J. Fac. Educ. Tottori Univ.
(Nat. Sci.), **44** (1995) 17–64

Mixed Robinson-Shensted Correspondence, Fomin Version, and Mixed Knuth Correspondence for (*A*, *B*)-Partially Strict Tableaux

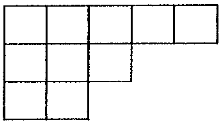
Masao ISHIKAWA

§1. Introduction and Elementary Definitions

In this paper we consider Haiman's mixed insertion in three different styles. First we consider the mixed Robinson-Schensted correspondences defined in [Ha] for preparations of latter sections. We present them in the most generalized form, i.e letters in both of top line and bottom line of a biword may have circles. Secondly we consider Fomin's generalization of Robinson-Schensted correspondences and construct the mixed version of Fomin's generalization. It will be needed to extend Fomin's generalization as fixing *R*-correspondences cell by cell. Thirdly as an applicatin of the mixed insertion we consider the mixed Knuth corresondences for (*A*, *B*)-partially strict tableaux. (*A*, *B*)-partially strict tableaux enable us to threat the Knuth and dual Knuth correspondences simultaniously. In each cases we treat ordinary and skew insertions. In the rest of this section we give terminology and elementary definitions and theorems which are well known. In Section 2 we treat Haiman's mixed insertion. In Section 3 we consider the mixed version of Fomin's generalization. In Section 4 we treat the mixed Knuth correspondence.

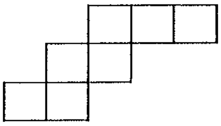
We denote the set of positive integers by **P**, the set of nonnegative integers by **N**, and the set of integers by **Z**. If $n \in \mathbf{P}$, we write $[n] := \{1, 2, \cdots, n\}$. And we use the notaion in the book [Mc] concerning partitions.

A (*skew*) *shape* (*skew diagram*) S is a finite subposet of $\mathbf{P}^2$ which is convex: i.e. $a, b \in S$ implies $[a, b] \subseteq S$. A *normal* shape is one with a unique minimum element (1, 1). We visualize a shape by a diagram in which points are designated by squares.



is an example of a normal shape.

A normal shape represents the partition whose parts are the length of its rows. For example the above normal shape is denoted by (5, 3, 2). A shape is designated by a set difference $\lambda \setminus \mu$ of two normal shapes λ and μ, wherein $\lambda \supseteq \mu$. For example the shape



is denoted by (5, 3, 2)\(2, 1). This is called "English" notation and we use only this notation unless otherwise mentioned. In the diagrams drawn in "English" notation we suppose that we take the axes down and right. We call each square a *cell* and the vertices of each square *vertices*.

Definition 1.1

Suppose we are given a finite totally ordered set $\mathscr{A}$. We use elements of $\mathscr{A}$ as letters in tableaux and biwords. A (*skew*) *reverse plane partition* π is a pair which consists of a shape $sh(\pi)$ and an order preserving map f: $sh(\pi) \to \mathscr{A}$. A *partial* (*skew*) *tableau* (resp. *standard* (*skew*) *tableau*) is by definition a (skew) reverse plane partition wherein f is injective (resp. bijective). If the shape $sh(\pi)$ is normal, we omit the word "skew" from these terminologies. We express (skew) reverse plane partitions or partial (skew) tableaux by filling each cell with the value of the function f.

Example 1.1

$$\pi = \begin{array}{cccccc} & & 1 & 1 & 2 & 3 \\ & 1 & 1 & 2 & 3 & \\ 1 & 2 & 3 & 3 & & \\ 1 & 3 & 4 & & & \end{array} \qquad \tau = \begin{array}{ccccc} & & 1 & 7 & 12 \\ & 5 & 6 & 10 & \\ 2 & 8 & 13 & & \\ 15 & & & & \end{array} \qquad \sigma = \begin{array}{cccc} 1 & 2 & 5 & 6 \\ 3 & 4 & 10 & \\ 7 & 9 & & \\ 8 & & & \end{array}$$

π is a skew reverse plane partition and τ a partial skew tableau. If $\mathscr{A} = [10]$, then σ is a standard tableau.

Definition 1.2

Let π be a tableau and $a \in \mathscr{A}$ be a letter not in π. We describe Schensted's *insertion algorithm* as follows. We insert a into the first row of π by replacing the leftmost element of elements which are greater than a. If every element in the row is smaller than a, then a is just added in the end of the row and this procedure terminates. The element replaced by a is inserted into the second row and so on. The resulting tableau is denoted by $\pi \leftarrow a$ and we call this procedure the *Robinson-Schensted insertion by rows.* If we change the word row into column in the foregoing definition, then we obtain the *Robinson-Schensted insertion by columns* and the resulting tableau is denoted by $a \to \pi$.

Example 1.2

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 8 & \\ 6 & 12 & \\ 10 & & \end{array} \leftarrow 4 = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & 8 & \\ 10 & 12 & \end{array} \qquad 4 \to \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 8 & \\ 6 & 12 & \\ 10 & & \end{array} = \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & 8 \\ 4 & 12 & \\ 10 & & \end{array}$$

Lemma 1.1

Let π be a tableau and $a, b \in \mathscr{A}$ be letters not in π. If we insert a into π by rows, let (s, t) denotes the newly added cell. If we insert b into $\pi \leftarrow a$ by rows, let (s', t') denotes the newly added cell.

(1) *If* $a < b$ *then we have* $s \geq s'$ *and* $t < t'$.

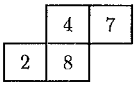
(2) *If* $a > b$ *then we have* $s < s'$ *and* $t \geq t'$. ■

Definition 1.3

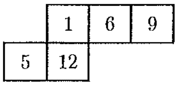
Let π and σ be partial skew tableaux which share no common parts. We use $\pi \oplus \sigma$ to denote a partial skew tableau constructed by placing translates of tableaux π, σ so that all cells of σ are above and to the right of π. If a is a letter, let the symbol a also stand for a one-cell tableau containing the letter a. So we sometimes write $\pi \oplus a$, where π is a tableau and a a letter.

Example 1.3

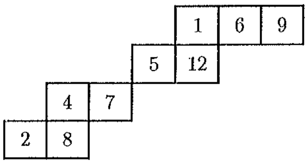
$\pi =$



$\sigma =$



$\pi \oplus \sigma =$



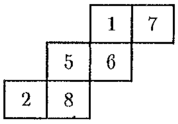
Definition 1.4

If π is a partial (skew) tableau, then $\mathrm{trunc}_{\leq a}\pi$ ($\mathrm{trunc}_{<a}\pi$) denotes the restriction of π to those cells containing letters $\leq a (< a)$.

Example 1.4

Let τ be the partial skew tableau given in Example 1.1. Then

$\mathrm{trunc}_{\leq 8}\,\tau =$



Definition 1.5

Fix another finite totally ordered set $\mathscr{A}'$. A *biword* w is by definition an injective map from a subset $\mathscr{B}'$ of $\mathscr{A}'$ into $\mathscr{A}$. Set $|w| = |\mathscr{B}'|$ to be the cardinality of $\mathscr{B}'$ and call it the *length* of w. If $\mathscr{B}' = \mathscr{A}'$ and w is a bijection of $\mathscr{A}'$ onto $\mathscr{A}$, then we call w a *permutation.* We denote w by the two-line array

$$w = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

where each letter $u_i \in \mathscr{A}'$ and $v_i \in \mathscr{A}$ appears at most once and $u_1 < u_2 < \cdots < u_m$. The *inverse word* w^{-1} of w is the inverse map w^{-1} from the image of w into $\mathscr{A}'$. If w is as above, we denote the top and bottom lines of w by $\hat{w} = u_1 u_2 \cdots u_m$ and $\check{w} = v_1 v_2 \cdots v_m$.

Example 1.5

The following w is a biword and the inverse word of w is the right one.

$$w = \begin{pmatrix} 1 & 3 & 5 & 6 & 7 & 8 \\ 7 & 2 & 6 & 9 & 1 & 4 \end{pmatrix} \qquad w^{-1} = \begin{pmatrix} 1 & 2 & 4 & 6 & 7 & 9 \\ 7 & 3 & 8 & 5 & 1 & 6 \end{pmatrix}$$

Definition 1.6

Suppose that w is given in two-line notation as

$$w=\begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

We construct a sequence of partial tableaux:

$$(\emptyset, \emptyset)=(\pi_0, \sigma_0), (\pi_1, \sigma_1), \dots, (\pi_m, \sigma_m)=(\pi, \sigma)$$

where $v_1, v_2, \dots, v_m$ are inserted by rows into the π's and $u_1, u_2, \dots, u_m$ are placed in the σ's so that π_k and σ_k have the same shape for all k. π is denoted by $\emptyset \leftarrow w$. σ is said to be the *recording tableau* and denoted by $R: \emptyset \leftarrow w$.

Example 1.6

Let w be as in Example 1.5. Then we have

$$\emptyset \leftarrow w = \begin{array}{|c|c|c|}\hline 1 & 4 & 9 \\ \hline 2 & 6 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \qquad R: \emptyset \leftarrow w = \begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 3 & 8 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array}$$

Theorem 1.1

Fix finite totally ordered sets $\mathscr{A}$ *and* $\mathscr{A}'$. *Fix* $n \in P$. *The map* $w \mapsto (\emptyset \leftarrow w,\ R; \emptyset \leftarrow w)$ *just defined is a bijection between biword of length n and pairs of partial tableaux having the same shape which is a parition of n.*

Definition 1.7

If the top line of w is $1, 2, \dots, m$, then we denote it only by the bottom line of w and call it a word.

Let π be a partial skew tableau whose i-th row is designated by R_i for $i=1, 2, \dots, l$, where l is the number of rows of π. The *row word for* π is by definition

$$w_\pi = R_l R_{l-1} \cdots R_1 .$$

Example 1.7

The row word for τ in Example 1.1 is

$$w_\tau = 15\ \ 2\ \ 8\ \ 13\ \ 5\ \ 6\ \ 10\ \ 1\ \ 7\ \ 12.$$

Next we define the Schützenberger's Jeu de Taquin in accordance with [Sa].

Definition 1.8

Let π be a partial skew tableau of shape λ/μ. And let c be a cell which is at an outer corner of λ. We define a *forward slide* on π into c as follows.

Set $c=(i, j)$. Let c' be the cell of $\max\{\pi_{i-1,j}, \pi_{i,j-1}\}$. Then we slide $\pi_{c'}$ into c. Reset $c := c'$.

We continue this procedure until we reach an outer corner of μ. We denote the resulting tableau by $j_c(\pi)$.

Let c be a cell which is an inner corner of μ. Similarly we define a *backword slide* on π into c to produce the tableau $j^c(\pi)$ by the following.

Set $c = (i, j)$. Let c' be the cell of min $\{\pi_{i+1,j}, \pi_{i,j+1}\}$. Then we slide $\pi_{c'}$ into c. Reset $c := c'$.
We continue this procedure until we reach an inner corner of λ.

Example 1.8
If we perform the forward slide on τ defined in Example 1.1 into the cell (3, 4), then we obtain the following τ'. And if we perform the backward slide on τ into (1, 2), then we obtain the following τ''.

$$\tau' = \begin{array}{|c|c|c|c|c|} & & 1 & 7 & 12 \\ & & 6 & 10 & \\ 2 & 5 & 8 & 13 & \\ 15 & & & & \end{array} \qquad \tau'' = \begin{array}{|c|c|c|c|c|} & 1 & 6 & 7 & 12 \\ & 5 & 10 & & \\ 2 & 8 & 13 & & \\ 15 & & & & \end{array}$$

Definition 1.9
Given a partial skew tableau π, we play *jeu de taquin* by choosing an arbitrary sequence of slides that brings π to normal shape ad then applying the slides. The resulting tableau is denoted by $j(\pi)$.

The following theorem is proven in [Sc].

Theorem 1.2 (Schützenberger)
Let π be a partial skew tableau. Let π' be a partial tableau of a normal shape obtained from π by a sequence of slides. Then π' is unique-in fact. π' is the insertion tablesu for w_π, i.e. $\pi' = \emptyset \leftarrow w_\pi$. ■

Example 1.9
Let τ be as in Example 1.1. Then

$$j(\tau) = \emptyset \leftarrow w_\tau = \begin{array}{|c|c|c|c|c|} 1 & 5 & 6 & 7 & 12 \\ 2 & 10 & & & \\ 8 & 13 & & & \\ 15 & & & & \end{array}$$

As an easy corollary of the theorem we obtain the following.

Corollary 1.1
Let π be a partial tableau and a a letter not in π. Then we have

$$j(\pi \oplus a) = \pi \leftarrow a \quad \text{and} \quad j(a \oplus \pi) = a \rightarrow \pi. \quad ■$$

Definition 1.10
Let P be any finite poset. For k a positive integer, set $c_k(P)$ (resp. $a_k(P)$) to be the size of the largest number of elements which is the union of k chains (resp. antichains). Now, let

$\lambda_k(P) = c_k(P) - c_{k-1}(P)$ and $\mu_k(P) = a_k(P) - a_{k-1}(P)$. Then $\lambda(P) = (\lambda_1(P), \lambda_2(P), \ldots)$ and $\mu = (\mu_1(P), \mu_2(P), \ldots)$ are partitions.

The following theorem is proven in [GK].

Theorem 1.3 [GK]
Let P be any finite poset. Then $\mu(P)$ is equal to the cojugate of $\lambda(P)$. ■

Definition 1.11
Let w be a biword. Suppose that w is given in two-line notation as

$$w = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

A poset $P(w)$ induced from w is by definition a subposet of $\mathbf{N}^2$ composed of (u_i, v_i) for $i = 1, 2, \ldots m$.

We cite a theorem from [Gr].

Theorem 1.4 (Greene)
Let w be a biword. Let λ denote the shape of $\emptyset \leftarrow w$. Then for each k, we have

$$c_k(P(w)) = \lambda_1 + \lambda_2 + \cdots + \lambda_k$$

$$a_k(P(w)) = \lambda'_1 + \lambda'_2 + \cdots + \lambda'_k. \quad ■$$

§2 Mixed Robinson-Schensted Correspondence

We treat Haiman's mixed insertion in this section. We cite a lemma (Lemma 2.1) from [Ha] but the proof will be different. Corollary 2.5 will be important to prove a theorem in Section 3. In the latter part of this section we consider Stanley-Sagan's skew insertion in mixed version. The best reference for this section is [Ha].

Fix a finite totally ordered set $\mathscr{A}$ throughout this section. A pair (U, C) of subsets of $\mathscr{A}$ is called a *division* of $\mathscr{A}$ if it satisfies

$$U \uplus C = \mathscr{A}. \qquad \text{(disjoint union)}$$

Henceforth, we fix a division (U, C) of $\mathscr{A}$, and we call elements of U *uncircled letters* and elements of C *circled letters*.

Example 2.1

$$U = \{1, 2, 3, 4, 5, 6, 14, 15, 16, 17, 18, 19\}$$

$$C = \{{}^\circ 7, {}^\circ 8, {}^\circ 9, {}^\circ 10, {}^\circ 11, {}^\circ 12, {}^\circ 13, {}^\circ 20, {}^\circ 21, {}^\circ 22\}$$

is a division of [22]. As in this example we express the elements of C with circles since it is easy to distinguish them at first glance.

Example 2.2
The definition of (skew) reverse plane partitions, partial (skew) tableaux and standard (skew)

tableaux are as in Definition 1.1. We designate ciricled letters with circles as well. For example

$\pi =$

1	2	3	6	°10	17
°7	°8	°9	°11	18	
°12	15	16	19		
°13	°20	°21	°22		
14					

is a partial tableau.

Next we define Haiman's mixed insertion.

Definition 2.1 [Ha]
Let π be a partial tableau, and let $x \in \mathscr{A}$ be a letter which is not in π. We define $\text{INSERT}_{(U,C)}(x)$ as follows.

If $x \in U$, insert x into the first row of π; if $x \in C$, insert x into the first column of π. If the bumped element is uncircled, then we insert the element into the row immediately below, or if the bumped element is circled, then we insert the element into the column immediately to its right. Continue until an insertion takes place at the end of a row or column, bumping no new element. This procedure terminates in a finite number of steps.

Similarly we define $\overline{\text{INSERT}}_{(U,C)}(x)$ by swapping U and C in the foregoing definition. Namely uncircled letters are inserted into the column immediately to its right and circled letters are inserted into the row immediately below.

If we apply $\text{INSERT}_{(U,C)}(x)$ to π, then we denote the resulting partial tableau by $\pi \leftarrow^{m} x$. Similarly if we apply $\overline{\text{INSERT}}_{(U,C)}(x)$ to π, we denote the resulting partial tableau by $x \rightarrow^{m} \pi$.

Example 2.3
Let π be the partial tableau given in Example 2.2. Then $\pi \leftarrow^{m} 4$ and $5 \rightarrow^{m} \pi$ are as follows.

