J. Fac. Educ. Tottori Univ.
(Nat. Sci.), **44** (1995) 1–6

Chromatic E_1-terms — up to April 1995

Katsumi SHIMOMURA*

(*Received April 20, 1995*)

§1. Introduction

Throughout this note, everything is localized at a prime number p. For each p, in their paper [9], Miller, Ravenel and Wilson introduced the chromatic spectral sequence to compute an E_2-term of the Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(V(n))$ of the Toda-Smith spectrum $V(n)$ for each n. The intention of this note is to show in which paper an E_1-term of a chromatic spectral sequence is computed, not to claim any original result. The chromatic spectral sequence was invented to compute an E_2-term of the Adams-Novikov spectral sequence. Its E_1-term is also an E_2-term of the Adams-Novikov spectral sequence for computing homotopy groups of a localized spectrum. Besides, it is the homotopy groups under some condition. In this sense, these E_1-terms refrect some properties of homotopy groups, and the structures of E_1-terms would have more value than the way how to compute them. Unfortunately, most of the E_1-terms are too complicated to be written down explicitly. So this will show you where the E_1-terms are as well as how much they have been known by April 1995. Many of them are done in an unfamiliar journal like this.

In the next section, which is devoted for a novice, I will explain the chromatic world. Then I introduce notations in the section 3. You find the locations of E_1-terms in §4. In the last section, I will explain some additional results.

I would like to thank Professors Nori Minami and Haynes Miller. Nori encouraged me to publish this for a person who has some interests in the chromatic spectral sequence. Haynes advised me that this journal is suitable for this sake. I also thank those co-authors of mine for instigating me to go into this fields.

§2. The chromatic world

This section is designed for a novice to this field.

Working in the homotopy category $\mathscr{S}$ of spectra, an E-based Adams spectral sequence

$$E_1^{s,t} = \pi_t(E \wedge \bar{E}^s \wedge X) \Longrightarrow \pi_{t-s}(E^\wedge X).$$

plays an important role for a suitable ring spectrum E. Here $E^\wedge X$ denotes the E-nilpotent completion of a spectrum X (*cf.* [15]). The pair $(E_*, E_*(E))$ is a Hopf

* Faculty of Education, Tottori University, Tottori, 680, Japan

algebroid. In particular, if $E_*(E)$ is flat over E_*, then the category of $E_*(E)$-comodules has enough injectives and then the E_*-term is computed to be $\mathrm{Ext}^*_{E_*(E)}(E_*, E_*(X))$ (*cf.* [16]).

One of the typical examples for E is the Brown-Peterson spectrum BP. This is defined in the homotopy category $\mathscr{S}_{(p)}$ of spectra localized at each prime number p to have the coefficient ring $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$. So we work in the category $\mathscr{S}_{(p)}$. We call the BP-based Adams specal sequence the *Adams-Novikov spectral sequence.* In their paper [9], Miller, Ravenel and Wilson introduces the chromatic spectral sequence to compute the E_2-terms of the Adams-Novikov spectral sequences:

$$E_1^{s,t,*} = \mathrm{Ext}^{t,*}_{BP_*BP}(BP_*, M_n^s) \Longrightarrow E_2^{s+,*} = \mathrm{Ext}^{s+t,*}_{BP_*BP}(BP_*, BP_*/I_n).$$

Here the abutment is the E_2-term of the Adams-Novikov spectral sequence computing the homotopy groups $\pi_*(V(n-1))$ for the Toda-Smith spectrum $V(n-1)$. Furthermore, in the above, $BP_*BP = BP_*[t_1, t_2, \cdots]$ is the Hopf algebroid, I_n the ideal of BP_* generated by $p, v_1, v_2, \cdots, v_{n-1}$ ($p = v_0$, $I_0 = 0$), and M_n^s is a comodule denoted by $v_{n+s}^{-1}BP_*/(p, v^1, \cdots, v_{n-1}, v_n^\infty, \cdots, v_{n+s-1}^\infty)$ (see [9] for explicit definition, see also §3).

