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On homotopy of a finite spectrum at the prime 3

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§1. Introduction

Let $T(1)$ denote Ravenel's spectrum characterized by $BP_*(T(1)) = BP_*[t_1]$ as a subcomodule algebra of $BP_*(BP) = BP_*[t_1, t_2, \cdots]$, and $V(1)$ the Toda-Smith spectrum characterized by $BP_*(V(1)) = BP_*/(p, v_1)$. Here BP denotes the Brown-Peterson spectrum at a prime p with coefficient ring $BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$, where v_i's are the Hazewinkel generators with $|v_i| = 2(p^i - 1)$ (*cf.* [7]). Also consider the Bousfield localization functor L_2 from the category of p-local spectra to itself with respect to the spectrum $v_2^{-1}BP = \mathrm{dirlim}_{v_2} BP$ (*cf.* [6]). Put

$$X = S^0 \cup_{\alpha_1} e^q \cup_{\alpha_1} e^{2q} \cup_{\alpha_1} \cdots \cup_{\alpha_1} e^{(p-1)q}$$

$(q = 2p - 2)$ which is the $(p-1)q$ skeleton of $T(1)$, which is the $(p-1)q$ skeleton of BP as well. Here α_1 is the generator of $\pi_{q-1}(S^0) = \boldsymbol{Z}/p$. For $p > 3$, the homotopy groups of $\pi_*(L_2 X \wedge V(1))$ is computed from the result of the cohomology of the second Morava stabilizer algebra $S(2)$. In Ravenel's book [7], he also computed the cohomology of the Morava stabilizer algebra $S(2)$ at the prime 3 (*cf.* [5]). In this paper, we compute the homotopy groups of the spectrum $L_2 X$ at the prime 3 based on Ravenel's results [7, Th. 6.3.23]. The answer is

THEOREM. *The homotopy groups $\pi_*(L_2 V(1) \wedge X)$ is isomorphic to the E_2-term of the Adams-Novikov spectral sequence, which is the tensor product of $\Lambda(\zeta_2, \xi)$ and the free $K(2)_*$-module*

$$K(2)_*\{1, h_{11}, h_{20}\}.$$

Here $K(2)_* = (\boldsymbol{Z}/3)[v_2, v_2^{-1}]$, $|v_2| = 16$, $|\zeta| = -1$, $|\xi| = 54$, $|h_{11}| = 11$, $|h_{20}| = 15$ *and* $|b_{11}| = 34$, *in which* $|x| = r$ *means* $x \in \pi_r(L_2 V(1) \wedge X)$.

Unhappily Ravenel's result on which this theorem is based is found to have an error and so this is not a correct answer in the sense that it is not based on a correct result. But this paper shows how $\pi_*(L_2 V(1) \wedge X)$ is computed from the results on the E_2-term for $\pi_*(L_2 V(1))$, which we need in the forthcoming paper. In this sense, I believe that this has a worth to publish. The corrected answer will be found in [8].

On the cohomology $H^*S(2)$ of the Morava stabilizer algebra, around the end of 1993, H-W. Henn found a contradiction between his results and Ravenel's in [7].

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Then Ravenel admited his error and corrected it by the middle of 1994. By that time, H-W. Henn [2], V. Gorbounov, S. Siegel and P. Symonds [1] and N. Yagita [11] had computed the cohomology groups in their own ways. They express thier results in the language of the cohomology of groups. In the homotopy theorical language, the cohomology groups $H^*S(2)$ yields the E_2-term of the Adams-Novikov spectral sequence for computing the homotopy groups of $\pi_*(L_2V(1))$. Along this line, the author computes not only the E_2-term but the homotopy groups in [8].

§2. The Adams-Novikov E_2-term

Following [4] and [7], we will denote

$$V = V(1),\ X = T(1)^{(11)} = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8,\ M = T(1)^{(7)} = S^0 \cup_{\alpha_1} e^4,$$
$$VM = V(1) \wedge M \text{ and } VX = V(1) \wedge X.$$

In this section, we compute the E_2-term of the Adams-Novikov spectral sequence computing the homotopy groups $\pi_*(L_2VX)$. We denote

$$H^*N = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, N)$$

for a $BP_*(BP)$-comodule N. Then our E_2-term is

$$H^*BP_*(L_2VX).$$

Using the notation of [3],

$$BP_*(L_2V) = M_2^0,\ BP_*(L_2VM) = M_2^0 \otimes \Lambda(t_1) \text{ and}$$
$$BP_*(L_2VX) = M_2^0 \otimes Z/3[t_1]/(t_1^3).$$