$\pi \leftarrow^{m} 4 =$

1	2	3	4	°10	17
6	°7	°8	°9	°11	
°12	15	16	18	°22	
°13	19	°20	°21		
14					

$5 \rightarrow^{m} \pi =$

1	2	3	6	°10	17
5	°8	°9	°11	18	
°7	14	15	16	19	
°12	°20	°21	°22		
°13					

Next we define the conversion.

Definition 2.2 [Ha]
Let π be a partial tableau, and x any letter in π. Let $y \in A$ be a letter not in π. The operation of *converting* x into y in π is defined as follows.

First we replace x by y. The resulting tableau is not in general a partial tableau, so we repeat the following procedure until it becomes a partial tableau.
Let the letters in the cells adjecent to the cell of y, and which are immediately to its feft,

above, the its right and beneath, be y_1, y_2, y_3 and y_4, respectively.

$$\begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & y_2 & \multicolumn{1}{c}{} \\ \hline y_1 & y & y_3 \\ \hline \multicolumn{1}{c|}{} & y_4 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

Then one of the following two cases can occur.

(1) $y_1 > y$ or $y_2 > y$ If $y_1 > y_2$, we swap y and y_1, otherwise we swap y and y_2.

(2) $y > y_3$ or $y > y_4$ If $y_3 < y_4$, we swap y and y_3, otherwise we swap y and y_4.

Once case (1) occurs, only case (1) can continue to occur, once case (2) occurs, only case (2) can continue to occur.

The resulting partial tableau in which x is converted into y in π is denoted by $\pi(x \rightarrow y)$.

Example 2.4

Let π be a partial tableau given in Example 2.2. Then

$$\pi({}^\circ 22 \rightarrow 4) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & {}^\circ 10 & 17 \\ \hline 4 & {}^\circ 8 & {}^\circ 9 & {}^\circ 11 & 18 \\ \cline{1-5} {}^\circ 7 & {}^\circ 12 & 16 & 19 \\ \cline{1-4} {}^\circ 13 & 15 & {}^\circ 20 & {}^\circ 21 \\ \cline{1-4} 14 \\ \cline{1-1} \end{array}$$

It is easy to see that the procedure of conversion is reversible. i.e.

$$[\pi(x \rightarrow y)](y \rightarrow x) = \pi$$

The following lemma is from [Ha], but the proof is different. We use a result of Schützenberger or Thomas which is Theorem 3.9.7 in [Sa].

Lemma 2.1 [Ha]

Let π be a partial tableau with exactly one circled letter ${}^\circ x$ which is the greatest letter in π. Let a be a uncircled letter which is not in π, and $-\infty$ an uncircled letter less than a and all letters of π. Then we have

$$[\pi \leftarrow^m a]({}^\circ x \rightarrow -\infty) = [\pi({}^\circ x \leftarrow -\infty)] \leftarrow^m a.$$

Proof.

It is enough to show

$$\pi \leftarrow^m a = [\{\pi({}^\circ x \rightarrow -\infty)\} \leftarrow^m a](-\infty \rightarrow {}^\circ \mathrm{x}).$$

Set b to be the greatest letter of a and all uncircled letters in π. Let c denote the cell which contains ${}^\circ x$. We remove the cell c from π, and we obtain a partial tableau which is denoted by α. Then we slide α into c and we obtain a partial skew tableau which is denoted by α'. Obviously if we slide α' into the cell (1, 1), then we obtain α again.

$$\pi = \begin{array}{|c|c|c|c|}\hline 1&3&4&9\\ \hline 2&6&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&10&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3}\end{array} \qquad \alpha = \begin{array}{|c|c|c|c|}\hline 1&3&4&9\\ \hline 2&6&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&\multicolumn{1}{c|}{10}\\ \cline{1-2}\end{array} \qquad \alpha' = \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{}&1&4&9\\ \hline 2&3&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&6&\multicolumn{1}{c|}{10}\\ \cline{1-3}\end{array}$$

We treat two cases. When we insert a into the partial tableau α, the new cell added to α is (1) above and to the right of c, (2) equal to c or below and to the left of c.

Case (1):

In α' we place ${}^{\circ}x$ in the cell which is immediately below c and we supplement an appropriate number of cells containing uncircled letters $x_1, x_2,\ldots,x_m$ to its left if needed. Here we suppose that $b < x_1 < x_2 < \cdots < x_m < {}^{\circ}x$. The resulting partial skew tableau is denoted by φ'. If we slide φ' into the cell (1, 1), the resulting partial tableau is denoted by φ. π is the same as φ except the letters $x_1, x_2,\ldots,x_m$. If we fill the cell (1, 1) of α' with $-\infty$, we obtain $\pi({}^{\circ}x \to -\infty)$. We write this as α_0. Similary if we fill the cell (1, 1) of φ' with $-\infty$, we obtain a partial tableau and we denote this partial tableau by φ_0.

$$\varphi' = \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{}&1&4&9\\ \hline 2&3&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&6&\multicolumn{1}{c|}{10}\\ \cline{1-3} x_1&x_2&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3}\end{array} \qquad \varphi = \begin{array}{|c|c|c|c|}\hline 1&3&4&9\\ \hline 2&6&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&10&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3} x_1&\multicolumn{1}{c|}{x_2}\\ \cline{1-2}\end{array} \qquad \varphi_0 = \begin{array}{|c|c|c|c|}\hline -\infty&1&4&9\\ \hline 2&3&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&6&\multicolumn{1}{c|}{10}\\ \cline{1-3} x_1&x_2&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3}\end{array}$$

From the assumption $\varphi \leftarrow^{m} a$ is the same as $\pi \leftarrow^{m} a$ except the letters $x_1, x_2,\ldots,x_m$ and these letters are not bumped in the process of $\varphi \leftarrow^{m} a$. Set $\sigma = \varphi \oplus a$. By $j(\sigma) = \varphi \leftarrow^{m} a, j(\sigma)$ is the same as $\pi \leftarrow^{m} a$ except the letters $x_1, x_2,\ldots,x_m$. Set $\sigma' = \varphi' \oplus a$. If we slide σ' into the cell (2, 1), we obtain σ. So it is easy to see that $j(\sigma')$ is equal to $\pi \leftarrow^{m} a$ except the letters $x_1, x_2,\ldots,x_m$.

$$\sigma' = \begin{array}{|c|c|c|c|c|}\cline{5-5} \multicolumn{4}{c|}{}&7\\ \cline{2-5} \multicolumn{1}{c|}{}&1&4&\multicolumn{1}{c|}{9}\\ \cline{1-4} 2&3&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&6&\multicolumn{1}{c|}{10}\\ \cline{1-3} x_1&x_2&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3}\end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|}\cline{5-5} \multicolumn{4}{c|}{}&7\\ \hline 1&3&4&\multicolumn{1}{c|}{9}\\ \cline{1-4} 2&6&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&10&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3} x_1&\multicolumn{1}{c|}{x_2}\\ \cline{1-2}\end{array} \qquad \sigma_0 = \begin{array}{|c|c|c|c|c|}\cline{5-5} \multicolumn{4}{c|}{}&7\\ \hline -\infty&1&4&\multicolumn{1}{c|}{9}\\ \cline{1-4} 2&3&\multicolumn{1}{c|}{8}\\ \cline{1-3} 5&6&\multicolumn{1}{c|}{10}\\ \cline{1-3} x_1&x_2&\multicolumn{1}{c|}{{}^{\circ}x}\\ \cline{1-3}\end{array}$$

$\alpha_0 \leftarrow^{m} a$ is equal to $j(\alpha_0 \oplus a)$. If we put $\sigma_0 = \varphi_0 \oplus a, j(\sigma_0)$ is the same as $j(\alpha_0 \oplus a)$ except the letters $x_1, x_2,\ldots,x_m$ and ${}^{\circ}x$. If we slide σ' into the cells $(1, \lambda_1), (1, \lambda_1 - 1),\cdots,(1, 2)$ in this order and then slide into the cell (2, 1), it is easy to see the resulting tableau is the same as $j(\sigma_0)$ except the letter $-\infty$. Here we assume that the shape of π is λ. We denote the result after this procedure by σ'', then σ'' is the same as $j(\sigma_0)$ except the letter $-\infty$. So σ'' is the same as $\sigma_0 \leftarrow^{m} a$ except the letter $-\infty$.

Consequently if we slide σ'' into the cell (1, 1), the resulting tableau is the same as $(\alpha_0 \leftarrow a)(-\infty \to {}^{\circ}x)$ except the letters $x_1, x_2,\ldots,x_m$. Since $j(\sigma'')$ is independent of the order of slides, we have

$$\pi \leftarrow^{m} a = [\{\pi({}^{\circ}x \to -\infty)\} \leftarrow^{m} a](-\infty \to {}^{\circ}x).$$

And this prove the lemma in Case (1).
Case (2):

We place ${}^\circ x$ in the cell immediately to the right of the cell c in α' and supplement an appropriate number of cells containing uncircled letters $x_1, x_2, \ldots, x_m$ to the above of ${}^\circ x$ if needed. Here we also suppose that $b < x_1 < x_2 < \cdots < x_m < {}^\circ x$. We denote the resulting partial skew tableau by φ' and we can proceed in the same way as Case (1). ■

Remark 2.1
Under the same assumption as in Lemma 1, we have

$$[a \to^m \pi]({}^\circ x \to -\infty) = a \to^m [\pi({}^\circ x \to -\infty)].$$

And we conclude the following claim immediately from Lemma 1.
Let π be a partial tableau and ${}^\circ x$ be the least letter of the circled letters in π. Let a be an uncircled letter not in π and $-\infty$ an uncircled letter less than a and all letters in π. Then we have

$$[\pi \leftarrow^m a]({}^\circ x \to -\infty) = [\pi({}^\circ x \to -\infty)] \leftarrow^m a$$

$$[a \to^m \pi]({}^\circ x \to -\infty) = a \to^m [\pi({}^\circ x \to -\infty)].$$

Proof.
By the lemma we have

$$[(\mathrm{trunc}_{\leq {}^\circ x}\pi) \leftarrow^m a]({}^\circ x \to -\infty) = [(\mathrm{trunc}_{\leq {}^\circ x}\pi)({}^\circ x \to -\infty)] \leftarrow^m a.$$

And it is trivial that the other parts in $[\pi \leftarrow^m a]({}^\circ x \to -\infty)$ are the same as those in $[\pi({}^\circ x \to -\infty)] \leftarrow^m a$. ■

Fix another set of alphabets $\mathscr{A}'$ and its division (U', C'). We defined biwords and permutations in Definition 1.4.

Example 2.5

$$w = \begin{pmatrix} 1 & {}^\circ 2 & 3 & {}^\circ 4 & 5 & 6 & {}^\circ 7 & 8 & 9 & 10 & {}^\circ 11 & 12 & {}^\circ 13 & {}^\circ 14 \\ 14 & 13 & {}^\circ 5 & 12 & {}^\circ 6 & 2 & 11 & {}^\circ 9 & 1 & {}^\circ 8 & {}^\circ 4 & {}^\circ 7 & {}^\circ 10 & 3 \end{pmatrix}$$

is a permutation if $\mathscr{A} = \mathscr{A}' = [14]$. And

$$w^{-1} = \begin{pmatrix} 1 & 2 & 3 & {}^\circ 4 & {}^\circ 5 & {}^\circ 6 & {}^\circ 7 & {}^\circ 8 & {}^\circ 9 & {}^\circ 10 & {}^\circ 11 & 12 & 13 & 14 \\ 9 & 6 & {}^\circ 14 & {}^\circ 11 & 3 & 5 & 12 & 10 & 8 & {}^\circ 13 & 7 & {}^\circ 4 & {}^\circ 2 & 1 \end{pmatrix}$$

Definition 2.3
Let

$$w = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

be a biword. We define the *mixed insertion tableau* of w as follows:
Construct a sequence of partial tableaux $\emptyset = \pi_0, \pi_1, \ldots, \pi_m = \pi$: for each $i = 1, 2, \ldots, m$ form π_i from π_{i-1} by performing $\mathrm{INSERT}_{(U,C)}(v_i)$ on π_{i-1} if u_i is a uncircled letter, or performing

$\overline{\mathrm{INSERT}}_{(U,C)}(v_i)$ on π_{i-1} if u_i is a circled letter. The resulting partial tableau π is denoted by $\emptyset \leftarrow^m w$. And the recording partial tableau is denoted by $R: \emptyset \leftarrow^m w$.

Example 2.6

Let w be the permutation in Example 2.5. Then

$$\emptyset \leftarrow^m w = \begin{array}{|c|c|c|c|c|c|}\hline 1 & {}^\circ 4 & {}^\circ 9 & {}^\circ 10 & 13 & 14 \\ \hline 2 & {}^\circ 5 & 12 \\ \cline{1-3} 3 & {}^\circ 8 \\ \cline{1-2} {}^\circ 6 & 11 \\ \cline{1-2} {}^\circ 7 \\ \cline{1-1} \end{array} \qquad R: \emptyset \leftarrow^m w = \begin{array}{|c|c|c|c|c|c|}\hline 1 & {}^\circ 2 & {}^\circ 4 & {}^\circ 7 & {}^\circ 11 & {}^\circ 13 \\ \hline 3 & 6 & 12 \\ \cline{1-3} 5 & 9 \\ \cline{1-2} 8 & 10 \\ \cline{1-2} {}^\circ 14 \\ \cline{1-1} \end{array}$$

Theorem 2.1

Fix sets of alphabets $\mathscr{A}'$ *and their divisions* (U', C') *and* (U, C) *respectively. Fix a positive integer* $n \in \mathbf{P}$. *The map* $w \mapsto (\emptyset \leftarrow^m w, R: \emptyset \leftarrow^m w)$ *just defined is a bijection between biwords of length n and pairs of partial tableaux such that they have the same shape which is a partition of n.*

Let w be a biword. Let x be a letter in the bottom (resp. top) line of w and $y \in \mathscr{A}$ be a letter not in the bottom (resp. top) line of w. We indicate the word wherein x is replaced by y by $w(x \rightarrow^b y)$ (resp. $w(x \rightarrow^t y)$). In the case of top line we rearrange the biword so that the top line of w is in increasing order.

Theorem 2.2 (Haiman)

Let w be a biword and ${}^\circ x$ *be the least letter of the circled letters in the bottom line of w. Let* $-\infty$ *be an uncircled letter less than all letters in the bottom line of w. Then we have*

$$\emptyset \leftarrow^m [w({}^\circ x \rightarrow^b -\infty)] = [\emptyset \leftarrow^m w]({}^\circ x \rightarrow -\infty)$$
$$R: \emptyset \leftarrow^m [w({}^\circ x \rightarrow^b -\infty)] = R: \emptyset \leftarrow^m w$$

Proof.

We use induction on length of a word m. It is clear if $m = 0$. Put $w = w'a$, where

$$w' = \begin{pmatrix} u_1 & u_2 & \cdots & u_{m-1} \\ v_1 & v_2 & \cdots & v_{m-1} \end{pmatrix}$$

and $a = (u_m, v_m)$. And let $\pi = \emptyset \leftarrow^m w'$. First we assume that u_m is uncircled.

(Case 1) $v_m = {}^\circ x$:

Assume that in $\pi \leftarrow^m {}^\circ x$, ${}^\circ x$ is in the cell $(k, 1)$. Let $a_1 < a_2 < \cdots < a_{k-1}$ be the letters in the first row above it in π. Then these are uncircled letters since ${}^\circ x$ is the least letter of uncircled letters in π. The first k letters in the first column in $\pi \leftarrow^m {}^\circ x$ are $a_1, a_2, \ldots, a_{k-1}$, ${}^\circ x$ and the first k letters in the first column in $\pi \leftarrow^m -\infty$ are $-\infty, a_1, a_2, \ldots, a_{k-1}$. They have the same parts in other fields since the subsequent bumping process is identical in $\pi \leftarrow^m {}^\circ x$ and $\pi \leftarrow^m -\infty$. So it is easy to verify

$$\emptyset \leftarrow^m [w({}^\circ x \rightarrow -\infty)] = \pi \leftarrow^m -\infty = [\emptyset \leftarrow^m w]({}^\circ x \rightarrow -\infty).$$

And the other identity is trivial.

(Case 2) $a \neq {}^\circ x$:

If v_m is circled, then $v_m > {}^\circ x$. So the process to insert v_m into π and the process to convert ${}^\circ x$ to $-\infty$ in π have no influence to each other. So the claim is clear. If v_m is uncircled, then we can easily prove the claim by Remark 1.

