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Note on the β family in the homotopy of spheres

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§1. Introduction

In the stable homotopy groups of spheres, there are α, β and γ families. These families are now recognized as v_1-, v_2- and v_3-periodic maps, respectively. In this sense, there seems to be δ family, or v_4-periodic maps, but so far we have no proof of the existence. The α family is closely related to the Im J for $J: \pi_*(SO) \to \pi_*(S^0)$, and Im J is now well understood. Our next problem is to understand Coker J. They consists of v_n-periodic maps for $n > 1$, and our first target must be v_2-periodic maps, say, the β family. The existence of the β family shows that the Coker J in the homotopy groups of spheres have non-zero groups at a dimension greater than any large dimension. Besides, products of them add more elements in the 4-th line of the Adams-Novikov spectral sequence. This shows the complexity of the groups. For example, we have not determined the ring structure of the subring of the stable homotopy groups of spheres generated by β family. Note that we know the ring structure of the subring generated by α family. Originally, these families are obtained by constructing a suitable spectrum, but after their cerebrated paper [4], these elements are characterized by the words of the Adams-Novikov spectral sequence. This idea leads to more general chromatic theory in the stable homotopy.

Here we discuss how much we have known about the β family, and so I make no claims of originality.

§2. Definition of the β family

First β-elements β_t are introduced by Toda [25] for $0 < t < p$ in the p-primary component of the stable homotopy of spheres for an odd prime p. Then L. Smith [24] defined β_t for any $t > 0$ by giving an essential self-map $\beta: \Sigma^{2p^2-2}V(1) \to V(1)$ for a prime $p \geq 3$. Here $V(n)$ for $n \geq -1$ are the so-called Toda-Smith spectrum defined inductively by $V(-1) = S^0$ and the cofiber sequences

(2.1) $V(-1) \xrightarrow{p} V(-1) \xrightarrow{i} V(0) \xrightarrow{j} \Sigma V(-1)$, and $\Sigma^{2p-2}V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1}V(0)$.

Here $p \in \boldsymbol{Z} \cong \pi_0(S^0)$ and α is an essential map known as Adams map. Using these notation, at the prime $p > 3$, β elements are defined by

(2.2) $$\beta_t = jj_1\beta^t i_1 i.$$

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We also consider the β family in the track groups $[V(0), V(0)]_*$:

(2.3) $$\beta_{(t)} = j_1 \beta^t i_1.$$

Then

$$\beta_t = j\beta_{(t)} i.$$

It is useful to use the language of BP, the Brown-Peterson spectrum at a prime number p. The coefficient ring of the Brown-Peterson spectrum BP is

$$BP_* = BP_*(S^0) = \pi_*(BP) = Z_{(p)}[v_1, v_2, \dots]$$

for generators $|v_n| = 2p^n - 2$. Using the ring structure of the Brown-Peterson spectrum we can construct an Adams-type spectral sequence, which we call the Adams-Novikov spectral sequence:

$$E_2^{s,t}(X) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*(X)) \Rightarrow \pi_*(X)$$

for a p-local spectrum X. Here the Ext group is the derived functor of Hom on $BP_*(BP)$-comodules, where $BP_*(BP)$ is a Hopf algebroid associated to the ring spectrum BP. By this, for a cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of p-local spectra with $f_* = BP_*(f)$ monomorphic, we have a long exact sequence

$$\cdots \to E_2^s(X) \xrightarrow{f_*} E_2^s(Y) \xrightarrow{g_*} E_2^s(Z) \xrightarrow{\delta} E_2^{s+1}(X) \to \cdots.$$

Therefore, the cofiber sequences of (2.1) yield the connecting homomorphisms

$$\delta\colon E_2^s(V(0)) \to E_2^{s+1}(V(-1)) = E_2^{s+1}(S^0) \quad \text{and} \quad \delta\colon E_2^s(V(1)) \to E_2^{s+1}(V(0)).$$