In [5] (*cf.* [7]), Ravenel determined the structure of $H^*M_2^0$ at the prime 3:

THEOREM 2.1. *$H^*M_2^0$ is isomorphic as an $K(2)_*$-algebra to*

$$E(\zeta_2, \xi) \otimes E(h_{10}, h_{11}) \otimes K(2)_*[b_{10}, b_{11}]/I,$$

where

$$I = (h_{10}h_{11},\ b_{10}^2 + b_{11}^2,\ h_{10}b_{10} - h_{11}b_{11},\ h_{11}b_{10} + h_{10}b_{11}),$$

$\zeta_2 = v_2^{-1}t_2 + v_2^{-3}(t_2^3 - t_1^{12}) - v_2^{-4}v_3t_1^3$ *and*

$$\xi = \left\langle (h_{10}, h_{11}), \begin{pmatrix} h_{10} & -h_{11} \\ h_{11} & h_{10} \end{pmatrix}, \begin{pmatrix} h_{11} & -h_{10} \\ h_{10} & h_{11} \end{pmatrix}, \begin{pmatrix} h_{10} \\ h_{11} \end{pmatrix} \right\rangle.$$

This implies immediately the following

COROLLARY 2.2. *$H^*M_2^0$ is isomorphic as a $K(2)_*$-module to the tensor product of $\Lambda(\zeta_2, \xi)$ and the $K(2)_*$-module*

$$K(2)_*\{1, h_{10}, h_{11}, b_{11}\} \otimes_{K(2)_*} K(2)_*[b_{10}].$$

PROOF. Put

$$K = E(h_{10}, h_{11}) \otimes K(2)_*[b_{10}, b_{11}]/I \text{ and}$$

$$L = K(2)_*\{1, h_{10}, h_{11}, b_{11}\} \otimes_{K(2)_*} K(2)_*[b_{10}].$$

We define a map $f: K \to L$ by

$$f(b_{10}^a b_{11}^{2b+\varepsilon}) = (-1)^b b_{11}^\varepsilon b_{10}^{a+2b}$$

$$f(h_{1\varepsilon} b_{10}^a b_{11}^{2b}) = (-1)^b h_{1\varepsilon} b_{10}^{a+2b}$$

$$f(h_{1\varepsilon} b_{10}^a b_{11}^{2b+1}) = (-1)^{b+\varepsilon+1} h_{1(\varepsilon+1)} b_{10}^{a+2b+1},$$

for $\varepsilon \in \boldsymbol{Z}/2 = \{0, 1\}$. Then we see easily that f is an isomorphism by a usual fashion. q.e.d.

By definition, we have the short exact sequence

$$0 \longrightarrow M_2^0 \xrightarrow{i} BP_*(L_2VM) \xrightarrow{j} \Sigma^4 M_2^0 \longrightarrow 0.$$

This gives rise to the long exact seqeunce

$$\text{(2.3)} \qquad \cdots \longrightarrow H^s M_2^0 \xrightarrow{i_*} H^s BP_*(L_2VM) \xrightarrow{j_*} H^s M_2^0 \xrightarrow{\delta} H^{s+1} M_2^0 \longrightarrow \cdots,$$

in which δ is the multiplication by h_{10}. Thus we see the following

COROLLARY 2.4. *The E_2-term $H^*BP_*(L_2VM)$ of the Adams-Novikov spectral sequence for $\pi_*(L_2VM)$ is the tensor product of $\Lambda(\zeta_2, \xi)$ and the $K(2)_*$-module*

$$K(2)_*\{1, h, h_{20}, b_{11}\} \otimes K(2)_*[b_{10}] \oplus K(2)_*\{h_{11}\}.$$

Here h denotes the cohomology class represented by

$$h = t_1^2 - at_1$$

for $a \in BP_*(L_2VM)$ corresponding to t_1.

For computing the E_2-term for $\pi_*(L_2VX)$, we prepare the following

LEMMA 2.5. *In the E_2-term $H^*BP_*(L_2VM)$, we have the relations*

$$b_{10} = -hh_{10}, \; b_{11} = hh_{11} \quad \textit{and} \quad h_{20}b_{10} = hb_{11}.$$

PROOF. Let a denote the generator of $BP_*(VM) = BP_*/(3, v_1) \otimes \Lambda(a)$ with coaction $\psi(a) = a + t_1$. Noticing that t_1 is primitive, we compute $d_1(at_1^2)$ in the cobar complex and obtain

$$d_1(at_1^2) = t_1 \otimes t_1^2 + at_1 \otimes t_1 = -b_{10} - hh_{10},$$

which gives the first relation.