In the case that u_m is circled, we can prove in the same way. ■

Corollary 2.1 [Ha]

Let w be a biword and ${}^\circ x$ *be a circled letter in the bottom line of w. Let y be a circled letter which is greater or less than the same circled letters in the bottom line of w as* ${}^\circ x$*, but may differ from* ${}^\circ x$ *in its order relation to the uncircled letters. Then*

$$\emptyset \leftarrow^m [w({}^\circ x \rightarrow^b {}^\circ y)] = [\emptyset \leftarrow^m w]({}^\circ x \rightarrow {}^\circ y)$$
$$R: \emptyset \leftarrow^m [w({}^\circ x \rightarrow^b {}^\circ y)] = R: \emptyset \leftarrow^m w$$

Proof.

Let ${}^\circ b_1 < {}^\circ b_2 < \cdots < {}^\circ b_k$ be the circled letters less than ${}^\circ x$ in the bottom line of w. Successively convert $({}^\circ b_1 \rightarrow -b_1)$, $({}^\circ b_2 \rightarrow -b_2), \dots, ({}^\circ b_k \rightarrow -b_k)$, where $-b_1 > -b_2 > \cdots > -b_k$ are less than all elements in the bottom line of w. Theorem 2.1 applies at each stage. Now convert $({}^\circ x \rightarrow -\infty)(-\infty \rightarrow {}^\circ y)$. Finally convert $({}^\circ -b_k \rightarrow b_k)$, $({}^\circ -b_{k-1} \rightarrow b_{k-1}), \dots, ({}^\circ -b_1 \rightarrow b_1)$. Since these conversions involve only letters less than $(x \rightarrow)$ and $(y \rightarrow)$, they commute with $({}^\circ x \rightarrow {}^\circ y)$. ■

Corollary 2.2

Let w be a biword. Let ${}^\circ x_1, {}^\circ x_2, \dots, {}^\circ x_k$ *(resp.* ${}^\circ y_1, {}^\circ y_2, \dots, {}^\circ y_l$*) be all the circled letters in the top (resp. bottom) line of w. Set*

$$w' = w({}^\circ x_1 \rightarrow^t -x_1)({}^\circ x_2 \rightarrow^t -x_2)\cdots({}^\circ x_k \rightarrow^t -x_k)({}^\circ y_1 \rightarrow^b -y_1)({}^\circ y_2 \rightarrow^b -y_2)\cdots({}^\circ y_l \rightarrow^b -y_l),$$

where $-x_1 > -x_2 > \cdots > -x_k$ *(resp.* $-y_1 > -y_2 > \cdots > -y_k$*) are uncircled letters which are less than all the uncircled letters in the top (resp. bottom) line of w. Then*

$$[\emptyset \leftarrow^m w]({}^\circ y_1 \rightarrow -y_1)({}^\circ y_2 \rightarrow -y_2)\cdots({}^\circ y_k \rightarrow -y_k) = \emptyset \leftarrow w'$$
$$[R: \emptyset \leftarrow^m w]({}^\circ x_1 \rightarrow -x_1)({}^\circ x_2 \rightarrow -x_2)\cdots({}^\circ x_k \rightarrow -x_k) = R: \emptyset \leftarrow w'$$

Proof.

Set $w'' = w({}^\circ y_1 \rightarrow^b -y_1)({}^\circ y_2 \rightarrow^b -y_2)\cdots({}^\circ y_k \rightarrow^b -y_k)]$. By Theorem 2.1 we have

$$\emptyset \leftarrow^m w'' = [\emptyset \leftarrow^m w]({}^\circ y_1 \rightarrow -y_1)({}^\circ y_2 \rightarrow -y_2)\cdots({}^\circ y_k \rightarrow -y_k).$$

w'' may have circled letters only in the top line. So $\emptyset \leftarrow^m w''$ corresponds to the Haiman's left-right insertion. Set

$$w'' = \begin{pmatrix} u''_1 & u''_2 & \cdots & u''_m \\ v''_1 & v''_2 & \cdots & v''_m \end{pmatrix}.$$

For each k, if u''_k is uncircled, then v''_k is inserted by ordinary Schensted's row insertion, or if u''_k is circled, then v''_k is inserted by ordinary Schensted's column insertion. Set $a_1, a_2, \dots, a_r$ (resp. $b_1, b_2, \dots, b_{m-r}$) to be the v''_k's in which the corresponding u''_k's are uncircled (resp.

circled). Then we have

$$\emptyset \leftarrow^{m} w'' = j(b_{m-r} \oplus b_{m-r-1} \oplus \cdots \oplus b_1 \oplus a_1 \oplus a_2 \oplus \cdots \oplus a_r).$$

And the right hand corresponds to $\emptyset \leftarrow^{m} w'$. This prove the first identity. And the second identity will be an easy consequence of Theorem 2.3. In the proof of Theorem 2.3 we use the first identity. ■

Example 2.7

Set w to be as in Example 2.5. Then

$$w'' = \begin{pmatrix} -14 & -13 & -11 & -7 & -4 & -2 & 1 & 3 & 5 & 6 & 8 & 9 & 10 & 12 \\ 3 & -10 & -4 & 11 & 12 & 13 & 14 & -5 & -6 & 2 & -9 & 1 & -8 & -7 \end{pmatrix}$$

and

$$\emptyset \leftarrow w'' = \begin{array}{|c|c|c|c|c|c|} \hline -10 & -9 & -8 & -7 & 13 & 14 \\ \hline -6 & 1 & 12 \\ \cline{1-3} -5 & 2 \\ \cline{1-2} -4 & 11 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad \emptyset \leftarrow w = \begin{array}{|c|c|c|c|c|c|} \hline -14 & -11 & -7 & -4 & -2 & 1 \\ \hline -13 & 6 & 12 \\ \cline{1-3} 3 & 9 \\ \cline{1-2} 5 & 10 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

Theorem 2.3

Let w be a biword. Then

$$\emptyset \leftarrow^{m} w^{-1} = R : \emptyset \leftarrow^{m} w$$

$$R : \emptyset \leftarrow^{m} w^{-1} = \emptyset \leftarrow^{m} w$$

Proof.

Suppose that w is given in two-line notation as

$$w = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

Let a (resp. b) be the greatest letter in the top (resp. bottom) line of w. It suffices to show that

$$\mathrm{trunc}_{<a} \emptyset \leftarrow^{m} w^{-1} = \mathrm{trunc}_{<a} R : \emptyset \leftarrow^{m} w$$

$$\mathrm{trunc}_{<b} R : \emptyset \leftarrow^{m} w^{-1} = \mathrm{trunc}_{<b} \emptyset \leftarrow^{m} w$$

and $\emptyset \leftarrow^{m} w$ and $\emptyset \leftarrow^{m} w^{-1}$ have the same shape. We can assume the first two equations above by induction on the length of w, since deleting a from the bottom line of w^{-1} correspond to deleting the last number from the top line of w and deleting b from the bottom line of w correspond to deleting the last number from the top line of w^{-1}.

So we have to prove the equality of shapes. As in the above corollary let ${}^{\circ}x_1, {}^{\circ}x_2, \ldots, {}^{\circ}x_k$ (resp. ${}^{\circ}y_1, {}^{\circ}y_2, \ldots, {}^{\circ}y_l$) be all the circled letters in the top (resp. bottom) line of w. Set

$$w' = w({}^{\circ}x_1 \to -x_1)({}^{\circ}x_2 \to -x_2) \cdots ({}^{\circ}x_k \to -x_k)({}^{\circ}y_1 \to -y_1)({}^{\circ}y_2 \to -y_2) \cdots ({}^{\circ}y_l \to -y_l),$$

where $-x_1 > -x_2 > \cdots > -x_k$ (resp. $-y_1 > -y_2 > \cdots > -y_k$) are uncircled letters which are less than all the uncircled letters in the top (resp. bottom) line of w. Then we have

$$[w']^{-1} = [w^{-1}](^\circ x_1 \to -x_1)(^\circ x_2 \to -x_2)\cdots(^\circ x_k \to -x_k)(^\circ y_1 \to -y_1)(^\circ y_2 \to -y_2)\cdots(^\circ y_l \to -y_l).$$

$\emptyset \leftarrow^m w$ and $\emptyset \leftarrow w'$ have the same shape by the first identity of the above corollary, and $\emptyset \leftarrow^m w^{-1}$ and $\emptyset \leftarrow [w']^{-1}$ have the same shape by the first identity of the above corollary. In addition $\emptyset \leftarrow w'$ and $\emptyset \leftarrow [w']^{-1}$ have the same shape by properties of ordinary Schensted's insertion. This prove the theorem. ■

Corollary 2.3

Let w be a word and $^\circ x$ be the least letter of the circled letters in the top row of w. Let $-\infty$ be an uncircled letter less than all letters in the top row of w. Then we have

$$\emptyset \leftarrow^m [w(^\circ x \to^t -\infty)] = \emptyset \leftarrow^m w$$

$$R : \emptyset \leftarrow^m [w(^\circ x \to^t -\infty)] = [R : \emptyset \leftarrow^m w](^\circ x \to -\infty). \quad ■$$

Fix a set of alphabets $\mathscr{A}$ and its division (U, C). Then an involution of $\mathscr{A}$ is a bijection from a subset of $\mathscr{A}$ into itself such that $w \circ w = id$.

Example 2.8

$$w = \begin{pmatrix} 1 & ^\circ 2 & ^\circ 3 & 4 & 5 & ^\circ 6 \\ 5 & ^\circ 3 & ^\circ 2 & 4 & 1 & ^\circ 6 \end{pmatrix}$$

is an involution. And the number of fixed points of w is two.

Corollary 2.4

The above mixed Robinson-Schensted correspondence gives the bijection between involutions and partial tableaux. And moreover in the above correspondence the number of fixed points in an involution is equal to the number of odd length columns in its partial tableau.

Proof.

The first part of the corollary is clear from the last theorem. And the proof of the second part is quite similar to the proof given in pp. 44, [Ro]. So we ommit the proof.

Example 2.9

Let w be as in Example 2.7. Then

$$\emptyset \leftarrow^m w = \begin{array}{|c|c|c|} \hline 1 & 2 & ^\circ 6 \\ \hline ^\circ 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{1}{|c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Definition 2.4

Let

$$w = \begin{pmatrix} u_1 & u_2 & \cdots & u_m \\ v_1 & v_2 & \cdots & v_m \end{pmatrix}$$

be a biword. We define a poset $P(w)$ and $\bar{P}(w)$ have the same underlying set which is the subset of $\mathbf{Z}^2$ composed of vetices $V_1, V_2, \dots, V_m$: for $i = 1, 2, \dots, m$ each vertex $V_i = (\bar{u}_i, \bar{v}_i) \in \mathbf{Z}^2$ is defined from (u_i, v_i) as follows.

$$\bar{u}_i = \begin{cases} u_i & \text{if } u_i \in U' \\ -u_i & \text{if } u_i \in C' \end{cases}$$

$$\bar{v}_i = \begin{cases} v_i & \text{if } v_i \in U \\ -v_i & \text{if } v_i \in C \end{cases}$$

The order of $P(w)$ is the ordinary product order: i.e.

$$(\bar{u}_i, \bar{v}_i) \le (\bar{u}_i', \bar{v}_i') \text{ if and only if } \bar{u}_i \le \bar{u}_i' \text{ and } \bar{v}_i \le \bar{v}_i'.$$

The order of $\bar{P}(w)$ is a dual order: i.e.

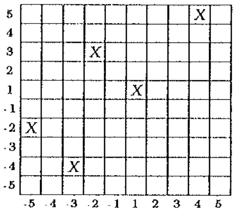
$$(\bar{u}_i, \bar{v}_i) \le (\bar{u}_i', \bar{v}_i') \text{ if and only if } \bar{u}_i \le \bar{u}_i' \text{ and } \bar{v}_i \ge \bar{v}_i'.$$

Example 2.10

Let w be as follows.

$$w = \begin{pmatrix} 1 & {}^\circ 2 & {}^\circ 3 & 4 & {}^\circ 5 \\ 1 & 3 & {}^\circ 4 & 5 & {}^\circ 2 \end{pmatrix}$$

The underlying set of $P(w)$ and $\bar{P}(w)$ is in following diagram. Here the axes are pointing right and upward.



In $P(w)$ one cell is greater than another if it is above and to the right of that cell. In $\bar{P}(w)$ one cell is greater than another if it is below and to the right of that cell.

From Theorem 1.3 and Corollary 1.2 we immediately obtain the following theorem.

Corollary 2.5

Let w be a biword. And set λ to be the shape of $\emptyset \leftarrow^m w$. Then for each k, we have

$$\begin{aligned} c_k(P(w)) &= \lambda_1 + \lambda_2 + \cdots + \lambda_k \\ a_k(P(w)) &= \lambda_1' + \lambda_2' + \cdots + \lambda_k' \, . \end{aligned}$$

We will use this corollary to prove the main theorem of Section 2.

From now we treat a skew version. Let $\mathrm{PST}(\lambda/\mu)$ denote the set of all partial skew tableaux having the shape λ/μ. The following theorem is an easy extension of [SS], pp. 175, Theorem 5.1. Let $|\lambda|$ denote the weight of a partition λ and let $|w|$ denote the length of a biword w. The weight of a skew partion λ/μ is $|\lambda| - |\mu|$ and denoted by $|\lambda/\mu|$. If λ/μ is a skew partition of weight n, then we write $\lambda/\mu \vdash n$.

Theorem 2.4

Fix sets of alphabets $\mathscr{A}'$ and $\mathscr{A}$ and their divisions (U', C') and (U, C) respectively. Fix positive integers $n, m \in P$. Let α and β be fixed partitions. (Here we assume $|\alpha| + m = |\beta| + n$.) Then the map

$$(w, \tau, \kappa) \leftrightarrow (\pi, \sigma)$$

defined below is a bijection between biword w with $\tau \in \mathrm{PST}(\alpha/\mu)$, $\kappa \in \mathrm{PST}(\beta/\mu)$, such that $\check{w} \uplus \tau = \pi$, $\hat{w} \uplus \kappa = \sigma$, on the one hand, and $\pi \in \mathrm{PST}(\lambda/\alpha)$ and $\sigma \in \mathrm{PST}(\lambda/\beta)$ such that $\lambda/\beta \vdash n$, $\lambda/\alpha \vdash m$ on the other.

Proof.

Let $A = \hat{w} \bigcup \kappa = \{a_1 < a_2 < \cdots < a_n\}$, where $n = |A|$. We construct a sequence of partial tableaux:

$$(\tau, \kappa) = (\pi_0, \sigma_0), (\pi_1, \sigma_1), \dots, (\pi_n, \sigma_n) = (\pi, \sigma)$$

by the following rule. Set the shape of π_k to be λ_k/β_k, then $\lambda_0 = \alpha$ and $\beta_0 = \mu$. At each step σ_k is obtained from σ_{k-1} by placing a_k on λ_k/λ_{k-1}. Next we explain how to construct π_k from π_{k-1} for $k = 1, 2, \dots n$. At k-th step we see whether $a_k \in \hat{w}$ or $a_k \in \sigma$.

Case 1: $a_k \in \hat{w}$

Let the corresponding letter in $\check{w}$ be b_k. Perform $\pi_{k-1} \leftarrow^{m} b_k$ is a_k is uncircled, or perform $b_k \rightarrow^{m} \pi_{k-1}$ if a_k is circled. Here we think as if the cells of β_{k-1} is filed with $-\infty$ and perform the mixed insertion. The resulting tableau is π_k. We have $\beta_k = \beta_{k-1}$. And let λ_k be the shape of π_k.

Case 2: $a_k \in \sigma$ and the cell containing a_k in σ contains a letter b_k in τ.

Let (i, j) denote the cell containing a_k in σ. Remove b_k from the cell (i, j) in π_{k-1}. Insert b_k into the $(i+1)$-th row if a_k is uncircled, or insert a_k into $(j+1)$-th column if a_k is circled. And set π_k to be the resulting tableau. Let β_k be the parition added the cell (i, j) on β_{k-1}. And let λ_k be the shape of π_k.

Case 3: $a_k \in \sigma$ but the cell containing a_k in σ contains no letter in τ.