Their theory contains more than that. For a spectrum E, let $L_E\colon \mathscr{S} \to \mathscr{S}$ denote the Bousfield localiztion functor and $\langle E \rangle$ the Bousfield class ([2], [3], *cf.* [15]). By $E(n)$ and $K(n)$, we denote the n-th Johnson-Wilson and the n-th Morava spectrum, respectively, whose coefficient groups are $E(n)_* = \mathbf{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$ and $K(n)_* = \mathbf{Z}/p[v_n, v_n^{-1}]$. Following to [15], we write L_n for $L_{E(n)}$. Since $L_nL_m = L_n$ for $n \le m$, we have $L_nX \to L_mX$ and the chromatic tower

$$L_0\mathscr{S}_{(p)} \longleftarrow L_1\mathscr{S}_{(p)} \longleftarrow \cdots \longleftarrow L_n\mathscr{S}_{(p)} \longleftarrow \cdots \longleftarrow \mathscr{S}_{(p)}$$

of functors. Every category $L_n\mathscr{S}_{(p)}$ is classified by the homotopy functor π_*. The homotopy groups $\pi_*(X)$ is computed by the $E(n)$-based Adams spectral sequence. For a large prime p, say $p > n^2 + n$ or so, the E_2-term for a finite spectrum agrees with the homotopy groups, since the spectral sequence collapses from the E_2-term. This is the key point. In fact, the E_2-terms are computable algebraically. Besides, some algebraic theory can be applied to understand some phenomina behind the homotopy theory.

For the relation between $L_{n-1}\mathscr{S}_{(p)}$ and $L_n\mathscr{S}_{(p)}$, $\langle E(n) \rangle = \vee_{k=0}^n \langle K(k) \rangle$ implies that the difference of $L_nS_{(p)} \to L_{n-1}S_{(p)}$ is evaluated by $L_{K(n)}S_{(p)}$. For $X = V(n)$ and a large prime nimber p, the E_1-term of a chromatic spectral sequence is the E_2-term of the Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(L_{K(n)}X)$ of $L_{K(n)}X$, which will be collapse to give the homotopy groups. For instance, the homotopy groups $\pi_*(L_nV(n-1))$ is isomorphic to $\mathrm{Ext}^*_{BP_*BP}(BP_*, M_n^0)$ that is the chromatic E_1-term.

§3. Notation (the chromatic SS)

In this note we state how much we know about the E_1-terms of the chromatic spectral sequences.

First we would like to set up notations:

$$BP_* = E(\infty)_*, \ \Gamma = \Gamma(\infty) = BP_*(BP) \quad \text{and} \quad \Gamma(n) = E(n)_*(E(n)).$$

and denote

$$H^{s,t}(M) = H(\infty)^{s,t}(M) = \mathrm{Ext}_\Gamma^{s,t}(BP_*, M)$$

for a Γ-comodule M and

$$H(n)^{s,t}(M) = \mathrm{Ext}_{\Gamma(n)}^{s,t}(E(n)_*, M)$$

for an $\Gamma(n)$-comodule M.

Comodules $N(n)_j^i$ for $n \geq 0$:
First set

$$N(n)_j^0 = E(n)_*/I_j$$

for the ideal $I_j = (p, v_1, \cdots, v_{j-1})$. Suppose the comodule $N(n)_j^i$ is defined. Then we define the comodules by

$$M(n)_j^i = v_{i+j}^{-1} N(n)_j^i$$

and $N(n)_j^{i+1}$ to fit in the exact sequence

$$0 \longrightarrow N(n)_j^i \longrightarrow M(n)_j^i \longrightarrow N(n)_j^{i+1} \longrightarrow 0.$$

Then the chromatic spectral sequence is set up by the exact couple induced from the above short exact sequences:

$$E_1^{s,t} = H(n)^t M(n)_j^s \Longrightarrow H(n)^{s+t} N(n)_j^0.$$

The target of the spectral sequence is the E_2-term of the $E(n)$-based Adams spectral sequence converging to the homotopy groups $\pi_*(L_n V(j-1))$ of a spectrum $L_n V(j-1)$ whose existence is studied in [27]. Note that

$$M(n)_j^i = N(n)_j^i \text{ if } i+j=n, \text{ and } = 0 \text{ if } i+j>n.$$

Furthermore,

$$H(n)^s M(m)_j^i = H(n)^s M(n)_j^i$$

if $m \geq n$. We will denote this simply by

$$H(n)^s M_j^i = H(n)^s M(m)_j^i.$$

§4. E_1-terms (the Bockstein SS)

The chromatic E_1-terms are computed inductively by the v_j-Bockstein spectral sequences

$$E_1^{s,t} = H(n)^* M_{j+1}^{i-1} \Longrightarrow H(n)^* M_j^i$$

obtained from the short exact sequence

$$0 \longrightarrow M_{j+1}^{i-1} \xrightarrow{1/v_j} M_j^i \xrightarrow{v_j} M_j^i \longrightarrow 0.$$

Here the first step of the induction is given as follows:

$H(n)^s M_n^0 = 0$ if $s > n^2$ and $p - 1 \nmid n$ by Morava [11],
$H(n)^s M_n^0$ is known if $n \leq 3$ by Ravenel [14],
$H(2)^* M_2^0$ at the prime 3 is given by Henn [5],
Gorbounov, Siegel and Symonds [4],
Yagita [28], and Shimomura [23], independently
$H(n)^0 M_n^0 = K(n)_*$ by Ravenel [14], and Moreira [12], and
$H(n)^i M_n^0$ is known if $i = 1$ and $= 2$ by Ravenel [14].

