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A Necessary Condition for Monotone (p, μ)-u.d. mod 1 Sequences II

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1 Abstract

In this paper we improve the proof, which is more simpler, in [GK] and give a related result.

2. Definition and Results

Let $p(n)$ be the non-negative sequence with $p(1) > 0$, $s(n) = p(1) + \cdots + p(n)$ and μ be a Borel measure such that $\int_0^1 e^{2\pi i h x} d\mu(x) \neq 1$ for some $h \in N$ or μ not a point measure.

Let k be an integer $k \geq 2$. Let $\vec{a} = (a_1, a_2, \cdots, a_k)$ and $\vec{b} = (b_1, b_2, \cdots, b_k)$ be two vectors with real components. We say that $\vec{a} < \vec{b}\,(\vec{a} \leq \vec{b})$ if $a_j < b_j (a_j \leq b_j)$ for all $j = 1, 2, \cdots, k$. The set of points $\vec{x} \in R^k$ with $\vec{a} \leqq \vec{x} < \vec{b}$ denotes by $[\vec{a}, \vec{b})$.

The fractional part of $\vec{x}$ denotes $\{\vec{x}\} = (\{x_1\}, \cdots, \{x_k\})$ and $|\vec{b} - \vec{a}| = \prod_{j=1}^{k} (b_j - a_j)$. $l\vec{g}(n) = (lg_1(n), lg_2(n), \cdots, lg_k(n))$.

Definition 1. *The sequence* $(\vec{g}(n))$, $n = 1, 2, \cdots$, *is said to be u.d. mod* 1 *in* R^k *if*

$$\lim_{N \to \infty} \frac{1}{s(N)} \# \{n: \vec{a} \leq \{\vec{g}(n)\} < \vec{b},\ 1 \leq n \leq N\} = |\vec{b} - \vec{a}|,$$

for all integers $[\vec{a}, \vec{b}) \subseteq [0, 1]^k$.

Definition 2. *The sequence* $(\vec{g}(n))$ *said to be* (p, μ)*-u.d. mod* 1 *in* R^k *if*

$$\lim_{N \to \infty} \frac{1}{s(N)} \sum_{n=1}^{N} p(n) C_J(\{g(n)\}) = \int_{I^k} C_J(\vec{x}) d\mu(\vec{x}),$$

for all interval $J \subseteq [0, 1]^k$.

If $p(n) = 1$ and $\mu(\vec{x}) = d\vec{x}$, then we have ordinary u.d. mod 1 in Multi-dimension case.

Theorem 1. (Weyl). $(\vec{g}(n))$ is (p, μ)-u.d. mod 1 iff

$$\lim_{N \to \infty} \frac{1}{s(N)} \sum_{n=1}^{N} p(n) \exp(2\pi i \vec{h} \cdot \vec{g}(n)) = \int_{I^k} \exp(2\pi i \vec{h} \cdot \vec{g}(n))\, d\vec{x},$$

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for every lattice point $\vec{h} \in Z^k$, $\vec{h} \neq 0$, where $C_J(\{\vec{x}\})$ denotes the characteristic function of $J \subseteq [0, 1]^k$.

We simplify and improve the proof of [N] or [GK].

Theorem 2. *Let $(g(n))$ be a non-decreasing real sequence and μ be a Borel measure such that $\int_0^1 e^{2\pi ihx} d\mu(x) \neq 1$ for some $h \in N$ or μ not a point measure. If $(g(n))$ is (p, μ)-u.d. mod 1, then $\lim_{n\to\infty} g(n)/\log s(n) = \infty$.*

Proof. Since $s(n) = p(1) + p(2) + \cdots + p(n)$, $p(1) > 0$, for all real number t, we define $s(t)$ and $g(t)$ as follows: $s(t) = s(n)$ if $t = n \in N$, $s(t) = (t-n)s(n+1) + (n+1-t)s(n)$, otherwise. Then $s(t)$ is monotone and continuous. $g(t) = g(n)$ if $n \leq t < n+1$.

Without loss of generality, we may assume $h = 1$ and $g(n) > 0$.

By the theorem 1 (Weyl), we have

$$\lim_{t\to\infty} \frac{1}{s(t)} \sum_{j=1}^{t} p(j) e^{2\pi i g(j)} = \int_0^1 e^{2\pi i x} d\mu(x) = w, \tag{1}$$

where $|w| < 1$, because of the assumption of theorem.