Let (i, j) denote the cell containing a_k in σ. Let β_k (resp. λ_k) be the parition added the cell (i, j) on β_{k-1} (resp. λ_{k-1}). Let π_k be the same as π_{k-1} except the shape of π_k being λ_k/β_k. ■

Example 2.11

Set $\alpha = (211)$ and $\beta = (41)$. Let $w = \begin{pmatrix} 2 & {}^{\circ}4 \\ 4 & {}^{\circ}2 \end{pmatrix}$. Let $\tau \in \mathrm{PST}(\alpha/(1))$ and $\kappa \in \mathrm{PST}(\beta/(1))$ be as follows.

$$\tau = \begin{array}{|c|c|} \hline & {}^{\circ}3 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1} \multicolumn{1}{|c|}{{}^{\circ}5} \\ \cline{1-1} \end{array} \qquad \kappa = \begin{array}{|c|c|c|c|} \hline & 1 & {}^{\circ}3 & {}^{\circ}5 \\ \hline \multicolumn{1}{|c|}{6} \\ \cline{1-1} \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & {}^{\circ}2 & {}^{\circ}3 & {}^{\circ}5 \\ \hline 1 & {}^{\circ}4 \\ \cline{1-2} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|}\hline & & {}^{\circ}3 & {}^{\circ}5 \\ \hline & {}^{\circ}1 & 4 & 6 \\ \hline & 2 \\ \cline{1-2} \end{array}$$

Definition 2.5

To simulate the skew insersion we prepare $|\alpha|$ uncircled letters $a_1 < a_2 < \cdots < a_{|\alpha|}$ which are less than all the letters in $\mathscr{A}'$, and $|\beta|$ uncircled letters $b_1 < b_2 < \cdots < b_{|\beta|}$ which are less than all the letters in $\mathscr{A}$. Let $\bar{\alpha}$ (resp. $\bar{\beta}$) denote the partial tableau whose shape is α (resp. β) and whose j-th row contains the letters $a_{\alpha'_{j-1}+1}, a_{\alpha'_{j-1}+2}, \dots, a_{\alpha'_j}$ (resp. $b_{\beta'_{j-1}+1}, b_{\beta'_{j-1}+2}, \dots, b_{\beta'_j}$) from top to bottom.

For example, if $\alpha = (211)$ and $\beta = (41)$, then we have

$$\bar{\alpha} = \begin{array}{|c|c|}\hline a_1 & a_4 \\ \hline a_2 \\ \cline{1-1} a_3 \\ \cline{1-1} \end{array} \qquad \bar{\beta} = \begin{array}{|c|c|c|c|}\hline b_1 & b_3 & b_4 & b_5 \\ \hline b_2 \\ \cline{1-1} \end{array}$$

Definition 2.6

We now define a bracketing operation on the triple (w, τ, κ) defined in the the above theorem. We will denote the image of (w, τ, κ) by $[w, \tau, \kappa]$. $w' = [w, \tau, \kappa]$ is a biword of length $|\lambda| = |\alpha| + m = |\beta| + n$ whose top line includes the letters $a_1, a_2, \dots, a_{|\alpha|}$ and letters in $\mathscr{A}'$, and whose bottom line includes the letters $b_1, b_2, \dots, b_{|\beta|}$ and letters in $\mathscr{A}$. So the top line of w' is composed of the letters in $\{a_1, a_2, \dots, a_{|\alpha|}\} \cup \hat{w} \cup \kappa$ in increasing order. The bottom line of w' is constructed as follows.

Step 1: The pairs $\binom{i}{j} \in w$ are transfered to w' unchanged.

Step 2: We consider the pair $(\bar{\beta}, \kappa)$ of partial tableaux having the same shape β. (Here the shape of κ is actually β/μ.) We perform the (mixed) delition procedures on $(\bar{\beta}, \kappa)$ so that we obtain a biword of length $|\beta/\mu|$ and a partial tableau of the shape μ. The pairs in this biword are transfered to w' unchanged.

Step 3: Recall that the shape of τ is α/μ. We place the partial tableau of the shape μ obtained in Step 2 into τ so that we have a partial tableau τ' of the shape α. We have the pair $(\tau', \bar{\alpha})$ of the same shape α. Again we perform the (mixed) delition procedures on $(\tau', \bar{\alpha})$ and obtain a biword of length $|\alpha|$. Finally the pairs in this biword are transfered to w' unchanged and we obtain a word of length $|\lambda|$.

We define one more bracketing operation on the pair (π, σ) defined in the above theorem. Recall that the shape of π (resp. σ) is λ/β (resp. λ/α). The image of (π, σ) is denoted by $[\pi, \sigma]$ and $[\pi, \sigma]$ is a pair (π', σ') of partial tableaux having the same shape λ defined as follows. We place $\bar{\beta}$ (resp. $\bar{\alpha}$) into π (resp. σ) so that we obtain a partial tableau π' (resp. σ') of the shape λ.

Example 2.12

Let (w, τ, κ) be as Example 2.11. In Step 2, we perform the mixed deletion procedure on the pair

$$\bar{\beta}=\begin{array}{|c|c|c|c|}\hline b_1 & b_3 & b_4 & b_5 \\ \hline b_2 \\ \cline{1-1}\end{array} \qquad \kappa=\begin{array}{|c|c|c|c|}\hline \; & 1 & {}^\circ 3 & {}^\circ 5 \\ \hline 6 \\ \cline{1-1}\end{array}$$

We obtain a biword $\begin{pmatrix} {}^\circ 1 & {}^\circ 3 & {}^\circ 5 & 6 \\ b_4 & b_3 & b_2 & b_1 \end{pmatrix}$ and a partial tableau

$$\begin{array}{|c|}\hline b_5 \\ \hline\end{array}$$

of the shape $\mu = (1)$. In Step 3, we perform the mixed deletion procedure on the pair

$$\tau'=\begin{array}{|c|c|}\hline b_5 & {}^\circ 3 \\ \hline 1 \\ \cline{1-1} {}^\circ 5 \\ \cline{1-1}\end{array} \qquad \bar{\alpha}=\begin{array}{|c|c|}\hline a_1 & a_4 \\ \hline a_2 \\ \cline{1-1} a_3 \\ \cline{1-1}\end{array}$$

Then we obtain a biword

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & {}^\circ 3 & {}^\circ 5 & b_5 \end{pmatrix}.$$

Consequently we have

$$w' = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & {}^\circ 1 & 2 & {}^\circ 3 & 4 & {}^\circ 5 & 6 \\ 1 & {}^\circ 3 & {}^\circ 5 & b_5 & b_4 & {}^\circ 4 & b_3 & {}^\circ 2 & b_2 & b_1 \end{pmatrix}.$$

On the other hand, the iamge of (π, σ) by the bracket operation is

$$\pi'=\begin{array}{|c|c|c|c|}\hline b_1 & b_3 & b_4 & b_5 \\ \hline b_2 & {}^\circ 2 & {}^\circ 3 & {}^\circ 5 \\ \hline 1 & {}^\circ 4 \\ \cline{1-2}\end{array} \qquad \sigma'=\begin{array}{|c|c|c|c|}\hline a_1 & a_4 & {}^\circ 3 & {}^\circ 5 \\ \hline a_2 & {}^\circ 1 & 4 & 6 \\ \hline a_3 & 2 \\ \cline{1-2}\end{array}$$

Lemma 2.2

We use the same notation as in Theorem 2.3. *The bracket operations defined above are injections and the following diagram commutes.*

$$\begin{tikzcd} (w, \tau, \kappa) \arrow[r, leftrightarrow] \arrow[d, "{[\cdot]}"] & (\pi, \sigma) \arrow[d, "{[\cdot]}"] \\ w' \arrow[r, leftrightarrow] & (\pi', \sigma') \end{tikzcd}$$

where the top and bottom bijections are the skew and ordinary mixed Robinson-Schensted maps, respectively.

Proof.
It is clear that $(\pi, \sigma) \mapsto (\pi', \sigma')$ is an injection from the definition. It is easy to see that $(w, \tau, \kappa) \mapsto w'$ is an injection since we can construct the inverse of this map.

Now we prove the above diagram commute. We construct $(\emptyset \leftarrow^m w', R: \emptyset \leftarrow^m w')$ and show this pair is equal to (π', σ'). Let $(\pi_0, \sigma_0), (\pi_1, \sigma_1), \ldots, (\pi_{|w|}, \sigma_{|w|})$ be the tableaux pairs constructed by applying skew mixed Robinson-Schensted to (w, τ, k). Let $(\pi'_0, \sigma'_0), (\pi'_1, \sigma'_1), \ldots, (\pi'_r, \sigma'_r)$ be the tableaux pairs constructed by applying skew mixed Robinson-Schensted to w', where $r = |w| + |\alpha|$. Let

$$w' = \begin{pmatrix} a_1 & a_2 & \cdots & a_{|\alpha|} & u_1 & u_2 & \cdots & u_{|w|} \\ v_1 & v_2 & \cdots & v_{|\alpha|} & v_{|\alpha|+1} & v_{|\alpha|+2} & \cdots & v_r \end{pmatrix}.$$

Set

$$w_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_{|\alpha|} \\ v_1 & v_2 & \cdots & v_{|\alpha|} \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} u_1 & u_2 & \cdots & u_{|w|} \\ v_{|\alpha|+1} & v_{|\alpha|+2} & \cdots & v_r \end{pmatrix}.$$

irst we construct $(\emptyset \leftarrow^m w_1, R: \emptyset \leftarrow^m w_1)$. It is easy to see that the resulting pair is $(\tau', \check{\alpha})$ in Definition 2.5 from the definition. Now we insert w_2 into this tableau τ'. Suppose that we constuct $\pi'_{i+|\alpha|}$ from $\pi'_{i-1+|\alpha|}$ by inserting $v_{i+|\alpha|}$.
Case 1: $u_i \in \hat{w}$

In this case $v_{i+|\alpha|}$ is an ordinary letter (i.e. $v_{i+|\alpha|} \neq b_j$ for any j). Since b_j's in $\pi'_{i-1+|\alpha|}$ are less than $v_{i+|\alpha|}$, the process to insert $v_{i+|\alpha|}$ into $\pi'_{i-1+|\alpha|}$ is the same insertion process as skew mixed Robinson-Schensted.
Case 2: $u_i \in \kappa$

In this case $v_{i+|\alpha|} = b_j$ for some j. If we insert $v_{i+|\alpha|}$ into $\pi'_{i-1+|\alpha|}$, then the bumping process continues until it reaches the cell in $\pi'_{i-1+|\alpha|}$ which correspond to the cell containing u_i in κ. If the cell contains no letter then the bumping process terminates. If the process contains an ordinary letter then the subsequent bumping process is the same as the internal skew mixed insertion procedure. This proves the lemma. ■

Theorem 2.5
Assume that $\mathscr{A} = \mathscr{A}'$, $(U, C) = (U', C')$, $m = n$ *and* $\alpha = \beta$ *in Theorem* 2.4.

If $(w, \tau, \kappa) \leftrightarrow (\pi, \sigma)$ *by the correspondence in Theorem* 2.4, *then* $(w^{-1}, \kappa, \tau) \leftrightarrow (\sigma, \pi)$. ■

Corollary 2.6
Under the same assumption of Theorem 2.5, *if* w *is an involution then the correspondence in Theorem* 2.4 *gives a bijection* $(w, \tau) \leftrightarrow \pi$ *between* $\pi \in \mathrm{PST}(\alpha/\mu)$ *such that* $\check{w} \uplus \tau = \pi$ *on the one hand and* $\pi \in \mathrm{PST}(\lambda/\alpha)$ *such that* $\lambda/\alpha \vdash n$ *on the other. In this bijection we have*

$$\mathrm{fix}(w) + \mathrm{odd}(\mu) = \mathrm{odd}(\lambda)$$

Here fix (w) *denotes the number of fixed points of a biword* w *and* odd (λ) *denotes the number of odd length columns in a partition* π. ■

The first claim is clear from Theorem 2.5. The proof of the second claim is quite similar to that of [Ro], pp. 53, Corollary 3.3.8 and we ommit the proof.

Example 2.13

Let $w = \begin{pmatrix} {}^{\circ}3 & 4 \\ {}^{\circ}3 & 4 \end{pmatrix}$ and let

$$\pi = \begin{array}{|c|c|c|}\hline \ & \ & 6 \\ \hline {}^{\circ}1 & 2 \\ \cline{1-2} \end{array}$$

Then

$$\tau = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & {}^{\circ}3 \\ \hline \ & \ & 4 \\ \cline{1-3} {}^{\circ}1 & 2 & 6 \\ \cline{1-3} \end{array}$$

§3 The Fomin Version of The Mixed Robison-Schensted Correspondence

In this section we give an extension of Fomin's method and present the Fomin version of the mixed R-S correspondence as an application of it. To define Fomin's generalization we first introduce some terminology from [Ro] and prove a theorem on the Fomin version of the mixed insertion. The only difference of this section from [Fo2] is that we define R-correspondences cell by cell and this is an easy extension of [Fo2]. The best reference for this section is [Fo2] and [Ro]. The author expresses special thanks to T. Roby for his helpful discussions with the author about this section. If he was not in Japan, this section would not be added to this paper.

Definition 3.1 [St]
Let r be a positive integer. A poset P is called *r-differential* if it satisfies the following three conditions:

(D1) P is locally finite, graded poset and has a $\hat{0}$ element.

(D2) If $x \neq y$ in P and there are exactly k elements of P which are covered by both x and y, then there are exactly k elements of P which cover x and y.

(D3) If $x \in P$ and x covers exactly k elements of P, then x is covered by exactly $k + r$ elements of P.

When $r = 1$, we will sometimes omit the r in r-differentialand say simply differential.

Proposition 3.1 [St]
If P is a poset satisfying (D1) *and* (D2), *then for $x \neq y$ in P the integer k of* (D2) *is equal to zero or one.*

Proof.
Suppose the contrary. Let x and y be elements of minimal rank for which $k > 1$. Then x and y both cover elements $x_1 \neq y_1$ of P. But x_1 and y_1 are elements of smaller rank with $k > 1$, a contradiction. ■

Remark 3.1
By the above proposition, if $x \neq y \in P$, there is at most one element which covers (is covered by) both x and y. And there exists a unique element which covers both x and y if and only if there exits a unique element which is coverd by both x and y.

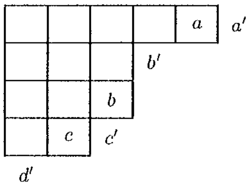
For a lattice L satisfying (D1), condition (D2) is equivalent to L being modlar.

Example 3.1
The Young's lattice $\mathbf{Y}$ is the set of all normal shapes and the order is defined by set inclusion. $\mathbf{Y}$ is a distributive lattice and a differential poset.

Definition 3.2
Fix a differential poset P. For each $x \in P$, set $C^+(x) := \{y \in P \mid y \text{ covers } x\}$ and $C^-(x) := \{y \in P \mid x \text{ covers } y\}$. Then by the definition of differential posets, $C^+(x)$ and $C^-(x) \bigcup \{x\}$ have the same cardinalities. A *R-correspondence* $\varphi = \{\varphi_x\}_{x \in P}$ is a collection of maps $\varphi_x : C^+(x) \bigcup \{x\} \to C^-(x) \bigcup \{x\}$ such that for each $x \in P$, the restriction of the map φ_x on $C^+(x)$ is a bijection and $\varphi_x(x) = x$.

Definition 3.3
In the Young's lattice $\mathbf{Y}$ there are two natural R-correspondences. If $y \in C^+(x)$, then y and x differs by exactly one cell. Let $a := y \setminus x$ be this cell. And let the coordinates of a be (i, j). If $i \neq 1$, we can remove the rightmost cell of $(i-1)$-th row from x and let z denote the resulting diagram. We associate z with y if $i \neq 1$, and x with y if $i = 1$. We call this R-correspondence the *natural R-correspondence by rows* and denote it by φ_R. Another one is as follows If $j \neq 1$, we can remove the downmost cell of $(j-1)$-th column from x and let z denote the resulting diagram. We associate z with y if $j \neq 1$, and x with y if $j = 1$. We call this R-correspondence the *nalural R-correspondence by columns* and denote it by φ_C.

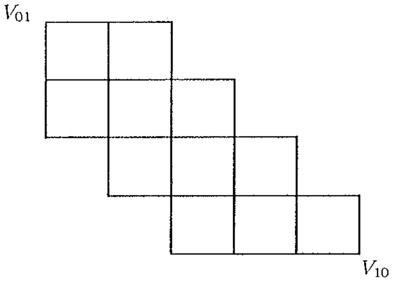
Example 3.2



Let $x \in \mathbf{Y}$ be as above. There are four elements which covers x and three elements which is covered by x. The each element which covers x is obtained by adding each cell a', b', c', d' to x in the above diagram. And each element which is covered by x is obtained by removing each cell a, b, c from x. We indicate these elements by the cells added to or removed from x. Then φ_R, natural R-correspondence by rows, maps b' to a, c' to b, d' to c, a' to x to x. And φ_C, natural R-correspondence by columns, maps a' to a, b' to b, c' to c, d' to x and x to x.

We fix a connected skew diagram S throughout this section. We use "French" notation only for this fixed skew diagram S. In diagrams written in "French" notation we suppose that the axes are pointing right and upwards.

For an example



is the skew diagram $(5, 4, 3, 2)\backslash(2, 1)$.

Definition 3.4

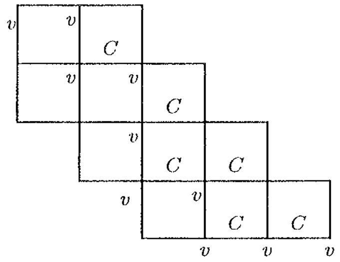
Set $C(S)$ to be the set of all cells in S and $V(S)$ to be the set of all vertices in S. And let ROW (S) denote the set of all rows of S and COL (S) the set of all columns of S. The most left and up vertex in $V(S)$ is denoted by V_{01} and the most right and below vertex in $V(S)$, by V_{10}. $V(S)$ is considered to be a poset wherein one vertex is greater than another if it is upper and to the right of another.