For the second, recall the homology

$$t_1^9 = v_2^2 t_1$$

raised by $d_0(v_3)$. More precisely, we had better work with the Hopf algebroid $(E(2)_*, E(2)_*(E(2)))$ instead of $BP_*(BP)$ as in [9]. Then we compute to obtain

$$\begin{aligned} d_1(t_1t_2) &= v_2t_1 \otimes \zeta_2 - v_2^{-2}(t_1 \otimes t_2^3 + t_2 \otimes t_1^9) + t_1 \otimes t_2 - t_1^2 \otimes t_1^3, \\ -d_1(v_2^{-2}t_3) &= v_2^{-2}(t_1 \otimes t_2^3 + t_2 \otimes t_1^9) + v_2^{-1}b_1, \\ -d_1(v_2a\zeta_2) &= -v_2t_1 \otimes \zeta_2 \text{ and} \\ -d_1(at_2) &= -t_1 \otimes t_2 + at_1 \otimes t_1^3. \end{aligned}$$

Summing up these we have the second homologous relation.

Note that i_* in (2.3) is an isomorphism for odd s. Then the last relation follos from

$$i_*(h_{20}b_{10}) = h_{11}b_{10} = h_{10}b_{11} = i_*(hb_{11}).$$

q.e.d.

In the above corollary, this lemma does not imply the equation $b_{10}^2 = 0$, since $hh_{10} \neq h_{10}h$.

Now consider the long exact sequence

(2.6)

$$\cdots \longrightarrow H^sBP_*(L_2VM) \longrightarrow H^sBP_*(L_2VX) \longrightarrow H^sM_2^0 \xrightarrow{\delta} H^{s+1}(L_2VM) \longrightarrow \cdots,$$

associated to the short exact sequence

$$0 \longrightarrow BP_*(L_2VM) \subset BP_*(L_2VX) \longrightarrow \Sigma^8M_2^0 \longrightarrow 0.$$

Let a and b denote the elements of $BP_*(L_2VX)$ corresponding to t_1 and t_1^2, respectively. Then we see that

$$\psi(b) = b + 2at_1 + t_1^2$$

for the structure map

$$\psi = (BP \wedge i \wedge L_2VX)_*: BP_*(L_2VX) \longrightarrow BP_*(L_2VX) \otimes_{BP_*} BP_*(BP),$$

where $i: S^0 \to BP$ denotes the unit map of BP. Hence by the definition of the connecting homomorphism, we have

$$\delta(x) = hx$$

for $h = \{t_1^2 - at_1\}$ in the exact sequence (2.6). These together with Lemma 2.5 give rise to the following

THEOREM 2.7. *The E_2-term $H^*BP_*(L_2VX)$ of the Adams-Novikov spectral sequence for $\pi_*(L_2VX)$ is the tensor product of $\Lambda(\zeta_2, \xi)$ and the free $K(2)_*$-module*

$$K(2)_*\{1, h_{20}, h_{11}\}.$$

This result indicates that the result of Ravenel's is incorrect. The correction will be found in [8], see also [2]. In fact, the result is supposed to have a kind of Poincaré duality. But this theorem does not show this result to satisfy the duality.

§ 3. The homotopy groups

The Adams-Novikov spectral sequence $\{E_r^{s,t}\}$ has differentials d_r: $E_r^{s,t} \to E_r^{s+r,t+r-1}$. Furthermore, $E_r^{s,t} = 0$ unless $t \equiv 0 \bmod 2p-2$, which is 4 in our case. Thus, $d_r = 0$ for $r \leq 4$. On the other hand, in Theorem 2.7, the bidegrees of the generators of the E_2-term are:

$$|\zeta_2| = (1, 0),\ |\xi| = (2, 56),\ |h_{20}| = (1, 16),\ |h_{11}| = (1, 12).$$

Therefore, $E_r^{s,t} = 0$ if $s > 4$, which indicates $d_r = 0$ for $r > 4$. Hence we deduce that $d_r = 0$ for all $r > 1$ and the spectral sequence collapses. Furthermore, $V(1)$ is an M-module spectrum, where M is the mod 3 Moore spectrum. Therefore there's no algebraic extensions. Hence the E_2-term is the homotopy groups of $L_2X \wedge V(1)$.

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