Composing these, we obtain a map

$$\eta = \delta\delta\colon E_2^s(V(1)) \to E_2^{s+2}(S^0).$$

H. Miller, D. Ravenel and S. Wilson [4] defined β family in the E_2-term of the Adams-Novikov spectral sequence by

(2.4) $$\beta_t = \eta(v_2^t) \in E_2^2(S^0) \quad \text{and} \quad \beta_{(t)} = \delta(v_2^t) \in E_2^1(V(0)).$$

Here note the following

LEMMA 2.5. *The element v_2^t is a cocycle, that is, it is in $E_2^0(V(1))$.*

PROOF. Note that $BP_*(V(1)) = BP_*/(p, v_1)$ and recall the Landweber's formula $\eta_R(v_2) \equiv v_2 \bmod(p, v_1)$. Besides, $x \in \mathrm{Ext}_{BP_*(BP)}^0(BP_*, BP_*/(p, v_1))$ if and only if $\eta_R(x) \equiv x \bmod(p, v_1)$, by the definition of the Ext groups. Thus the lemma follows. q.e.d.

Similarly, we obtain the following:

LEMMA 2.6. ([4]) *At the prime number $p > 2$, the element x_n^s is a cocycle of $\mathrm{Ext}_{BP_*(BP)}^0(BP_*, BP_*/(p^i, v_1^j))$ if $p^{i-1} | j \le a_{n-i+1}$ for $n \ge 0$ and $i, j > 0$.*

Here the integer a_n is defined by

$$a_0 = 1 \qquad \text{and} \qquad a_n = p^n + p^{n-1} + 1 \qquad \text{for } n > 0,$$

and the element x_n satisfies

$$x_n \equiv v_2^{p^n} \qquad \bmod (p, v_1),$$

which is defined inductively by

$$x_0 = v_2; \qquad x_1 = v_2^p - v_1^p v_2^{-1} v_3;$$

$$x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_3;$$

$$x_n = x_{n-1}^p - 2v_1^{b_n} v_2^{p^n - p^{n-1}+1} \qquad \text{for } n \geq 3$$

with $b_n = (p+1)(p^{n-1}-1)$ for $n > 1$. At the prime 2, we have another

LEMMA 2.7. ([16]) *At the prime number 2, the element x_n^s is a cocycle of $\mathrm{Ext}^0_{BP_*}(BP_*, BP_*/(2^i, v_1^j))$ if $p^{i-1}|j \leq a_{n-i+1}$ for $n \geq 0$ and $i, j > 0$. Furthermore, x_n^s is a cocycle of $\mathrm{Ext}^0_{BP_*(BP)}(BP_*, BP_*/(2^{i+2}, y_i^m))$ if $2^i m \leq a_{n-i-1}$ for $n > i+2$ and $m, i > 0$.*

Here

$$y_0 = v_1; \qquad y_1 = v_1^2 - 4v_1^{-1} v_2; \qquad \text{and} \qquad y_i = y_{i-1}^2 \qquad \text{for } i > 1.$$

Consider the short exact sequences:

$$0 \to BP_* \xrightarrow{p^i} BP_* \to BP_*/(p^i) \to 0,$$

$$0 \to BP_*/(p^i) \xrightarrow{v_1^j} BP_*/(p^i) \to BP_*/(p^i, v_1^j) \to 0 \qquad \text{for } j = mp^{i-1} \text{ with } j \leq a_{n-i+1}, \text{ and}$$

$$0 \to BP_*/(2^{i+2}) \xrightarrow{y_i^m} BP_*/(2^{i+2}) \to BP_*/(2^{i+2}, y_i^m) \to 0 \qquad \text{for } mp^i \leq a_{n-i-1} \text{ with } n > i+2.$$

We denote the connecting homomorphisms associated to this short exact sequences by

$$\delta_i\colon \mathrm{Ext}^s_{BP_*(BP)}(BP_*, BP_*/(p^i)) \to \mathrm{Ext}^{s+1}_{BP_*(BP)}(BP_*, BP_*),$$

$$\delta'_{j,i}\colon \mathrm{Ext}^s_{BP_*(BP)}(BP_*, BP_*/(p^i, v_1^j)) \to \mathrm{Ext}^{s+1}_{BP_*(BP)}(BP_*, BP_*/(p^i)), \qquad \text{and}$$

$$\delta''_{m,i}\colon \mathrm{Ext}^s_{BP_*(BP)}(BP_*, BP_*/(2^{i+2}, y_i^m)) \to \mathrm{Ext}^{s+1}_{BP_*(BP)}(BP_*, BP_*/(2^{i+2})).$$