A *path* $\mathscr{P}$ of S is by definitiom a path (a set of edges) from V_{01} to V_{10} in S which goes right and down. For a path $\mathscr{P}$ let $V(\mathscr{P})$ (resp. $E(\mathscr{P})$) denote the set of vertices (resp. edges) included in $\mathscr{P}$. Notice that $V(\mathscr{P})$ is regarded as a subposet of $V(S)$. Let $C(\mathscr{P})$ denote the set of cells which are above and to the right of $\mathscr{P}$. We define *upper* and *lower boundaries* of S as the paths whose vertices are defined by

$$\begin{aligned} V(\partial^+(S)) &:= \{(x, y) \in V(S) : (x+1, y+1) \notin V(S)\} \\ V(\partial^-(S)) &:= \{(x, y) \in V(S) : (x-1, y-1) \notin V(S)\} \end{aligned}.$$

Example 3.3

In the following diagram the vertices with v are on a path $\mathscr{P}$ and $C(\mathscr{P})$ is the set of the cells containing C.



Definition 3.5

Fix a connected skew diagram S and a differential poset P. A *system of R-correspondences* on S is by definition a family of R-correspondences $\Phi = \{\varphi^{(c)}\}_{c \in C(S)}$ wherein, for each cell $c \in C(S)$, $\varphi^{(c)}$ is a R-correspondence.

Definition 3.6.
Set $P = \mathbf{Y}$ to be the Young's lattice. We regard ROW (S) and COL (S) as the set of the row numbers and column numbers respectively. Fix divisions (U', C') of ROW (S) and (U, C) of COL (S).

Let $\mathbf{Z}_2$ denote the cyclic group of order 2. Set $\sigma = (\sigma_i)_{i \in \mathrm{ROW}(S)} \in \mathbf{Z}_2{}^{\mathrm{ROW}(S)}$ and $\tau = (\tau_j)_{j \in \mathrm{COL}(S)} \in \mathbf{Z}_2{}^{\mathrm{COL}(S)}$ by

$$\sigma_i = \begin{cases} 0 & \text{if } i \in U' \\ 1 & \text{if } i \in C' \end{cases} \qquad \tau_j = \begin{cases} 0 & \text{if } j \in U \\ 1 & \text{if } j \in C \end{cases}$$

Let c be a cell in S whose cordinates are given by (i, j). We attach φ_R to c if $\sigma_i + \tau_j = 0$, or φ_C to c if $\sigma_i + \tau_j = 1$. In this way we obtain a system of R-correspondences. We call this system the *mixed system of R-correspondences* induced from (U, C, U', C') and denote it by Φ_m.

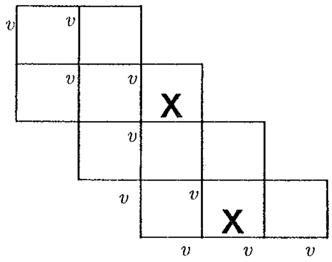
Example 3.4
Set $S = (5^5)$ and $R = \mathbf{Y}$.

°5	φ_R	φ_C	φ_R	φ_R	φ_C
4	φ_C	φ_R	φ_C	φ_C	φ_R
°3	φ_R	φ_C	φ_R	φ_R	φ_C
°2	φ_R	φ_C	φ_R	φ_R	φ_C
1	φ_C	φ_R	φ_C	φ_C	φ_R
	°1	2	°3	°4	5

For the divison of rows and columns shown in the diagram, the mixed system of R-correspondences, Φ_m, attach the above R-correspondences to each cell.

Definition 3.7
Given a connected skew diagram S and a path $\mathscr{P}$ of S, a *generalized permutation* on $C(\mathscr{P})$ is a subset w of $C(\mathscr{P})$ which does not share any row or column. Other commonly used terms include *nontaking rook placement* or *permutations with restricted positions.* We call the cardinality of w *length* of w and denote it by $|w|$.

Example 3.5
Let $\mathscr{P}$ be as in Example 3.3.



The set of cells containing X in the above diagram is a generalized permutation on $C(\mathscr{P})$.

In a graded poset, if y covers x, then we denote this relation by "$x \lessdot y$". And if y covers x or x is equal to y, then we write "$x \underset{\cdot}{\leq} y$".

Definition 3.8 [Fo2]
Let P and Q be any graded posets. A map $g: P \mapsto Q$ is called a *growth* if it preserve the relation $\underset{\cdot}{\leq}$: i.e.

$$x \underset{\cdot}{\leq} y \Rightarrow g(x) \underset{\cdot}{\leq} g(y).$$

A growth is an order preserving map but an order preserving map is not always a growth.

Example 3.6
(1) The rank function $\rho: P \mapsto \mathbf{N}$ is a growth.
(2) The composition of two growthes is a growth.

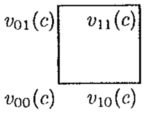
Let $g: P \mapsto Q$ be a growth and $\rho: Q \mapsto \mathbf{N}$ the rank function of Q. By composing these we get a new growth called the *modulus* of g and written $|g|: P \mapsto \mathbf{Z}$.

Definition 3.9
Fix a connected skew diagram S and a differential poset P. A growth $g: V(S) \to P$ on $V(S)$ is called a *two-dimensional* growth.

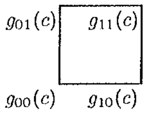
Example 3.7
Set $S = (5^5)$ and $P = \mathbf{Y}$. The following is a two-dimensional growth.

$\emptyset$	1	2	21	31	311
$\emptyset$	1	2	21	21	211
$\emptyset$	1	1	11	11	111
$\emptyset$	1	1	11	11	11
$\emptyset$	$\emptyset$	$\emptyset$	1	1	1
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

let $g: V(S) \to P$ be a two-dimensional growth. Let $c \in C(S)$ be a cell in S. Let $v_{00}(c)$, $v_{01}(c)$, $v_{10}(c)$ and $v_{11}(c)$ denote each vertex of c as in the following diagram.



Let $g_{00}(c)$, $g_{01}(c)$, $g_{10}(c)$ and $g_{11}(c)$ denote the values of g at the vertices $v_{00}(c)$, $v_{01}(c)$, $v_{10}(c)$ and $v_{11}(c)$, respectively.



Definition 3.10

Fix a connected skew diagram S and a differential poset P. Let $\Phi = \{\varphi^{(c)}\}_{c \in c(S)}$ be a system of R-correspondences on S.

A two-dimensional growth $g\colon V(S) \to P$ is said to be *consistent* (or *compatible*) with Φ if g satisfies

$$g_{00}(c) = \varphi^{(c)}{}_{g_{01}(c)}(g_{11}(c)).$$

for each cell $c \in C(S)$ such that $g_{01}(c) = g_{10}(c)$. We sometimes call it simply *Φ-consistent* (or *Φ-compatible*).

Example 3.8

The two-dimensional growth in Example 3.7 is consistent with the system of R-correspondences in Example 3.4.

Definition 3.11

Fix a connected skew diagram S and a differential poset P. And suppose we are given a path $\mathscr{P}$ of S. Let (h, w) be a pair of a growth $h\colon V(\mathscr{P}) \to P$ on $V(\mathscr{P})$ and a generalized permutation w on $C(\mathscr{P})$. The pair (h, w) is said to be *admissible* if it satisfies the following conditions:

1. For each row $R \in \mathrm{ROW}(S)$, if h is strict on $R \cap E(\mathscr{P})$, then w has no cell in $R \cap C(\mathscr{P})$.
2. For each column $C \in \mathrm{COL}(S)$, if h is strict on $C \cap E(\mathscr{P})$, then w has no cell in $C \cap C(\mathscr{P})$.

Set

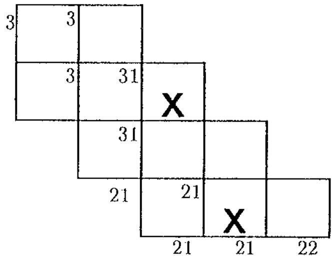
$$m' = \text{the cardinality of } \{C \in \mathrm{COL}(S) \mid h \text{ is strict on } C \cap E(\mathscr{P})\}$$

$$n' = \text{the cardinality of } \{R \in \mathrm{ROW}(S) \mid h \text{ is strict on } R \cap E(\mathscr{P})\}.$$

The pair $(m, n) = (m' + |w|, n' + |w|)$ of integers is called the *weight* of the pair (h, w). If $\mathscr{P} = \partial^+(S)$, then $C(\mathscr{P})$ has no cell so that $|w|$ is always zero and we say h has weight (m, n).

Example 3.9

Set $P = \mathbf{Y}$. Let $\mathscr{P}$ be as in Example 3.3.



Then the above pair (h, w) of a growth $h\colon V(\mathscr{P}) \to P$ on $V(\mathscr{P})$ and a generalized permutation w on $C(\mathscr{P})$ is admissible and has the weight $(4, 3)$.

The author and Roby obtained the following easy extension of the Fomin's theorem.

Theorem 3.1

Fix a connected skew diagram S and a differential poset P. Set Φ to be a system of R-correspondences on S. Let $\mathscr{P}$ be a path of S.

If (h, w) is an admissible pair of a growth $h\colon V(\mathscr{P}) \to P$ on $V(\mathscr{P})$ and a generalized permutation w on $C(\mathscr{P})$, then there exists one and only one Φ-compatible two-dimensional growth $g\colon V(S) \to P$ which satisfies:

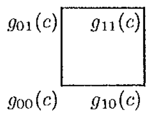
- (i) $g|_{\mathscr{P}} = h$
- (ii) *For each cell $c \in C(\mathscr{P})$, if $g_{00}(c) = g_{01}(c) = g_{10}(c)$, then we have*

$$g_{11}(c) = \begin{cases} x & \text{if } c \notin w \\ \{\varphi_x^{(c)}\}^{-1}(x) & \text{if } c \in w \end{cases}$$

where x is equal to $g_{00}(c) = g_{01}(c) = g_{10}(c)$.

Proof.

We prove this theorem in the same method as [Ro]. We construct $g\colon V(S) \to P$ as follows. First we construct g on cells which are above and to the right of $\mathscr{P}$. Recall that the value of g at each vertex of $c \in C(S)$ is denoted as follows.



Here we abbreviate $g_{ij}(c)$ to g_{ij} for $i, j = 0, 1$, and $\varphi^{(c)}$ to φ. We construct g_{11} from given g_{00}, g_{01} and g_{10} so that the resulted g become a growth.

(Case 1): $|g_{01}| < |g_{10}|$.

The definition of growth forces $g_{00} = g_{01}$ and $g_{00} \lessdot g_{10}$. Then by $g_{00} = g_{01} \lessdot g_{10} \leq g_{11}$ we have $g_{11} = g_{10}$.

(Case 2): $|g_{01}| > |g_{10}|$.

In the same way as Case 1 we obtain $g_{00} = g_{10} \lessdot g_{10} = g_{11}$.

(Case 3): $|g_{01}| = |g_{10}|$ but $g_{01} \neq g_{10}$.

By the definition of growth we have $g_{00} \lessdot g_{01}$ and $g_{00} \lessdot g_{10}$. Then Remark 2.1 assures us that there exists one and only one element g_{11} which cover both g_{01} and g_{10}.

(Case 4): $g_{01} = g_{10}$

Set $g_{01} = g_{10} = x$. If $g_{00} \neq x$, then we have $g_{11} = \varphi_x^{-1}(g_{00})$ since g is consistent with Φ. If $g_{00} = x$, then g_{11} is completely determined by the assumption.

Next we construct g on cells which are below and to the left of $\mathscr{P}$. We can construct g_{00} from given g_{01}, g_{10} and g_{11} in the same way as above. ■

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be paths of S. Set $V = V(\mathscr{P}_1) \cap V(\mathscr{P}_2)$. Then V has at least two vetices: V_{01} and V_{10}. Fix any subsets P_v of P for each $v \in V$.

For example, set $\mathscr{P}_1 = \partial^-(S)$ and $\mathscr{P}_2 = \partial^+(S)$. We have $V = \{V_{01}, V_{10}\}$. Fix any $\alpha, \beta \in P$. $(P_{01}, P_{10}) = (\{\hat{0}\}, \{\hat{0}\})$, $(P_{01}, P_{10}) = (\{\alpha\}, \{\alpha\})$, $(P_{01}, P_{10}) = (\{\alpha\}, \{\beta\})$ and $(P_{01}, P_{10}) = (P, P)$ are examples.

The following theorem immediately follows from Theorem 3.1.

Theorem 3.2

Fix a connected skew diagram S and a differential poset P. Fix a system of R-correspondences on S and positive integers $(m, n) \in \mathbf{P}^2$. *Let* $\mathscr{P}_1$ *and* $\mathscr{P}_2$ *be paths of S. Set* $V = V(\mathscr{P}_1) \cap V(\mathscr{P}_2)$. *Fix an arbitrary family* $F = (P_v)_{v \in V}$ *of subsets of P. Theorem 3.1 gives a bijection*

$$(h_1, w_1) \leftrightarrow (h_2, w_2)$$

where (h_i, w_i) *are admissible pairs of a growth* $h_i: V(\mathscr{P}_i) \to P$ *on* $V(\mathscr{P}_i)$ *and a generalized permutation* w_i *on* $C(\mathscr{P}_i)$ *such that the weight of* (h_i, w_i) *is* (m, n) *and* $h_i(v) \in P_v$ *for each* $v \in V$. $(i = 1, 2)$ ■

Theorem 3.3

Fix a connectd skew diagram S and a differential poset P. Fix a system of R-correspondences on S and positive integers $(m, n) \in \mathbf{P}^2$. *Let* $\mathscr{P}_1$, $\mathscr{P}_2$ *and* $\mathscr{P}_3$ *be paths of S. Set* $V = V(\mathscr{P}_1) \cap V(\mathscr{P}_2) \cap V(\mathscr{P}_3)$. *Fix an arbitrary family* $F = (P_v)_{v \in V}$ *of subsets of P. Let* (h_1, w_1) *correspond to* (h_2, w_2) *in the correspondence of Theorem 3.2. Let* (h_2, w_2) *correspond to* (h_3, w_3) *in the above correspondence.* Then (h_2, w_2) *corresponds to* (h_3, w_3) *in the above correspondence. Here we assume that* (h_i, w_i) *are admissible pairs of a growth* $h_i: V(\mathscr{P}_i) \to P$ *on* $V(\mathscr{P}_i)$ *and a generalized permutation* w_i *on* $C(\mathscr{P}_i)$ *such that the weight of* (h_i, w_i) *is* (m, n) *and* $h_i(v) \in P_v$ *for each* $v \in V$. $(i = 1, 2, 3)$ ■

Set $\mathscr{P} = \partial^-(S)$, $\mathscr{P}' = \partial^+(S)$, $P_{V_{01}} = \{\hat{0}\}$ and $P_{V_{10}} = \{\hat{0}\}$. We obtain the following corollary from Theorem 3.2.

Corollary 3.1

Fix a connectd skew diagram S and a differential poset P. Fix a system of R-correspondences on S and a positive integer $n \in \mathbf{P}$. *Then we constructed a bijection between generalized permutations of length n and growths* $g^+: \partial^-(S) \to P$ *which satisfy* $g^+(V_{01}) = g^+(V_{10}) = \hat{0}$ *and has weight* (n, n). ■

Example 3.10

Here we give a big example which correspond to the mixed insertion algorithm of Example 2.5. Set P be the Young's lattice $\mathbf{Y}$ and S to be (14^{14}). Let (U, C, U', C') be as in Example 1.5 and Φ_m be the mixed system of R-correspondences determined by (U, C, U', C'). Set $\mathscr{P} = \partial^-(S)$ and suppose that the growth h on $V(\mathscr{P})$ has value $\emptyset$ at each vertex. Set w to be the cells containing X in the following diagram. Then the two-dimensional growth extended from (h, w) is as follows.

