N(j)_n^s \hookrightarrow M(j)_n^s \to N(j)_n^{s+1} \to 0,$$

so that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N(j-1)^s_n & \hookrightarrow & M(j-1)^s_n & \longrightarrow & N(j-1)^{s+1}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle v_n} & & \downarrow{\scriptstyle v_n} & & \\
0 & \longrightarrow & N(j)^s_n & \hookrightarrow & M(j)^s_n & \longrightarrow & N(j)^{s+1}_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_j} & & \downarrow{\scriptstyle p_j} & & \downarrow{\scriptstyle p_j} & & \\
0 & \longrightarrow & N^s_n & \hookrightarrow & M^s_n & \longrightarrow & N^{s+1}_n & \longrightarrow & 0
\end{array}$$

commutes, in which v_n's are monomorphisms induced from v_n: $N(j-1)^0_n \to N(j)^0_n$ and the other vertical arrows p_j's are epimorphisms induced from the one p_j: $N(j)^0_n \to N^0_n$ inductively. Note that

$$N(1)^s_n = N^s_n \quad \text{and} \quad M(1)^s_n = M^s_n.$$

Furthermore, these definitions give us inclusions

$$\iota\colon N(j)^{s-1}_{n+1} \subset N^s_n \quad \text{and} \quad \iota\colon M(j)^{s-1}_{n+1} \subset M^s_n$$

defined by

$$\iota(x) = x/v^j_n.$$

We also introduce a notation:

$$d_s(x) \equiv y \mod I(j)_n \qquad \text{for } x \in \Omega^s M^{s-1}_{n+1},$$

if $d_s(\tilde{x}) = y$ in $\Omega^{s+1} M(j)^{s-1}_{n+1}$ for some $\tilde{x}$ such that $p_j(\tilde{x}) = x$.

Now we will present how to compute the Bockstein spectral sequences. Suppose first that we have determined the structure of $H^i M^{s-1}_{n+1}$. Then find elements $\tilde{x} \in \Omega^i M^s_n$, $y(x) \in \Omega^{i+1} M^s_n$ with $v_n \nmid y(x)$, and an integer $a(n) > 0$ for each $[x] \in H^i M^{s-1}_{n+1}$ so that

$$p_j(\tilde{x}) = x,\ d_i(\tilde{x}) \equiv v_n^{a(x)} y(x) \mod I(a(x)+1)_{n+1}.$$

and

Condition 6.1. The set $\{[y(x)] \mid [x] \in H^i M^{s-1}_{n+1},\ y(x) \neq 0\}$ is linearly independent.

Note that there may be some x such that $d_i(\tilde{x}) \equiv 0 \bmod I(j)_n$ for any $j > 0$. In this case we put

$$a(x) = \infty.$$

Besides, $y(x)$ is a cocycle, that is,

$$d_{i+1}(y(x)) = 0 \in \Omega^{i+2} M^{s-1}_{n+1},$$

since $0 = d_{i+1} d_i(\tilde{x}) \equiv v_n^{a(x)} d_{i+1}(y(x)) \bmod I(a(x)+1)_{n+1}$, and $I(1)_{n+1} = I_{n+1}$. The construction of an element $\tilde{x}$ will be stated in the next section.

Now suppose again that we find $\tilde{x}$ for all x, satisfying the condition 6.1, and put

$$B^i = \bigoplus_{j>0} C(a(x))\langle[\tilde{x}]|[x] \in H^iM_{n+1}^{s-1}\rangle, \tag{6.2}$$

where

$$C(j) = k(n)_*/(v_n^j) \qquad \text{if } j < \infty, \text{ and } C(\infty) = K(n)_*/k(n)_*,$$

and $C(j)\langle x_\alpha\rangle$ denotes a direct sum of modules generated by x_α/v_n^j isomorphic to $C(j)$. For the generator of $C(\infty)\langle y_\beta\rangle$, we use a convention that it consists of y_β/v_n^k for all $k>0$. Here, $K(n)_*$ and $k(n)_*$ are well known symbol relating to the Morava K-theories such as

$$k(0)_* = \boldsymbol{Z}_{(p)},\ k(n)_* = F_p[v_n] \quad \text{and} \quad K(n)_* = v_n^{-1}k(n)_* \quad (v_0 = p).$$

LEMMA 6.3. *B^i defined above fits into the commutative diagram of Lemma* 5.1.

PROOF. We will show the following statements:
1. $B^i \subset H^iM_n^s$,
2. $\operatorname{Im}\varphi_* \subset H^iM_n^s$,
3. v_n: $B^i \to B^i$.

The first one follows from $d_i(\tilde{x}/v_n^{a(x)}) = d_i(\tilde{x})/v_n^{a(x)} = v_n^{a(x)}y(x)/v_n^{a(x)} = 0$. For the second, we compute $\varphi_*([x]) = [x]/v_n = v_n^{a(n)-1}[\tilde{x}]/v_n^{a(x)}$, since $p_j(\tilde{x}) = x$ implies $\tilde{x} \equiv x \bmod I_{n+1}$. The last one is obvious by definition. q.e.d.