Using these results, they defined the general β-elements:

(2.8)
$$\beta_{sp^n/j,i} = \delta_i \delta'_{j,i}(x_n^{sp^n}) \in \mathrm{Ext}^2_{BP_*(BP)}(BP_*, BP_*), \qquad \text{and}$$

$$\beta_{2^n s/2^i m, i+2} = \delta_{i+2} \delta''_{m,i}(x_n^{sp^n}) \in \mathrm{Ext}^2_{BP_*(BP)}(BP_*, BP_*).$$

§ 3. Existence of the β family

For the β elements defined in the E_2-term of the Adams-Novikov spectral sequence, we have not yet determined which detects an essential homotopy element. Here we write down some known results. In order to explain these by modern language, we introduce the Morava K-theory. For each non-negative integer n, there is a homology theory $K(n)_*(-)$ with coefficient ring $K(n)_* = K(n)_*(S^0) = \boldsymbol{Z}/p[v_n, v_n^{-1}]$ for $n > 0$ and $K(0)_* = \boldsymbol{Q}$. We call a p-local finite spectrum X a type n spectrum if n is the smallest integer such that $K(n)_*(X) \neq 0$. If X is contractible, then X has type ∞. The Toda-Smith spectrum $V(n)$ is a typical example of type $n+1$ spectrum if it exists. $V(0) = M$ is the mod p Moore spectrum, and $V(1)$ is a cofiber of the generator α of $[V(0), V(0)]_{2p-2}$ for an odd prime p. Then in [24], Smith gave an essential self-map $\beta\colon \Sigma^{2p^2-2}V(1) \to V(1)$ for $p > 3$. A cofiber of β is denoted by $V(2)$. So far, it is known that the Toda-Smith spectrum $V(n)$ exists if and only if $2n < p$ for $n < 4$ ([28], [14]). On type n spectra, we have the following

THEOREM 3.1. (Hopkins and Smith [1]) *Let X be a p-local type n finite spectrum. Then there is a self-map $f\colon \Sigma^d X \to X$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is trivial for $m > n$. Here $d = 0$ when $n = 0$ and d is a multiple of $2p^n - 2$.*

The map f of this theorem is called v_n-map. The maps p, α and β above are v_0-, v_1- and v_2-maps, respectively. Using the v_2-map β, we can define β family (2.2) in the homotopy groups of $\pi_*(S^0)$. Let

$$X' \to X \to X'' \tag{3.2}$$

be a cofiber sequence, and let

$$\partial\colon \pi_i(X'') \to \pi_{i-1}(X')$$

be the associated "geometric" boundary homomorphism, induced from $h\colon X'' \to \Sigma X'$. Suppose that $BP_*(h) = 0$. Then (3.2) induces a short exact sequence of BP_*-homology and then the boundary homomorphism

$$\delta\colon \mathrm{Ext}^s_{BP_*(BP)}(BP_*, BP_*(X'')) \to \mathrm{Ext}^{s+1}_{BP_*(BP)}(BP_*, BP_*(X')),$$

which is

$$\delta\colon E_2^s(X'') \to E_2^{s+1}(X').$$

In fact, the E_2-term of the Adams-Novikov spectral sequence is $E_2^s(W) = \mathrm{Ext}^s_{BP_*(BP)}(BP_*, BP_*(W))$.