For any $v > 1$ and any $\varepsilon > 0$, we can choose an N_0 in such a way that for any $t \geq N_0$

$$\frac{1}{s(t)} \sum_{j=1}^{t} p(j) e^{2\pi i g(j)} = w + \varepsilon(t), \qquad |\varepsilon(t)| < (v-1)\varepsilon. \tag{2}$$

Now we define a non-decreasing real sequence (N_k), which not always integers,

$$s(N_k)v \leq s(N_{k+1}) < s(N_k)v^2. \tag{3}$$

From (2) and (3), we have

$$\begin{aligned} &\frac{1}{s(N_{k+1}) - s(N_k)} \left| \sum_{j=N_k+1}^{N_{k+1}} p(j) e^{2\pi i g(j)} \right| \\ &= \frac{1}{s(N_{k+1}) - s(N_k)} |(w + \varepsilon(N_{k+1}))s(N_{k+1}) - (w + \varepsilon(N_k))s(N_k)| \\ &\leq |w| + \frac{1}{s(N_{k+1}) - s(N_k)} |\varepsilon(N_{k+1})s(N_{k+1}) - \varepsilon(N_k)s(N_k)| \\ &= |w| + \frac{s(N_k)}{s(N_{k+1}) - s(N_k)} \left| \varepsilon(N_{k+1}) \frac{s(N_{k+1})}{s(N_k)} - \varepsilon(N_k) \right| \\ &\leq |w| + \frac{v^2+1}{v-1} \cdot \frac{(v-1)\varepsilon}{v^2+1} = |w| + \varepsilon, \end{aligned} \tag{4}$$

Since $\varepsilon > 0$ is arbitrary, we can choose ε and $\delta > 0$ such that $\cos 2\pi\delta > |w| + \varepsilon$.

To prove $g(N_{k+1}) - g(N_k) \geq \delta$ for all (N_k), assume on the contrary, that there exists an N_k such that $0 \leq g(N_{k+1}) - g(N_k) < \delta$. If we consider the real part of (4), then we have

$$\sum_{j=N_k+1}^{N_{k+1}} p(j) \cos 2\pi\delta \leq (|w| + \varepsilon)(s(N_{k+1}) - s(N_k)), \qquad \cos 2\pi\delta \leq |w| + \varepsilon.$$

This contradicts to the definition of δ.

Thus we obtain $g(N_{k+1}) - g(N_k) > \delta$ for $k = 0, 1, \cdots,$. So we have $g(N_k) > k\delta$.

Furthermore from the definition of (N_k), we get $s(N_k) < v^{2k} s(N_0)$.

Therefore we have for $N_k \leq n < N_{k+1}$

$$\liminf_{n\to\infty} \frac{g(n)}{\log s(n)} \geq \liminf_{k\to\infty} \frac{g(N_k)}{\log s(N_{k+1})} \geq \liminf_{k\to\infty} \frac{k\delta}{2(k+1)\log v + \log s(N_0)} = \frac{\delta}{2\log v}.$$

Since $v > 1$ is arbitrary, we obtain $\lim_{n\to\infty} \dfrac{g(n)}{\log s(n)} = \infty$, which proves the theorem.

Remark. The ideas of this proof are owed to P. Scatte who sent us the letter, which corrects our proof, and to [GK]. We would like to thank him for valuable comments.

Theorem 3. *Let $(\vec{g}(n))$ be a non-decreasing real sequence in R^k and μ be a Borel measure such that $\int_{I^k} e^{2\pi i \vec{h}\vec{x}} d\mu(\vec{x}) \neq 1$ for some $\vec{h} \in \mathbf{N}$ or μ not being a point measure. If $(\vec{g}(n))_{n=1}^{\infty}$ is (p, μ)-u.d. mod 1, then*

$$\lim_{n\to\infty} \frac{\vec{g}(n)}{\log s(n)} = \infty,$$

which means the each components tends to infinite.

Proof. The proof runs the same lines as theorem 2. By theorem 1 (Weyl), we have

$$\lim_{t\to\infty} \frac{1}{s(t)} \sum_{j=1}^{t} p(j) \exp(2\pi i \vec{h} \cdot \vec{g}(n)) = \int_{I^k} \exp(2\pi i \vec{h} \cdot \vec{x}) d\mu(\vec{x}) = w_{\vec{h}},$$

where $|w_{\vec{h}}| < 1$, $\vec{h} = (h_1, h_2, \cdots, h_k)$, because of the assumption of the theorem.

For $\vec{g}(n) = (g_1(n), g_2(n), \cdots, g_k(n))$, Then we have, for all $l = 1, 2, \cdots, k$,

$$\lim_{t\to\infty} \frac{1}{s(t)} \sum_{j=1}^{t} p(j) \exp(2\pi i g_l(j)) = w_l,$$

where $|w_l| < 1$ for all l because of $|w_h| < 1$.

By the same argument as in Theorem 2, we have, for all $l = 1, 2, \cdots, k$,

$$\lim_{n\to\infty} \frac{\vec{g}_l(n)}{\log s(n)} = \infty.$$

Thus

$$\lim_{n\to\infty} \frac{\vec{g}(n)}{\log s(n)} = \infty.$$

Reference

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[N] H. Niederreiter: Distribution mod 1 of monotone sequences, Indag. Math. Vol. 46, No. 3, (1984) 315–327.