LEMMA 6.4. *The sequence*

$$H^iM_{n+1}^{s-1} \xrightarrow{\varphi_*} B^i \xrightarrow{v_n} B^i \xrightarrow{\delta} H^{i+1}M_{n+1}^{s-1}$$

is exact for each $i \geq 0$.

PROOF. By Lemma 6.3, we have a commutative diagram

$$\begin{array}{ccccccc}
H^kM_{n+1}^{s-1} & \xrightarrow{\varphi_*} & B^k & \xrightarrow{v_n} & B^k & \xrightarrow{\delta} & H^{k+1}M_{n+1}^{s-1} \\
\| & & \downarrow g & & \downarrow g & & \| \\
H^kM_{n+1}^{s-1} & \xrightarrow{\varphi_*} & H^kM_n^s & \xrightarrow{v_n} & H^kM_n^s & \xrightarrow{\delta} & H^{k+1}M_{n+1}^{s-1},
\end{array}$$

an easy diagram chase shows the exactness of $H^iM_{n+1}^{s-1} \xrightarrow{\varphi_*} B^i \xrightarrow{v_n} B^i$, and that the sequence $B^i \xrightarrow{v_n} B^i \to H^{i+1}M_{n+1}^{s-1}$ is zero.

Now suppose that $\delta(\xi) = 0$ for $\xi \in B^i$, and put

$$\xi = \sum_x \lambda_x[\tilde{x}]/v_n^{a(x)} + \sum_{y,j} \mu_{y,j}[\tilde{y}]/v_n^j$$

for $\lambda_x \in k(n)_*$ and $\mu_{y,j} \in \boldsymbol{Z}_{(p)}$. Furthermore, by the definition of δ, $d_i(\tilde{x}) = v_n^{a(x)}y(x)$ gives rise to

$$\delta([\tilde{x}]/v_n^{a(x)}) = [y(x)].$$

Then,

$$\delta(\xi) = \sum_x [\lambda_x y(x)].$$

Here $[\lambda_x y(x)] = 0$ if $v_n | \lambda_x$. Otherwise, $\lambda_x \in \boldsymbol{Z}_{(p)}$, and so $\lambda_x = 0$ by Condition 6.1. Therefore,

$$\xi = \sum_{v_n|\lambda_x} \lambda_x[\tilde{x}]/v_n^{a(x)} + \sum_{y,j} \mu_{y,j}[\tilde{y}]/v_n^j,$$

and so we have an element

$$\zeta = \sum_{v_n|\lambda_x} (\lambda_x/v_n)[\tilde{x}]/v_n^{a(x)} + \sum_{y,j} \mu_{y,j}[\tilde{y}]/v_n^{j+1}$$

satisfying $v_n(\zeta) = \xi$. q.e.d.

Lemmas 6.3 and 6.4 with Lemma 5.1 shows the following

THEOREM 6.5. *Let B^i be the submodule obtained from $H^iM_{n+1}^{s-1}$ by* (6.2). Then,

$$H^iM_n^s = B^i.$$

By this theorem, if we determine $H^*M_n^0$, then we could obtain $H^*M_{n-s}^s$ and so $H^*M_0^n$ theoretically. For $H^*M_n^0$, it is computed by Ravenel [4] only for $n < 3$, and for $n = 3$ and $p > 3$ (*cf.* [6]).

§7. Construction on elements $\tilde{x}$

Now suppose that $[x] \in H^iM_{n+1}^{s-1}$. Then

$$d_i(x) \equiv 0 \mod I_{n+1}.$$

First find elements $[x]$ such that $[x]$ is not written as $[x] = [y^p]$ for any cochain $y \in \Omega^iM_{n+1}^{s-1}$. We denote a set of such elements by R^i. That is,

$$R^i = H^iM_{n+1}^{s-1} - S^i,$$

for

$$S^i = \{[x] \in H^iM_{n+1}^{s-1} | \exists y \in \Omega^iM_{n+1}^{s-1}, \text{ such that } [x] = [y^p]\}.$$

Suppose further that a computation for $[x] \in R^i$ shows

$$d_i(x) \equiv v_n^{a(x)}y(x) \not\equiv 0 \mod I(1 + a(x))_{n+1}. \tag{7.1}$$

Let G_{i+1} be a set of representatives of generators of $H^{i+1}M_{n+1}^{s+1}$, that is,

$$H^{i+1}M_{n+1}^{s+1} = \boldsymbol{Z}_{(p)}\langle [g] | g \in G_{i+1} \rangle.$$