THEOREM 3.3. (Geometric Boundary Theorem [2]) *If $\bar{x} \in E_2^s(X'')$ survives to $x \in \pi_*(X'')$, then $\delta(\bar{x}) \in E_2^{s+1}(X')$ survives to $\partial(x) \in \pi_*(X')$.*

By virtue of this theorem, we have the following

THEOREM 3.4. *Let $p > 3$. The elements $\beta_t \in E_2^2(S^0)$ given in (2.4) survives to $\beta_t \in \pi_*(S^0)$ in (2.2).*

We want to prove similar theorem for the elements of (2.8), but we know the following nonexistence theorems:

THEOREM 3.5. ([13]) *Let p be greater than 2. Then the elements β_{p^i/p^i} in (2.8) do not survive to homotopy elements.*

This theorem is closely related to the Kervaire invariant one problem. On this problem, we have

THEOREM 3.6. ([23]) *Let X be the 7-skeleton of the Mahowald spectrum $X\langle 1\rangle$, and L_2 the Bousfield localization functor with respect to $v_2^{-1}BP$. Then $\beta_{2^i/2^i}$ in* (2.8) *survives to $\pi_*(L_2X)$.*

We also have nonexistence theorem:

THEOREM 3.7. ([22]) *The element β_t in* (2.4) *at the prime* 3 *does not survive to a homotopy element if $t \equiv 4$, 7, 8 mod 9.*

Corresponding to this theorem,

THEOREM 3.8. ([12]) *The element β_t in* (2.4) *at the prime* 3 *survives to a homotopy element if $t = 1$, 2, 3, 5 and 6.*

This theorem seems to hold for $t \equiv 1$, 2, 3, 5 and 6 mod 9, but so far, the author has not got any information.

Hopkins and Smith theorem 3.1 certifies the existence:

THEOREM 3.9. *Let $p > 2$. Then for each i, j with $p^{i-1}|j$, the element $\beta_{sp^n/j,i}$ in* (2.8) *survives to a homotopy element for a large $s \gg 0$.*

In fact, their theorem suggests the existence of a v_2-map f on a type 2 spectrum $X(i,j)$ whose BP_*-homology is $BP_*/(p^i, v_1^j)$ for an odd prime p. Furthermore, $BP_*(f) = v_2^{sp^n}$ for a large s.

At a prime >3, S. Oka constructed many β-elements:

THEOREM 3.10. ([5], [7], [8]) *Let $s > 0$ and $0 < r \leq p$. Then $\beta_{sp/r}$ in* (2.8) *survives to a homotopy element if $r < p$, or if $s > 1$ and $r = p$. Besides, $\beta_{sp^2/p,2}$ in* (2.8) *survives to a homotopy element for $s > 1$.*

THEOREM 3.11. ([9], [10]) *$\beta_{sp^n/j}$ in* (2.8) *survives to a homotopy element for $s > 0$ and $j \leq 2^{n-1}p$ for each odd prime p.*

For more β-elements, we have Sadofsky's results:

THEOREM 3.12. *For j, $k \geq 1$, $s \geq 2$, and $n > \log_2(jp^{2^{k+1}-2})$, there is an element of π_*^S detected in the Adams-Novikov E_2-term by*

$$\beta_{sp^{2^{k+1}-1+n}/jp^{2^{k+1}-1},2^{k+1}} + p^{2^k}(\textit{sum of } \beta\textit{'s involving smaller powers of } v_2).$$

§ 4. Ring structure of the subring generated by the β family

Let B denote the subring of the stable homotopy groups $\pi_*(S^0)$ of spheres generated by the β family. In this section, we state some relations on this ring.

Consider the subalgebra B_1 of B generated by β_t's for $t > 0$. At a prime $p > 3$, H. Toda showed the relation in B_1:

THEOREM 4.1. ([29]) *Let r, s and t be integers* >0. *Then*

$$rs\beta_t\beta_{r+s-t} = t(r+s-t)\beta_r\beta_s.$$

COROLLARY 4.2. *A product* $\beta_{r_1}\beta_{r_2}\dots\beta_{r_k}$ $(k \geq 2)$ *is zero if* $r_1r_2\dots r_k \equiv 0 \bmod p$, *equals to* $\beta_1^{k-1}\beta_{r-k+1}$ *if* $r \not\equiv k-1 \bmod p$, *and equals to* $\beta_1^{k-2}\beta_2\beta_{sp-1}$ *if* $r = k-1+sp$ *up to some multiple of a unit. Here* $r = \sum_{i=1}^{k} r_i$.