14	0	1 X	2	21	31	311	321	421	4211	4221	4222	5222	5322	6322	63221
13	0	0	1 X	11	21	211	221	321	3211	3221	3222	4222	4322	5322	53221
12	0	0	0	1	11 X	111	211	221	2211	2221	2222	3222	3322	4322	43221
11	0	0	0	1	1	11	21	211 X	2111	2211	2221	2222	3222	4222	42221
°10	0	0	0	1	1	11	21	21	211	221	222	2221	3221	4221 X	42211
°9	0	0	0	1	1	11	21	21	211 X	221	222	2221	3221	3221	32211
°8	0	0	0	1	1	11	21	21	21	22	221 X	2211	2221	2221	22211
°7	0	0	0	1	1	11	21	21	21	22	22	221	2211 X	2211	22111
°6	0	0	0	1	1	11 X	21	21	21	22	22	221	221	221	2211
°5	0	0	0	1 X	1	1	2	2	2	21	21	22	22	22	221
°4	0	0	0	0	0	0	1	1	1	11	11	21 X	21	21	211
3	0	0	0	0	0	0	1	1	1	11	11	11	11	11	111 X
2	0	0	0	0	0	0	1 X	1	1	11	11	11	11	11	11
1	0	0	0	0	0	0	0	0	0	1 X	1	1	1	1	1
0		0	0	0	0	0	0	0	0	0	0	0	0	0	0
		1	°2	3	°4	5	6	°7	8	9	10	°11	12	°13	°14

Set $\mathscr{P} = \partial^-(S)$, $\mathscr{P}' = \partial^+(S)$, $n = m$, $P_{V_{01}} = \{\alpha\}$ and $P_{V_{10}} = \{\alpha\}$ for a fixed $\alpha \in P$. Then we obtain the following corollary from Theorem 3.2.

Corollary 3.2

Fix a connectd skew diagram S and a differential poset P. Fix a system of R-correspondences on S, a positive integer $n \in \mathbf{P}$, *and an element* $\alpha \in P$. *Then we have a bijection between admissible pairs* (g^-, w) *of a growth* $g^- : V(\partial^-(S)) \to P$ *and a generalized permutation w on C(S) such that the weight of* (g^-, w) *is* (n, n), $g^-(V_{01}) = g^-(V_{10}) = \alpha$, *on the one hand, and growths* $g^+ : V(\partial^+(S)) \to P$ *such that the weight of* g^+ *is* (n, n), $g^+(V_{01}) = g^+(V_{10}) = \alpha$, *on the other.* ■

Example 3.11

Set P to be the Young's lattice $\mathbf{Y}$ and S to be (6^6). Let (U, C, U', C') and w be as in the following diagram. And we suppose Φ_m to be the mixed system of R-correspondences determined by (U, C, U', C'). Set $\mathscr{P} = \partial^-(S)$ and suppose that the growth g^- on $V(\mathscr{P})$ has values as in the following diagram. Then the two-dimensional growth extended from (g^-, w) is as follows.

21 6	211	221	321 X	421	422	522
21 °5	211	221	221	321	322	422 X
21 4	211	221	221	321	322	322
2 °3	21	211	211	311 X	321	321
2 2	21	211	211	211	221	221
1 °1	11 X	111	111	111	211	211
1	1	11	11	11	21	21
	1	°2	3	°4	5	°6

Set $\mathscr{P} = \partial^-(S)$, $\mathscr{P}' = \partial^+(S)$, $P_{V_{01}} = \{\alpha\}$ and $P_{V_{10}} = \{\beta\}$ for a fixed $\alpha, \beta \in P$. Then we obtain the following corollary from Theorem 3.2.

Corollary 3.3

Fix a connectd skew diagram S and a differential poset P. Fix a system of R-correspondences on S, a positive integer $(n, m) \in \mathbf{P}^2$, and elements $\alpha, \beta \in P$. Then we have a bijection between admisible pairs (g^-, w) of a growth $g^- : V(\partial^-(S)) \to P$ and a generalized permutation w on $C(S)$ such that the weight of (g^-, w) is (m, n), $g^-(V_{01}) = \alpha$, $g^-(V_{10}) = \beta$, on the one hand, and growths $g^+ : V(\partial^+(S)) \to P$ such that the weight of g^+ is (m, n), $g^+(V_{01}) = \alpha$, $g^+(V_{10}) = \beta$, on the other. ■

Example 3.12

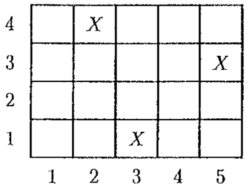
Set P to be the Young's lattice $\mathbf{Y}$ and S to be (6^5). Let (U, C, U', C') and w be as in the following diagram. And we suppose Φ_m to be the mixed system of R-correspondences determined by (U, C, U', C'). Set $\mathscr{P} = \partial^-(S)$ and suppose that the growth g^- on $V(\mathscr{P})$ has values as in the following diagram. Then the two-dimensional growth extended from (g^-, w) is as follows.

211 °5	221	222	322	332	432	442
21 °4	22	221 X	321	322	422	432
21 °3	22	22	32	321	421	431
11 °2	21	21	31	311 X	411	421
11 1	21	21	31	31	41	411
1	2	2	3	3	4	41
	°1	2	°3	4	°5	6

Corollary 3.4

Let P be a differential poset and S a connected skew diagram. Fix a system of R-correspondences on S. Then we have a bijection between admissible pairs (g^-, w) and growths $g^+ : \partial^-(S) \to P$.

Set P to be the Young's lattice $\mathbf{Y}$ and $S=(k^l)$ for fixed integers $k, l \in \mathbf{P}$. Set $\mathscr{A}'=[k]$ and $\mathscr{A}=[l]$. A word with bars $w\colon \mathscr{A}' \to \mathscr{A}$ is identified with a generalized permutation on S: A pair $\begin{pmatrix} i \\ j \end{pmatrix} \in w$ if and only if the cell (i, j) is included in the generalized permutation. For example, if $k=5$ and $l=4$, the biword $w=\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 3 \end{pmatrix}$ is identified with

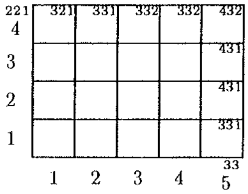


Let $V_{00}=(1, 1)$ and $V_{11}=(k, l)$ which are vertices in S.

Proposition 3.2

Let $g^+\colon V(\partial^+(S)) \to \mathbf{Y}$ be a growth. Set $\alpha=g^+(V_{01})$, $\beta=g^+(V_{10})$ and $\lambda=g^+(V_{11})$. The growth g^+ is identified with a pair (π, σ) of partial tableaux such that $\pi \in \mathrm{PST}(\lambda/\beta)$ and $\sigma \in \mathrm{PST}(\lambda/\alpha)$.

Let $g^-\colon V(\partial^-(S)) \to \mathbf{Y}$ be a growth. Set $\alpha=g^-(V_{01})$, $\beta=g^-(V_{10})$ and $\mu=g^-(V_{00})$. The growth g^- is identified with a pair (τ, κ) of partial tableaux such that $\tau \in \mathrm{PST}(\alpha/\mu)$ and $\sigma \in \mathrm{PST}(\beta/\mu)$.



For example, the above $g^+\colon V(\partial^+(S)) \to \mathbf{Y}$ is identified with the following pair.

$$\pi=\begin{array}{|c|c|c|c|}\hline & & 1 & 5 \\ \hline & & 2 \\ \cline{1-3} & 3 \\ \cline{1-2}\end{array} \qquad \sigma=\begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & & \\ \cline{1-3} 1 & 4 \\ \cline{1-2}\end{array}$$

From here we present the Fomin version of the Haiman's mixed correspondence which is given by the mixed system of R-correspondences. And prove the equivalence of the Fomin version and the mixed insertion procedure in a similar method given in [Fo2]. Notice that, in the "French" notation diagram S, the axes are pointing right and upward, while, in other "English" notation diagrams, the axes are pointing downward and right.

First we cite two lemmas which is proven in [Fo1] and [Fo2].

Lemma 3.1 [Fo1]

Let P be any finite poset. Let $e \in P$ be an extremal (i.e. maximal or minimal) element of

P. Then $\lambda(P \setminus \{e\}) \subset \lambda(P)$.

Let e_1 and e_2 be extremal elements of P. (i.e. maximal or minimal elements) Define $\lambda_{00} = \lambda(P \setminus \{e_1, e_2\})$, $\lambda_{01} = \lambda(P \setminus \{e_1\})$, $\lambda_{10} = \lambda(P \setminus \{e_2\})$ and $\lambda_{11} = \lambda(P)$. Then we have $\lambda_{00} \lessdot \lambda_{01} \lessdot \lambda_{11}$ and $\lambda_{00} \lessdot \lambda_{10} \lessdot \lambda_{11}$ by Lemma 3.1.

Lemma 3.2 [Fo2]

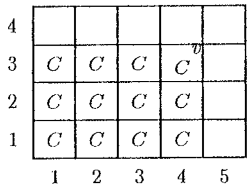
Assume $\lambda_{01} = \lambda_{10}$ *in the above situation. Let* $A = (i_A, j_A)$ *denote the cell* $\lambda_{01} \setminus \lambda_{00}$ *and* $B = (i_B, j_B)$ *denote the cell* $\lambda_{11} \setminus \lambda_{01}$.

Case 1: *If* e_1 *and* e_2 *are extremal elements of different types (i.e., one is maximal and the other minimal.) then* $(i_B, j_B) = (i_A + 1, j_A)$, *or,* $i_B \leq i_A$ *and* $j_B = j_A + 1$.

Case 2: *If* e_1 *and* e_2 *are extremal elements of the same type (i.e., both maximal or both minimal) then* $i_B > i_A$ *and* $j_B \leq j_A$.

The interested reader should consult with [Fo2] or appendex A of [Ro].

Henceforth we set P to be the Young's lattice $\mathbf{Y}$. And from now on we set $S = (k^l)$ for fixed integers $k, l \in \mathbf{P}$. We regard $\mathrm{COL}(S) = [k]$ and $\mathrm{ROW}(S) = [l]$, and fix divisions (U', C') of $[k]$ and (U, C) of $[l]$. Let Φ_m denote the mixed system of R-correspondences indeced from (U, C, U', C'). For any $v \in V(S)$, let S_v denote the skew diagram composed of the cells which are beneath and to the left of v in S. For example, in the following diagram S_v is composed of the cells containing C for given v.

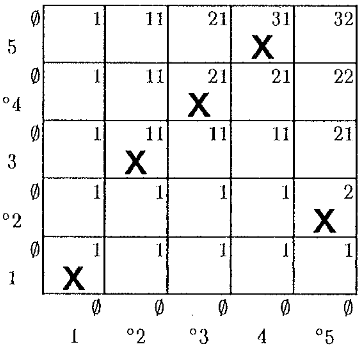


Definition 3.12

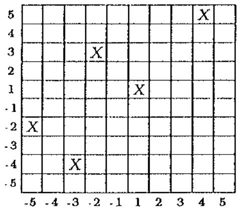
Let w be a generalized permutation on $C(S)$. For each $v \in V(S)$, set $w_v = C(S_v) \cap w$. Then w_v is a generalized permutation on S_v. The posets $P(w_v)$ and $\bar{P}(w_v)$ are defined from w_v in the same way as in Definition 2.4. Then chains in $P(w_v)$ are antichains in $\bar{P}(w_v)$ and antichains in $P(w_v)$ are chains in $\bar{P}(w_v)$. So by Theorem 1.2 we have $\lambda(P(w_v)) = \mu(\bar{P}(w_v))$. We associate $\lambda_v = \lambda(P(w_v))$ with each $v \in V(S)$, then we have a map $g \colon V(S) \to \mathbf{Y}$. By Lemma 3.1 g is a growth on $V(S)$. We call g the Greene-Kleitman two-dimensional growth defined by w.

Example 3.13

Let $S = (5^5)$. And let w be the cells containing X in the following diagram which is the same generalized permutation as in Example 1.7. The Greene-Kleitman two-eimensional growth $g \colon V(S) \to \mathbf{Y}$ $v \mapsto \lambda(P(w_v))$ defined by w is as follows.



The underlying set of $P(w)$ and $\bar{P}(w)$ is as in the following diagram.



Theorem 3.4

Fix $n \in \mathbf{P}$ and set $S = (k^l)$. Fix a division (U', C') of $R(S)$ and a division (U, C) of $C(S)$. Let Φ_m be the mixed system of R-correspondences induced from (U, C, U', C'). Let w be a generalized permutation on $C(S)$ and set $g_0^- : \partial^-(S) \rightarrow \mathbf{Y}$ to be the constant map defined by $g_0^-(v) = \emptyset$. Then the Φ-compatible two-dimensional growth induced from admissible pair (g_0^-, w) and the Greene-Kleitman two-dimensional growth defined by w have the same value.

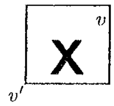
Proof.

Set g to be the Greene-Kleitman two-dimensional growth defined by w. By Lemma 3.1 it is easy to see that g is a growth on $V(S)$. So it is enough to prove the following claim.
For each cell $c \in V(C)$ such that $g_{01}(c) = g_{10}(c)$ we have

$$g_{11}(c) = \begin{cases} \{\varphi_x^{(c)}\}^{-1}(g_{00}(c)) & \textit{if } g_{00}(c) \neq x \textit{ or } c \in w \\ x & \textit{if } g_{00}(c) = x \textit{ and } c \notin w \end{cases}$$

where $g_{01}(c) = g_{10}(c) = x$.

Set $c = (i, j)$ and asume that $g_{01}(c) = g_{10}(c) = x$. (In this (i, j), the axes are pointing right and upword.) (Case 1) If $g_{00}(c) = x$ and $c \notin w$, then w have no cell eathier to the right of c or below c. So we have $g_{11} = x$.
(Case 2) Asuume that $g_{00}(c) = x$ and $c \in w$. Set $v = v_{11}(c)$ and $v' = v_{00}(c)$.

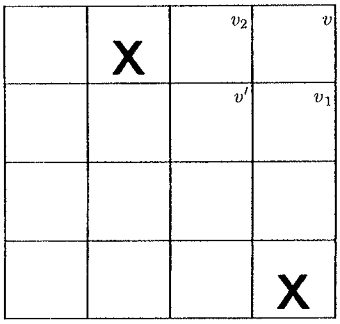


If $(i, j) \in U' \times U$, then $P(w_v) \setminus P(w_{v'})$ is a maximum element of $P(w_v)$. If $(i, j) \in C' \times C$, then $P(w_v) \setminus P(w_{v'})$ is a minimum element of $P(w_v)$. In either case $\lambda(P(w_v))$ is obtained from $\lambda(P(w_{v'}))$ by adding one cell in the first row. Similarly if $(i, j) \in U' \times C \cup C' \times U$, then $\bar{P}(w_v) \setminus \bar{P}(w_{v'})$ is either a minimum element or a maximum element of $\bar{P}(w_v)$. In either case $\lambda(\bar{P}(w_v))$ is obtained from $\lambda(\bar{P}(w_{v'}))$ by adding one cell in the first column.

(Case 3) Assume that $g_{00}(c) \neq x$.

Then w has cells both to the left of c and just below c. Let c_1 denote the cell which is to the left of c and containing X. Let c_2 denote the cell which is just below c and containing X. Set $v = v_{11}(c)$, $v_1 = v_{10}(c)$, $v_2 = v_{01}(c)$ and $v' = v_{00}(c)$. Set $e_1 = P(w_v) \setminus P(w_{v_1})$ and $e_2 = P(w_v) \setminus P(w_{v_2})$, then they are extremal elements of $P(w_v)$ and $\bar{P}(w_v)$.

And we have $P(w_{v'}) = P(w_v) \setminus \{e_1, e_2\}$. Define $\lambda_{00} = \lambda(P(w_{v'})) = \lambda(P(w_v) \setminus \{e_1, e_2\})$, $\lambda_{01} = \lambda(P(w_{v_1})) = \lambda(P(w_v) \setminus \{e_1\})$ and $\lambda_{11} = \lambda(P(w_v))$. Let $A = (x_A, y_A)$ denote the cell $\lambda_{01} \setminus \lambda_{00}$ and let $B = (x_B, y_B)$ denote the cell $\lambda_{11} \setminus \lambda_{01}$. (In this (x_A, y_A), the axes are pointing downword and right.)



Set the coordinates of c_1 (resp. c_2) to be $c_1 = (i', j)$ (resp. $c_2 = (i, j')$).

First assume that $(i, j) \in U' \times U \cup C' \times C$. If $(i, j) \in U' \times U$, then both e_1 and e_2 are maximal elements of $P(w_v)$. In addition e_1 is a minimal element of $\bar{P}(w_v)$ and e_2 is a maximal element of $\bar{P}(w_v)$. (For example if $j' \in C$, then we have $e_2 = (i, j')$. This is a maximal element of $P(w_v)$ and $\bar{P}(w_v)$.) If $(i, j) \in C' \times C$, then both e_1 and e_2 are minimal elennts of $P(w_v)$. In addition e_1 is a maximal element of $\bar{P}(w_v)$ and e_2 is a minimal element of $\bar{P}(w_v)$. In both of the above cases e_1 and e_2 are extremal elements of the same type in $P(w_v)$, but they are extremal elements of different types in $\bar{P}(w_v)$. By Lemma 3.2 we have $x_B > x_A$ and $y_B \leq y_A$. By Lemma 3.2 and Theorem 1.3 we obtain $(x_B, y_B) = (x_A, y_A + 1)$ or $x_B = x_A + 1$, $y_B \leq y_A$. Combining these we obtain $x_B = x_A + 1$ and $y_B \leq y_A$: i.e. $\lambda_{11} = (\varphi_R)^{-1}_{\lambda_{01}}(\lambda_{00})$.