These are found also in Ravenel's first book [16]. Note that if $p - 1 \nmid n$, then $H(n)^* M_n^0 = \pi_*(L_n V(n-1))$ by Morava's results. In [23], $\pi_*(L_2 V(1))$ at the prime 3 is also determined while $3 - 1 \mid 2$.

Using the following Lemma due to [9], we obtain $H(n)^* M_j^i$ from $H(n)^* M_{j+1}^{i-1}$:

LEMMA *Consider the commutative diagram*

$$\begin{tikzcd}
H(n)^s M_{j+1}^{i-1} \arrow[r] \arrow[dr] & H(n)^s M_j^i \arrow[r, "v_j"] & H(n)^s M_j^i \arrow[r, "\delta"] & H(n)^{s+1} M_j^i \\
& B \arrow[u, hook] \arrow[r, "v_j"] & B, \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

in which the upper sequence is the exact one associated to the above short exact sequence. If the lower sequence is also exact, then

$$H(n)^s M_j^i = B.$$

By this, we know the followings:

$H(0)^* M_0^0 = \boldsymbol{Q}$	
$H(1)^* M_0^1$	by Miller, Ravenel and Wilson [9]
$H(2)^s M_1^1$	by Miller, Ravenel and Wilson [9] ($s = 0$), Shimomura and Tamura [24] ($s = 1$, $p > 3$), Arita and Shimomura [1] ($s = 1$, $p = 3$) and Shimomura [18] ($s > 1$, $p > 3$)
$H(2)^0 M_0^2$	by Miller, Ravenel and Wilson [9] ($p > 2$), and Shimomura [17] ($p = 2$)
$H(2)^s M_0^2$	by Shimomura and Yabe [25], [26] ($s > 0$, $p > 3$)
$H(3)^0 M_2^1$	by Miller, Ravenel and Wilson [9]
$H(3)^1 M_2^1$	by Shimomura [19] ([21]) ($p > 3$)

$H(3)^0M_1^2$	by Kodama and Shimomura [7] ($p>3$)
$H(n)^0M_{n-1}^1$	by Miller, Ravenel and Wilson [9]
$H(n)^0M_{n-2}^2$	by Shimomura [20] ($p>2$ and $n\geq 4$).

Using the chromatic spectral sequence, we know the structures of

$\pi_*(L_0S^0)=\boldsymbol{Q}$	
$\pi_*(L_1S^0)$	by Ravenel [15]
$\pi_*(L_2S^0)$	by Shimomura and Yabe [26] ($p>3$).

§5. More with $T(k)$

Let $T(k)$ denote Ravenel's spectrum with

$$BP_*(T(k))=BP_*[t_1,t_2,\cdots,t_k]\subset BP_*(BP),\ \text{and}$$

put

$$M[k]_j^i=M_j^i\otimes_{BP_*}BP_*(T(k)).$$

Now we know the structure of

$H(n)^*M[k]_n^0$	by Ravenel [16]	$(n<2(p-1)(k+1)/p)$
$H(1)^*M[k]_0^1$	by Kodama and Shimomura [6]	($k>0$ and $k>1$ if $p=2$)
$H(2)^0M[k]_1^1$	by Kodama and Shimomura [6]	$k>3$ and $p>2$
$H(2)^*M[1]_2^0$	by Mahowald and Shimomura [8]	$p=2$
$H(2)^*M[1]_1^1$	by Shimomura [22]	$p=2$
$H(2)^*M[1]_1^0$	by Shimomura [22]	$p=2$
$H(3)^0M[1]_2^1$	by Mitsui and Shimomura [10]	$p>3$

Relating to $H(2)^*M[1]_2^0$, we have the structure

$$H(2)^*(E(2)_*(V(1)\wedge X))\quad\text{and}\quad H(2)^*(E(2)_*(V(0)\wedge X))$$

for $X=S^0\cup_{\alpha_1}e^4\cup_{\alpha_1}e^8$ at the prime $p=3$ by Simomura [23] and by Nakazawa and Shimomura [13], respectively. Since X is a $2p^2-2$ skeleton of the Ravenel's spectrum $T(1)$, we may consider $H(2)^*(E(2)_*(V(1)\wedge X))$ and $H(2)^*(E(2)_*(V(0)\wedge X))$ for any prime p, and the structure of them for $p>3$ is found in Ravenel's green book [16].