By this, in order to determine B_1, we have to know the relation on β_1^t. For this, H. Toda also gave the following

THEOREM 4.3. ([26], [27]) $\beta_1^6 = 0$ *and* $\beta_1^5 \neq 0$ *at the prime* $p = 3$ *in* B_1, *and* $\beta_1^{p^2+1} = 0$ *at a prime* $p > 5$.

S. Oka showed

THEOREM 4.4. ([6]] $\beta_1^{21} = 0$ *at the prime* $p = 5$.

Some time later, Ravenel determined the homotopy groups of spheres at the prime 5 up to dimension 1,000. Thus we read off the following:

THEOREM 4.5. ([14]) $\beta_1^{17} \neq 0$ *and* $\beta_1^{18} = 0$ *at the prime* 5.

Recently, at a prime >5, C-N. Lee and D. Ravenel showed

THEOREM 4.6. ([3]) *At the prime* $p > 5$, $\beta_1^{p^2-p-1} \neq 0$.

Furthermore, we have relations in B_1:

THEOREM 4.7. ([19]) *Let t be a positive integer prime to p. Then,*

$$\beta_1\beta_t \neq 0$$

if $t = kp^i - (p^{i-1}-1)/(p-1) - 1$ *for some* $i > 0$ *and* k *with* $p \nmid k+1$.

In the green book [14], Ravenel also showed

THEOREM 4.8. *At the prime* $p = 5$, $\beta_s\beta_t = 0$ *if and only if* $5|st$ *at long as* $s + t \leq p^2 - p + 1$.

Now turn to the product $\beta_s\beta_{tp/j}$ *with* $j \leq p$.

THEOREM 4.9. ([11], [17], [18], [20]) $\beta_s\beta_{tp/j} = 0$ *if* $j < p$ *or* $p|st$, *and* $\beta_s\beta_{tp/p} \neq 0$ *if* $p \nmid st(s-1)$, *or if* $s = rp + 1$ *and* $p \nmid tu(u+1)$ *for* $u = (r+t)/p^n$.

By this, we have determined whether or not the product of the form $\beta_s\beta_{tp/j}$ is trivial except for the case where

$$j = p,\ s = rp + 1,\ p \nmid t \quad \text{and} \quad p^{n+1}|r + t + p^n \quad \text{for some } n \geq 0.$$

On this case, if we consider in $\pi_*(L_2S^0)$, we have

THEOREM 4.10. ([21]) *Let s and t be positive integers. Then in the homotopy groups* $\pi_*(L_2S^0)$, $\beta_s\beta_{tp/p} = 0$ *if and only if one of the following condition holds:*
1) $p|st$ *or*
2) $s = rp + 1$ *and* $p^{n+1}|r + t + p^n$ *for some integers r and* $n \geq 0$.

Use the relation $\beta_s\beta_{tp^2/p,2} = \beta_{s+t(p^2-p)}\beta_{tp/p}$ given in [11] in the E_2-term of the Adams-Novikov spectral sequence, and we have the non-triviality theorem:

THEOREM 4.11. $\beta_s\beta_{tp^2/p,2} \neq 0$ *if* $p \nmid st(s-1)$ *or if* $s = rp + 1$ *and* $p \nmid tu(u+1)$ *for* $u = (r+t)/p^n$ *for some n.*

For the product of the form $\beta_{sp/i}\beta_{tp/j}$, we also see that

THEOREM 4.12. ([17], [20])

$$\beta_{sp/i}\beta_{tp/j} = \begin{cases} 0 & i + j \leq p, \\ 0 & i + j = p + 1,\ p \nmid s + t, \\ stx\beta_1^{p-1}\beta_2\beta_{(s+t-1)p-1} & i + j = p + 2, \\ \textit{nonzero} & p + 2 < i + j \leq 2p,\ p|s + t, \text{ and} \\ & p^2 \nmid t(s + t + p) \end{cases}$$

for some $x \neq 0$.

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