7.1. The case where $n > 0$

In this case, p^j-th power of (7.1) turns into

$$d_i(x^{p^j}) \equiv v_n^{p^ja(x)}y(x)^{p^j} \mod I(p^j + p^ja(x))_{n+1}.$$

Then we see that $[y(x)^{p^j}] \in H^{i+1}M_{n+1}^{s-1}$, since $d_{i+1}(y(x)^{p^j}) = d_{i+1}(y(x))^{p^j} = 0$. If $y(x)^{p^j} \in G_{i+1}$, just put

$$x_j' = x^{p^j}.$$

If $y(x)^{p^j} \notin G_{i+1}$, then there would be a generator $[y'] \in H^{i+1}M_{n+1}^{s-1}$ such that $[y'] = [y(x)^{p^j}]$. Then there exists a cochain $w \in \Omega^i M_{n+1}^{s-1}$ such that $d_i(w) = y(x)^{p^j} - y'$. So put

$$x_j' = x^{p^j} - v_n^{p^j a(x)} w.$$

Usually, you would find a cochain $x' \in \Omega^i M_{n+1}^{s-1}$ such that

$$d_i(x') \equiv v_n^{a(x')} y' \mod I(a(x') + 1)_{n+1}$$

with $a(x') \leq p^j a(x)$. Then, put

$$x_j'' = x_j' - v_n^{p^j a(x) - a(x')} x'.$$

If there is no such x', then put

$$x_j = x_j'.$$

Furthermore, if there is an element such that $d_i(x'') \equiv v_n^{a(x'')} y(x_j'') \mod I(1 + a(x''))_{n+1}$. Then put

$$x_j^{(3)} = x_j'' - v_n^{a(x_j'') - a(x'')} x''.$$

Here $d_i(x_j'') \equiv v_n^{a(x_j'')} y(x_j'') \mod I(1 + a(x_j''))_{n+1}$. Otherwise,

$$x_j = x_j''.$$

Continue this process, we will get a sequence of elements

$$x_j', x_j'', \ldots, x_j^{(k)}, \ldots,$$

for each $x \in R^i$ and j. If it ends at $x_j^{(m)}$, then we put

$$x_j = x_j^{(m)}.$$

We have, so far, no example that this process does not end.

We notice that if $y(x_j)$'s are dependent, this process must be continued. In fact, if there is an relation $\sum \lambda_j y(x_j) = 0$, then $y(x_j) = -\sum_{k \neq j} \lambda_k y(x_k)$, which indicates the existence of $x'' = \sum_{k \neq j} \lambda_k x_k$ such that $d_i(x'') = y(x_j)$. Therefore, if this process ends, then Condition 6.1 is automatically satisfied. Therefore, we just put

$$B^i = k(n)_* \langle x_j | x \in R^i, j \geq 0 \rangle.$$

Note that we also use another way, which is described below. It is the way starting from $i = 0$.

7.2. The case where $n = 0$

We begin with considering the case where $i = 0$. In this case, the p^j-th power of (7.1) must be

$$d_0(x^{p^j}) \equiv p^{j+a(x)} x^{p^j-1} y(x) \quad \mathrm{mod}(p^{1+j+a(x)}),$$

by the binomial theorem. In fact,

$$\begin{aligned} d_0(x^{p^j}) &= \eta_R(x^{p^j}) - x^{p^j} \\ &= (x + d_0(x))^{p^j} - x^{p^j} \\ &= (x + p^{a(x)} y(x))^{p^j} - x^{p^j} \\ &= \sum_{i=1}^{p^j} \binom{p^j}{i} x^{p^j - i} p^{ia(x)} y(x)^i. \end{aligned}$$

The other part of construction is the same as above, and we obtain x_j for $x \in R^0$.

For greater i, take $x \in H^i M_1^{s-1}$. Then there is a sequence Q: $p, a_1, \ldots, a_{s-1}$ for $a_i = v_i^{e_i}$ such that $f(w) = x$ for $w \in H^i v_s^{-1} B/(Q)$ and f: $H^i v_s^{-1} B/(Q) \to H^i M_1^{s-1}$. Suppose further that we have an element $h \in H^i M_s^0$ and $x' \in R^0$ such that $x = x'h'$ for $h' \in H^i v_s^{-1} B/(Q)$ with $pr(h') = h$, where pr: $H^i v_s^{-1} B/(Q) \to H^i M_s^0$ is the projection. Let Q_j denote the sequence $p^j, a_1, \ldots, a_{s-1}$. If there is an element $h'_j \in H^i v_s^{-1} B/(Q_j)$ such that $pr(h'_j) = h'$ for the projection pr: $H^i v_s^{-1} B/(Q_j) \to H^i v_s^{-1} B/(Q)$, we put

$$x_j = x'_j h'_{1+j+a(x)}.$$

Here x'_j is the element obtained from x' given as above. If we can not find such h'_j, just put

$$x_j = x'_j h'.$$

Note that if

$$d_i(h') \equiv p^{a(h')} y(h') \,\mathrm{mod}(p^{1+a(h')}),$$

then

$$d_i(x_j) = \begin{cases} p^{j+a(x')} x'^{p^j-1} y(x') h' & \text{if } j + a(x') < a(h'), \\ p^{j+a(x')} (x'^{p^j-1} y(x') h' + x'_j y(h')) & \text{if } j + a(x') = a(h'), \text{ and} \\ p^{a(h')} x'_j y(h') & \text{otherwise.} \end{cases}$$

Usually, this will yield the generators, but we cannot say that the elements constructed in this way satisfy the condition 6.1.

As noticed above, this method is also applied to the case where $n > 0$.

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