Next assume that $(i, j) \in U' \times C \cup C' \times U$. If $(i, j) \in U' \times C$, then e_1 is a minimal element of $P(w_v)$ and e_2 is a maximal element of $P(w_v)$. In addition both e_1 and e_2 are maximal elements of $\bar{P}(w_v)$. If $(i, j) \in C' \times U$, then e_1 is a maximal element of $P(w_v)$, e_2 is a minimal element of $P(w_v)$ and both e_1 and e_2 are minimal elements of $\bar{P}(w_v)$. In both of the above cases e_1 and e_2 are extremal elements of different types in $P(w_v)$ but they are extremal elements of the same type in $\bar{P}(w_v)$. By the same reason as above we have $\lambda_{11} = (\varphi_C)^{-1}_{\lambda_{01}}(\lambda_{00})$. This prove the theorem. ■

From this theorem we immediately obtain the following theorem.

Theorem 3.5

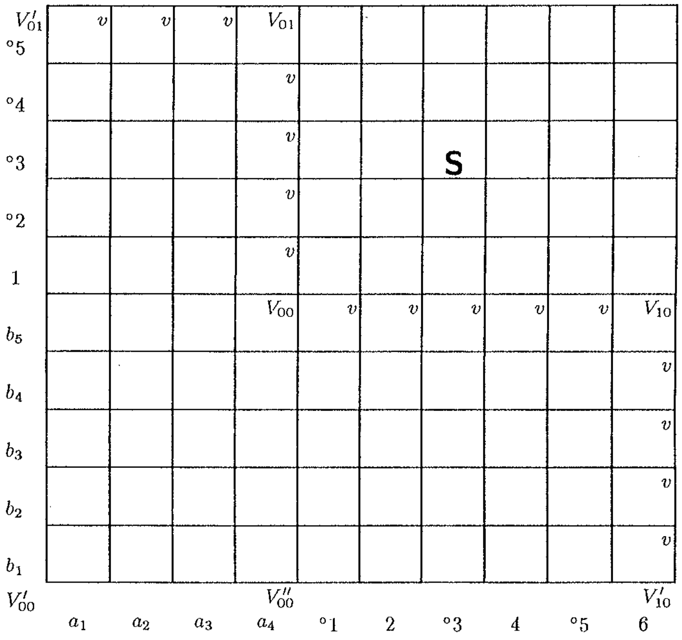
Fix $k \in \mathbf{P}$ *and set* $S = (k^l)$. *Fix a division* (U', C') *of* $[k]$ *and a division* (U, C) *of* $[l]$. *Let* Φ_m *be the mixed system of R-correspondences induced from* (U, C, U', C'). *Set* $P = Y$ *and set* Φ_m *to be the mixed system of R-correspondences induced from* (U, C, U', C') *in Corollary* 3.1. *Then the correspondence given in Corollary* 3.1 *and the correspondence given in Theorem* 2.1 *are identical.* ■

Theorem 3.6

Fix $k \in P$ *and set* $S = (k^l)$. *Fix a division* (U', C') *of* $[k]$ *and a division* (U, C) *of* $[l]$. *Set* $P = \mathbf{Y}$ *and set* Φ_m *to be the mixed system of R-correspondences induced from* (U, C, U', C') *in Corollary* 3.3. *Then the correspondence given in Corollary* 3.3 *and the correspondence given in Theorem* 2.4 *are identical.*

Proof.

Let $S' = ((|\alpha| + k)^{|\beta|+l})$. Place S into the right and above corner of S'. We prepare $|\alpha|$ uncircled letters $a_1 < a_2 < \cdots < a_{|\alpha|}$ which are less than all the letters in $[k]$, and $|\beta|$ uncircled letters $b_1 < b_2 < \cdots < b_{|\beta|}$ which are less than all the letters in $[l]$. We number the columns to the left of S with the letters $a_1, a_2, \ldots, a_{|\alpha|}$ and the rows below S with the letters $b_1, b_2, \ldots, b_{|\beta|}$. For example if we have S in Example 3.12, then S' is as follows.



Let V_{01} (resp. V_{10}) denote the most left and up (resp. right and down) vertex in $V(S)$ which is placed in $V(S')$. Let V_{00} denote the most left and down vertex in $V(S)$ which is placed in $V(S)$. Let V'_{01} (resp. V'_{10}) denote the most left and up vertex (resp. right and down) in $V(S')$. Let

V_{00}' denote the most left and down vertex in $V(S)'$. Let V_{00}'' denote the vertex which is just below V_{00} and the most down in $V(S)'$ as in the above diagram. We define the paths $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$ of S' as follows. Set $\mathscr{P}_1=\partial^+(S')$ and $\mathscr{P}_3=\partial^-(S')$. The $\mathscr{P}_2$ is defined as the path which goes right from V_{01}' to V_{01}, then goes down from V_{01} to V_{00}, then goes right from V_{00} to V_{10}, and finally goes down from V_{10} to V_{10}'. Let $\mathscr{Q}_1$ denote the subpath of $\mathscr{P}_2$ which goes right from V_{01}' to V_{01}. Let $\mathscr{Q}_2$ denote the subpath of $\mathscr{P}_2$ which goes down from V_{01} to V_{00}. Let $\mathscr{Q}_3$ denote the subpath of $\mathscr{P}_2$ which goes right from V_{00} to V_{10}. Let $\mathscr{Q}_4$ denote the subpath of $\mathscr{P}_2$ which goes down from V_{10} to V_{10}'. $\mathscr{Q}_2\bigcup\mathscr{Q}_3$ is denoted by $\partial^-(S)$. Given a growth $g^-: V(\partial^-(S))\to\mathbf{Y}$, we define a growth h: $V(\mathscr{P}_2)\to\mathbf{Y}$ as follows. g^- and h have the same value on $V(\partial^-(S))$. On $\mathscr{Q}_1$, h always has the value which correspond to $\bar{\alpha}$ defined in Definition 2.5. On $\mathscr{Q}_4$, h always has the value which correspond to $\bar{\beta}$ defined in Definition 2.5. This gives an injection

$$(g^-, w)\mapsto(h, w)$$

from the set of all pairs (g^-, w) such that $g^-: V(\partial^-(S))\to\mathbf{Y}$ is a growth on $V(S)$, $g^-(V_{01})=\alpha$, $g^-(V_{10})=\beta$ and w is a generalized permutation on $C(S)$, on the one hand, into the set of all pairs (h, w) such that h is a growth on $V(\mathscr{P}_2)$, $h({}_{01}')=h(V_{10}')=\emptyset$ and w is a generalized permutation on $C(\mathscr{P}_2)$, on the other. We denote this injection by F_1. From (h, w), we obtain a admisible pair (h', w') of a growth h': $V(\partial^-(S'))\to\mathbf{Y}$ and a generalized permutation w' on $C(S')$ by the corresondence of Theorem 3.2.

$$(h, w)\mapsto(h', w')$$

We denote this bijection by F_2. The composition of these maps $F=F_2\circ F_1$ is an injection. Now we define one more injection

$$g^+\mapsto g'^+$$

from the set of all growths g^+ on $V(\partial^+(S))$ such that $g^+(V_{01})=\alpha$ and $g^+(V_{10})=\beta$, on the one hand, into the set of all growths $g'^+(V_{10}')=\emptyset$, on the other. g'^+ and g^- have the same value on $V(\partial^+(S))$. On $\mathscr{Q}_1$, g'^+ always has the value which correspond to $\bar{\alpha}$. On $\mathscr{Q}_4$, g'^+ always has the value which correspond to $\bar{\beta}$. This injection is denoted by F'.

First we claim that F' can be identified with the bracket operation $(\pi, \sigma)\mapsto[\pi, \sigma]$ in Definition 2.5 by the identification in Proposition 3.1. But this is clear from difinition.

Next we claim that F can be identified with the bracket operation $(w, \tau, \kappa)\mapsto[w, \tau, \kappa]$ in Definition 2.5 by the identification in Proposition 3.1. If we prove this theorem then the theorem is an easy consequence of Lemma 2.2 and Theorem 3.3.

(Case 1) At first we consider the case: $\beta=\emptyset$

This forces us $\mu=\emptyset$ in Theorem 2.4. Let $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) denote the path which goes down from V_{01}' to V_{00}' *(resp. right from* V_{00}' to $V_{00}=V_{00}''$). Theorem 3.4 shows that the correspondence between pairs on $\mathscr{P}_1\bigcup\mathscr{P}_2$ and pairs on $\mathscr{R}_1\bigcup\mathscr{R}_2$ can be identified with the deletion process on $(\tau', \bar{\alpha})$ in Definition 2.5. This proves our claim in Case 1.

(Case 2) Next we consider general case.

Let $\mathscr{R}_1$ (resp. $\mathscr{R}_3$) denote the path which goes down from V_{01}' to V_{00}' (resp. from V_{00} to V_{00}''). Let $\mathscr{R}_2$ (resp. $\mathscr{R}_4$) denote the path which goes right from V_{00}' to V_{00}'' (resp. from V_{00}'' to

V'_{10}). Set S'' to be the skew diagram surrounded by $\mathscr{Q}_3, \mathscr{Q}_4, \mathscr{R}_3$ and $\mathscr{R}_4$. Then the correspondence between pairs on $\partial^+(S'') = \mathscr{Q}_3 \cup \mathscr{Q}_4$ and pairs on $\partial^+(S'') = \mathscr{R}_3 \cup \mathscr{R}_4$ can be identified with the mixed deletion procedure on $(\bar{\beta}, \kappa)$ in Definition 2.5 by Case 1. Set S''' to be the skew diagram surrounded by $\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{R}_3, \mathscr{R}_1$ and $\mathscr{R}_2$. Obtaining growths on $\mathscr{Q}_2 \cup \mathscr{R}_3$ from growths on $\mathscr{Q}_2$ and $\mathscr{R}_3$ is identified placing the resulting tableaus into τ to obtain τ'. Then Theorem 3.4 shows that the correspondence between pairs on $\mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{R}_3$ and pairs on $\mathscr{R}_1 \cup \mathscr{R}_2$ can be identified with the deletion procedure on $(\tau', \bar{\alpha})$ in Definition 2.5. This proves our claim. ■

§ 4 Mixed Knuth Correspondence for (A, B)-Partially Strict Tableaux

In this section we consider the mixed Kunuth correspondence and dual mixed Knuth correspondence simultaneously by using (A, B)-partialy strict tableaux.

Fix another division (A, B) of $\mathscr{A}$. Set $k = |A|$ and $l = |B|$ so that we have $|\mathscr{A}| = k + l$. We have two pairs (A, B) and (U, C) which are divisions of $\mathscr{A}$. We write

$$\begin{aligned} A_u &= A \cap U, & A_c &= A \cap C \\ B_u &= B \cap U, & B_c &= B \cap C \end{aligned}.$$

Example 4.1
Set $A = \{1, 3^\circ, 5, 7^\circ\}$, $B = \{\mathbf{2}, \mathbf{4}^\circ, \mathbf{6}, \mathbf{8}^\circ\}$, $U = \{1, \mathbf{2}, 5, \mathbf{6}\}$ and $C = \{3^\circ, \mathbf{4}^\circ, 7^\circ, \mathbf{8}^\circ\}$. Then (A, B) and (U, C) are divisions of $[8]$ and we have $A_u = \{1, 5\}$, $A_c = \{3^\circ, 7^\circ\}$, $B_u = \{\mathbf{2}, \mathbf{6}\}$, and $B_c = \{\mathbf{4}^\circ, \mathbf{8}^\circ\}$. As in this example we write elements of A in lightface and elements of B in boldface.

The folowing terminology is not so common but it is an easy extension of (k, l)-semistandard tableaux. For the definition of (k, l)-semistandard tableaux see [BR] or [Re].

Definition 4.1 (Okada)
Let π be a (skew) reverse plane partition. π is said to be (A, B)-partially strict if it satisfies the conditions:

(*i*) For any $m \in A$, m appears at most once in each column.

(*ii*) For any $m \in B$, m appears at most once in each row.

We call a (A, B)-partially strict (skew) reverse plane partition a (A, B)-partially strict (skew) tableau. A $(\mathbf{P}, \emptyset)$-partially strict skew tableau is usually called a column-strict skew tableau and a $(\emptyset, \mathbf{P})$-partially strict skew tableau, a row-strict skew tableau. If $A = \{1, 2, \dots, k\}$ and $B = \{1', 2', \dots, 1'\}$, where $1 < 2 < \cdots < k < 1' < 2' < \cdots < 1'$, then a (A, B)-partially strict tableau is called a (k, l)-semistandard tableau.

Example 4.2
Set (A, B) to be the division given in Example 4.1 Then

$$\pi = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \mathbf{2} & \mathbf{4}^\circ & 5 & 5 & 7^\circ \\ \hline 3^\circ & 3^\circ & 3^\circ & \mathbf{4}^\circ & \mathbf{6} & 7^\circ & 7^\circ \\ \cline{1-7} \mathbf{4}^\circ & 5 & 5 & \mathbf{6} \\ \cline{1-4} \mathbf{4}^\circ & 7^\circ & 7^\circ & \mathbf{8}^\circ \\ \cline{1-4} 5 & \mathbf{8}^\circ \\ \cline{1-2} \end{array}$$

is a (A, B)-partially strict tableau.

$$\pi' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 3 & \mathbf{2}' & \mathbf{3}' \\ \hline 4 & 4 & 1' & \mathbf{2}' \\ \cline{1-4} 1' & \mathbf{3}' \\ \cline{1-2} 1' \\ \cline{1-1} \end{array}$$

is an exmple of (4, 3)-semistandard tableau.

Definition 4.2

Let λ/μ be a skew diagram. Let $\mathscr{T}_{(A,B)}(\lambda/\mu)$ denote the set of all (A, B)-partially strict skew tableaux of shape λ/μ. For $\pi \in \mathscr{T}_{(A,B)}(\lambda/\mu)$ set the weight $wt(\pi)$ of π to be $\prod_{a\in\mathscr{A}} x_a^{m_a}$ where

$$m_a = \text{number of times } a \text{ occurs in } \pi$$

and x_a's are indeterminates. Set

$$HS^{(A,B)}_{\lambda/\mu}(x) = \sum_{\pi\in\mathscr{T}_{(A,B)}(\lambda/\mu)} wt(\pi).$$

It is clear from the definition that

$$HS^{(A,B)}_{\lambda/\mu}(x) = HS^{(B,A)}_{\lambda'/\mu'}(x).$$

In particular if $A = \{1, 2,\dots,k\}$ and $B = \{1', 2',\dots,l'\}$, where $1 < 2 < \cdots < k < 1' < 2' < \cdots < l'$, we write $HS^{(A,B)}_{\lambda/\mu}(x)$ as $HS^{(k,l)}_{\lambda/\mu}(x_1, x_2,\dots,x_k, y_1, y_2,\dots,y_l)$.

Proposition 4.1

Set $A = \{a_1, a_2,\dots,a_k\}$ *and* $B = \{\mathrm{b}_1, \mathrm{b}_2,\dots,\mathrm{b}_l\}$. *Then*

$$HS^{(A,B)}_{\lambda/\mu}(x) = HS^{(k,l)}_{\lambda/\mu}(x_{a_1}, x_{a_2},\dots,x_{a_k}, x_{\mathrm{b}_1}, x_{\mathrm{b}_2},\dots,x_{\mathrm{b}_l}).$$

Proof.

We can easily construct a bijection beteween $\mathscr{T}_{(A,B)}(\lambda/\mu)$ and $\mathscr{T}_{(\{1,2,\dots,k\},\{1',2',\dots,l'\})}(\lambda/\mu)$ using the jeu de taquin method in [Re], Section 3, pp. 266. For details se [Re]. ■

Definition 4.3

Let π be a (A, B)-partially strict tableau and let $x \in \mathscr{A}$. We define $\mathrm{INSERT}_{(A,B;U,C)}(x)$ as follow.

If $x \in U$, insert x into the first row of π; if $x \in C$, insert x into the first column of π. If the bumped element y is uncircled, then we insert y into the row immediately below or if the bumped element y is circled, then we insert y into the column immediately to its right by the following rules.

y replace the least element which is $> y$ if $y \in A_u \bigcup B_c$: or y replace the least element which is $\geq y$ if $y \in B_u \bigcup A_c$.

Continue until an insertion takes place at the end of a row or column, bumping no new element. This procedure terminates in a finite number of steps. Then set (s, t) to be the cell which is added to π.

Similarly we define $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x)$ by swapping U and C in the foregoing definition. If $x \in U$, insert x into the first column of π; if $x \in C$, insert x into the first row of π. The uncircled letters which are bumped are inserted into the column immediately to its right and circled letters are insertedinto the row immediately below by the following rule.

y replace the least element which is $> y$ if $y \in A_c \bigcup B_u$: or y replace the least element which is $\geq y$ if $y \in B_c \bigcup A_u$.