References

[1] Y. Arita and K. Shimomura, The chromatic E_1-term $H^1M_1^1$ at the prime 3, to appear in *Hiroshima Math. J.*

[2] A. K. Bousfield, The localization of spectra with respect to homology, *Topology*, **18** (1979), 257–281.

[3] A. K. Bousfield, The Boolean algebra of spectra, *Comment. Math. Helv.*, **54** (1979), 368–377.

[4] V. Gorbounov, S. Siegel and P. Symonds, The cohomology of the Morava stabilizer group S_2 at

the prime 3, preprint.
[5] H-W. Henn, On the mod p cohomology of profinite groups of positive p rank, preprint.
[6] N. Kodama and K. Shimomura, On the homotopy groups of a spectrum related to Ravenel's spectra $T(n)$, *J. Fac. Educ. Tottori Univ.* (*Nat. Sci.*) **42** (1993), 17–30.
[7] N. Kodama and K. Shimomura, The second line of the Adams-Novikov E_2-term for the Moore spectrum, preprint.
[8] M. Mahowald and K. Shimomura, The Adams-Novikov spctral sequence for the L_2-localization of a v_2-spectrum, *Contemporary Math.* **146**, A. M. S. (1993), 237–250.
[9] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in Adams-Novikov spectral sequence, *Ann. of Math.* **106** (1977), 469–516.
[10] H. Mitsui and K. Shimomura, The Ext groups $H^0M_2^1(1)$, *J. Fac. Educ. Tottori Univ.* (*Nat. Sci.*) **42** (1993), 85–101.
[11] J. Morava, Noetherian localizations of categories of cobordism comodules, *Ann. of Math.* **121** (1985), 1–39.
[12] M. R. F. Moreira, Primitives of BP_*BP modulo an invariant prime ideal, *Amer. J. Math.* **100** (1978), 1247–1273.
[13] Y. Nakazawa and K. Shimomura, The homotopy groups of the L_2-localization of a type one finite complex at the prime 3, preprint.
[14] D. C. Ravenel, The cohomology of the Morava stabilizer algebras, *Math. Z.* **152** (1977), 287–297.
[15] D. C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. J. Math.* **106** (1984), 351–414.
[16] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, 1986.
[17] K. Shimomura, Novikov's Ext^2 at the prime 2, *Hiroshima Math. J.* **11** (1981), 499–513.
[18] K. Shimomura, On the Adams-Novikov spectral sequence and products of β-elements, *Hiroshima Math. J.* **16** (1986), 209–224.
[19] K. Shimomura, The chromatic E_1-term $H^1M_2^1$ and its application to the homology of the Toda-Smith spectrum $V(1)$, *J. Fac. Educ. Tottori Univ.* (*Nat. Sci.*) **39** (1990), 63–83.
[20] K. Shimomura, The chromatic E_1-term $H^0M_2^n$ for $n \geq 2$, *J. Fac. Educ. Tottori Univ.* (*Nat. Sci.*) **39** (1990), 103–121.
[21] K. Shimomura, Correction to "The chromatic E_1-term $H^1M_2^1$ and its application to the homology of the Toda-Smith spectrum $V(1)$", *J. Fac. Educ. Tottori Univ.* (*Nat. Sci.*) **41** (1992), 7–11.
[22] K. Shimomura, The homotopy groups of the L_2-localized Mahowald spectrum $X\langle 1\rangle$, to appear in *Forum Mathematicum.*
[23] K. Shimomura, The homotopy groups of the L_2-localized Toda-Smith spectrum $V(1)$ at the prime 3, preprint.
[24] K. Shimomura and H. Tamura, Non-triviality of some compositions of β-elements in the stable homotopy of the Moore spaces, *Hiroshima Math. J.* **16** (1986), 121–133.
[25] K. Shimomura and A. Yabe, On the chromatic E_1-term $H^*M_0^2$, Topology and Representation Theory, *Contemporary Math.* **158**, (1994), 217–228.
[26] K. Shimomura and A. Yabe, The homotopy groups $\pi_*(L_2S^0)$, *Topology* **34** (1995), 261–289.
[27] K. Shimomura and M. Yokotani, Existence of the Greek letter elements in the stable homotopy groups of $E(n)_*$-localized spheres, *Publ. RIMS, Kyoto Univ.* **30** (1994), 139–150.
[28] N. Yagita, Small subalgebras of Steenrod and Morava stabilizer algebras, preprint.