It is easy to see that the resulting tableau is also (A, B)-partially strict. Let $\pi \leftarrow^m x$ (resp. $x \rightarrow^m \pi$) denote the tableau which is obtained after we applied $\mathrm{INSERT}_{(A,B;U,C)}(x)$ (resp. $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x)$) to π.

Example 4.3
Let π be the (A, B)-partially strict tableau in Example 4.2.

$\pi \leftarrow^m 4 =$

1	1	1	1	**4°**	5	5	7°
2	3°	3°	3°	**4°**	7°	7°	
4°	5	5	**6**	**8°**			
4°	**6**	7°	7°				
5	**8°**						

And we have $(s, t) = (4, 4)$.

$2 \rightarrow^m \pi =$

1	1	1	**2**	**4°**	5	5	7°
2	3°	3°	**4°**	**6**	7°	7°	
3°	5	5	5	**6**			
4°	7°	7°	**8°**				
4°	**8°**						

And we have $(s, t) = (6, 1)$.

Remark 4.1
In [Re] two insertion procedures are defined for (k, l)-semistandard tableaux. Set $A = \{1, 2, \dots, k\}$ and $B = \{1', 2', \dots, l'\}$, where $1 < 2 < \cdots < k < 1' < 2' < \cdots < l'$. If $U = A$ and $C = B$, then the insertion algorithm in Definition 4.2 is called RS1 insertion in [Re]. If $U = \mathscr{A}$ and $C = \emptyset$, then the insertion algorithm is called RS2 insertion.

Definition 4.4
Let π be a (A, B)-partially strict tableau. Set m_x to be the number of times x occurs in π

for each $x \in \mathscr{A}$. Let $m = \sum_{x \in \mathscr{A}} m_x$. We make a partial tableau $pt(\pi)$ with letters in $[m]$ from π as follows. If $x \in A$, then replace $m_x x$'s in π to $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2, \ldots, \sum_{y \leq x} m_x$ from left to right. If $x \in B$, then replace $m_x x$'s in π to $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2, \ldots, \sum_{y \leq x} m_x$ from top to bottom. If $x \in U$ then $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2, \ldots, \sum_{y \leq x} m_x$ are in U, and vice versa.

Example 4.4
If π is as in Example 4.2, then $pt(\pi)$ is as in Example 2.2.

Definition 4.5
A *word with repetition* is a sequence $w = w_1 w_2 \ldots w_m$ of letters in $\mathscr{A}$ wherein each $a \in \mathscr{A}$ can appear more than once. Given a word with repetition $w = w_1 w_2 \ldots, w_m$, we make the insertion tableau $\pi = \emptyset \leftarrow^m w$ for w as follows. For $i = 1, 2, \ldots m$ we define inductively $\pi_0 = \emptyset$ and $\pi_i = \pi_{i-1} \leftarrow^m w_i$. Let $\pi = \pi_m$.

Example 4.5

$$w = {}^{\circ}\mathbf{2} \quad {}^{\circ}\mathbf{2} \quad 1 \quad {}^{\circ}3 \quad \mathbf{4} \quad {}^{\circ}3 \quad {}^{\circ}3 \quad 1 \quad \mathbf{4} \quad {}^{\circ}\mathbf{2} \quad \mathbf{4}$$

is a word with repetition and the insertion tableau for w is as follows.

$$\emptyset \leftarrow^m w = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & {}^{\circ}\mathbf{2} & {}^{\circ}3 & \mathbf{4} \\ \hline {}^{\circ}\mathbf{2} & {}^{\circ}3 & {}^{\circ}3 & \mathbf{4} \\ \cline{1-4} {}^{\circ}\mathbf{2} & \mathbf{4} \\ \cline{1-2} \end{array}$$

Definition 4.6
For a given word with repetition $w = w_1 w_2 \ldots w_m$ we make a permutation $p(w)$ of $[m]$ as follows. For each $x \in \mathscr{A}$ let m_x denote the number of times x appears in w. For each $x \in \mathscr{A}$, if $x \in A_u \cup B_c$ then replace all x in w by $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2, \ldots, \sum_{y \leq x} m_x$ in increasing order. For each $x \in \mathscr{A}$, if $x \in \mathscr{A}$, if $x \in A_c \cup B_u$ then replace all x in w by $\sum_{y \leq x} m_x$, $\sum_{y \leq x} m_x - 1, \ldots,$ $\sum_{y<x} m_x + 1$ in decreasing order. If $x \in U$ then $\sum_{y<x} m_x + 1$, $\sum_{y<x} m_x + 2, \ldots, \sum_{y \leq x} m_x$ are in U, and vice versa.

Example 4.6
Let w be as in Example 4.5.

$$p(w) = {}^{\circ}3 \quad {}^{\circ}4 \quad 1 \quad {}^{\circ}8 \quad 9 \quad {}^{\circ}7 \quad {}^{\circ}6 \quad 2 \quad 10 \quad {}^{\circ}5 \quad 11$$

$$\emptyset \leftarrow^m p(w) = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & {}^{\circ}8 & {}^{\circ}3 & 9 \\ \hline {}^{\circ}4 & {}^{\circ}6 & {}^{\circ}7 & 10 \\ \cline{1-4} {}^{\circ}5 & 11 \\ \cline{1-2} \end{array}$$

The following proposition is easy to see from definitions.

Proposition 4.2
Let w be a word with repetition. Let π be the insertion tableau of w Then the following diagram commutes.

$$\begin{tikzcd} w \arrow[r, leftrightarrow] \arrow[d, "p"] & \pi \arrow[d, "pt"] \\ p(w) \arrow[r, leftrightarrow] & pt(\pi) \end{tikzcd}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively. ■

Lemma 4.1

Let π be a (A, B)-partially strict tableau and $c, x' \in \mathscr{A}$. If $\mathrm{INSERT}_{(A,B;U,C)}(x)$, determining s and t, is immediately followed by $\mathrm{INSERT}_{(A,B;U,C)}(x')$, determining (s', t'), then

(Case 1) $x, x' \in U$

(a) If $x < x'$ or $x = x' \in A$ then we have $s \geq s'$ and $t < t'$.

(b) If $x > x'$ or $x = x' \in B$ then we have $s < s'$ and $t \geq t'$.

(Case 2) $x, x' \in C$

(a) If $x > x'$ or $x = x' \in A$ then $s \geq s'$ and $t < t'$.

(b) If $x < x'$ or $x = x' \in B$ then $s < s'$ and $t \geq t'$.

(Case 3) $x \in U$ and $x' \in C$, we always have $s < s'$ and $t \geq t'$.

(Case 4) $x \in C$ and $x' \in U$, we always have $s \geq s'$ and $t < t'$.

Proof.

Choose arbitrary word with repetition w such that $\pi = \emptyset \leftarrow^{m} w$. Let $w' = wxx'$. Then it is easy to verify the lemma by using Proposition 4.2, Corollary 2.2, and Lemma 1.1. For example, we verigy Case 3. Assume that $x \in U$ and $x' \in C$. Then $x' \in C$. Then $x' \in C$ is changed into some negative letter $-x'$ which is less than x so that we obtain $s < s'$ and $t \geq t'$ immediately by Lemma 1.1. ■

Remark 4.2

In the foregoing lemma by changing $\mathrm{INSERT}_{(A,B;U,C)}(x)$ and $\mathrm{INSERT}_{(A,B;U,C)}(x')$ into $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x)$ and $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(x')$, respectively and swapping U and C, we obtain a similar result on $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(\cdot)$.

Fix another finite totally orderd set $\mathscr{A}'$ and its divisions (A', B') and (U', C') such that $|A'| = k'$ and $|B'| = l'$ We write

$$\begin{array}{ll} A'_u = A' \cap U', & A'_c = A' \cap C' \\ B'_u = B' \cap U', & B'_c = B' \cap C' \end{array}.$$

Definition 4.7

Let a be a $(k' + l') \times (k + l)$ matrix of nonnegative integers

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

whose rows are labeled by elements of $\mathscr{A}'$ and columns are labeled by elements of $\mathscr{A}$. a is said to be *addmissible* if it satisfies:

(1) If $(i, j) \in A' \times A \cup B' \times B$, $\quad a_{i,j} \in \mathbf{N}$.

(2) If $(i, j) \in A' \times B \cup B' \times A$, $\quad a_{i,j} \in \{0, 1\}$.

Let $\mathscr{M}(A', B', A, B)$ denote the set of all admissible $(k' + l') \times (k + l)$ matrices.

Example 4.7

Let $A' = \{2, {}^\circ 4\}$, $B' = \{1, {}^\circ 3\}$, $A = \{{}^\circ 3, 4\}$, and $B = \{1, {}^\circ 2\}$. Then

$$a = \begin{pmatrix} \mathit{2} & \mathit{0} & 0 & 0 \\ 0 & 1 & \mathit{1} & \mathit{2} \\ \mathit{0} & \mathit{3} & 1 & 0 \\ 1 & 0 & \mathit{1} & \mathit{2} \end{pmatrix}$$

is a admissibe 4×4 matrix.

Definition 4.8

Let $a \in \mathscr{M}(A', B', A, B)$. From a we make a two-line array

$$l(a) = \begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

as follows. We arrange $a_{u,v}$ pairs of row and column labels $\begin{pmatrix} u \\ v \end{pmatrix}$ by the following rule. First we assume that

$$u_1 \leq u_2 \leq \cdots \leq u_m.$$

(1) For each $u \in A'_u \cup B'_c$ we arrenge all labels $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ such that $u_i = u$ as follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \ldots, \ldots, v_{p_r},}_{\substack{\text{elements of } C\\ \text{in decreasing order}}} \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \ldots, \ldots, v_{p_{r+s}}}_{\substack{\text{elements of } U\\ \text{in increasing order}}}$$

(2) For each $u \in \mathscr{A}'_c \cup B'_u$ we arrenge all labels $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ such that $u_i = u$ is follows.

$$\underbrace{v_{p_1}, v_{p_2}, v_{p_3}, \ldots, \ldots, v_{p_r},}_{\substack{\text{elements of } U\\ \text{in decreasing order}}} \underbrace{v_{p_{r+1}}, v_{p_{r+2}}, \ldots, \ldots, v_{p_{r+s}}}_{\substack{\text{elements of } C\\ \text{in increasing order}}}$$

It is easy to see this gives a one to one correspondence between admissible matrices and two line arrays satisfying the above conditions. We call this two line array the *matrix word* of a and denote by $l(a)$. The top (resp. bottom) line of $l(a)$ is denoted by $\hat{l}(a) = u_1, u_2, \cdots, u_m$ (resp. $\check{l}(a) = v_1, v_2, \cdots, v_m$).

Example 4.8

The two line array which correspond to the matrix a in Example 4.7 is

$$l(a)=\begin{pmatrix} \mathbf{1} & \mathbf{1} & 2 & 2 & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ 4 & {}^\circ 4 & {}^\circ 4 & {}^\circ 4 \\ \mathbf{1} & \mathbf{1} & {}^\circ 3 & {}^\circ 2 & \mathbf{4} & \mathbf{4} & {}^\circ 3 & {}^\circ 2 & {}^\circ 2 & {}^\circ 2 & \mathbf{4} & \mathbf{4} & \mathbf{1} & {}^\circ 3 \end{pmatrix}.$$

Definition 4.9

Let $a \in \mathscr{M}(A', B', A, B)$. From a we make a two-line array $l(a)$ in Definition 4.8.

$$l=\begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

We construct a sequence of tableaux pairs:

$$(\emptyset, \emptyset)=(\pi_0, \sigma_0), (\pi_1, \sigma_1), \ldots, (\pi_m, \sigma_m)=(\pi, \sigma)$$

inductively as follows. For each $i=1, 2, \ldots, m$ form π_i from π_{i-1} by performing $\mathrm{INSERT}_{(A,B;U,C)}(v_i)$ on π_{i-1} if u_i is a uncircled letter, or performing $\overline{\mathrm{INSERT}}_{(A,B;U,C)}(v_i)$ on π_{i-1} if u_i is a circled letter. Form σ_i from σ_{i-1} by placing u_i on σ_{i-1} in the cell added to π_i. By Lemma 4.1 σ is a (A', B')-partially strict tableau and π and σ have the same shape.

Example 4.9

Let a be as in Example 4.7. Then

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & 4 & 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{2} & 4 \\ \cline{1-3} \mathbf{1} & {}^\circ 3 \\ \cline{1-2} {}^\circ\mathbf{2} & 4 \\ \cline{1-2} {}^\circ\mathbf{2} \\ \cline{1-1} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & 2 & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-3} 2 & {}^\circ\mathbf{3} \\ \cline{1-2} {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-2} {}^\circ 4 \\ \cline{1-1} \end{array}$$

Definition 4.10

Let $a \in \mathscr{M}(A', B', A, B)$. Let

$$l(a)=\begin{pmatrix} u_1 & u_2 & \cdots & \cdots & u_m \\ v_1 & v_2 & \cdots & \cdots & v_m \end{pmatrix}$$

be the two line array which correspond to a. We construct a biword w from l as follow. For each $x \in \mathscr{A}$ (resp. $x \in \mathscr{A}'$) let m_x (resp. m'_x) denote the numeber of times x occurs in the bottom (resp. top) line of l. Replace m'_x x's in the top line of l by $\sum_{y<x} m_x + 1, \sum_{y<x} m_x + 2, \ldots, \sum_{y \leq x} m_x$ from left to right. The circles are transfered unchanged in this replacement. For each $x \in \mathscr{A}$ let r_x (resp. s_x) be the number of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in U$ (resp. $u_i \in C$). So we have $r_x + s_x = m_x$. For each $x \in \mathscr{A}$ we replace v_j's such that $v_j = x$ by the following rules. The circles are transfered unchanged in this replacement.

(Case 1): $x \in A_u \cup B_c$

Replace the v_j's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in C$ by $\sum_{y<x} m_x + 1, \sum_{y<x} m_x + 2, \ldots,$

$\sum_{y<x} m_x + s_x$ from right to left. Then replace the v_j's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in U$ by $\sum_{y<x} m_x + s_x + 1, \sum_{y<x} m_x + s_x + 2, \ldots, \sum_{y \leq x} m_x$ from left to right.
(Case 2): $x \in A_c \cup B_u$
Replace the v_j's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in U$ by $\sum_{y<x} m_x + 1, \sum_{y<x} m_x + 2, \ldots,$ $\sum_{y<x} m_x + r_x$ from right to left. Then replace the v_j's of pairs $\begin{pmatrix} u_i \\ v_j \end{pmatrix}$ such that $v_j = x$ and $u_i \in C$ by $\sum_{y<x} m_x + r_x + 1, \sum_{y<x} m_x + r_x + 2, \ldots, \sum_{y \leq x} m_x$ from left to right.
Let $p(a)$ denote the resulting biword.

Example 4.10
Let a be as Examle 4.7 and $l(a)$ as Example 4.8. Then we have

$$p(a) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & {}^\circ 7 & {}^\circ 8 & {}^\circ 9 & {}^\circ 10 & {}^\circ 11 & {}^\circ 12 & {}^\circ 13 & {}^\circ 14 \\ 2 & 1 & {}^\circ 8 & {}^\circ 7 & 13 & 14 & {}^\circ 9 & {}^\circ 6 & {}^\circ 5 & {}^\circ 4 & 12 & 11 & 3 & {}^\circ 10 \end{pmatrix}.$$

The following proposition is easy to see from definitions.

Proposition 4.3
Let $a \in \mathscr{M}(A', B', A, B)$. *Then the following diagram commutes.*

$$\begin{tikzcd} a \arrow[r, leftrightarrow] \arrow[d, "p"] & (\pi, \sigma) \arrow[d] \\ p(a) \arrow[r, leftrightarrow] & (pt(\pi), pt(\sigma)) \end{tikzcd}$$

where the top and bottom bijections are the mixed Knuth and mixed Robinson-Schensted maps, respectively. ■

Example 4.11
Let $p(a)$ be as in Example 4.9. Then the insertion pair of $p(a)$ is as follows.

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline 1 & {}^\circ 4 & {}^\circ 9 & {}^\circ 10 & 13 & 14 \\ \hline 2 & {}^\circ 5 & 12 \\ \cline{1-3} 3 & {}^\circ 8 \\ \cline{1-2} {}^\circ 6 & 11 \\ \cline{1-2} {}^\circ 7 \\ \cline{1-1} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 4 & 5 & 6 & {}^\circ 7 & {}^\circ 14 \\ \hline 2 & {}^\circ 8 & {}^\circ 13 \\ \cline{1-3} 3 & {}^\circ 9 \\ \cline{1-2} {}^\circ 10 & {}^\circ 12 \\ \cline{1-2} {}^\circ 11 \\ \cline{1-1} \end{array}$$

From Proposition 4.3 we obtain the following theorem.

Theorem 4.1
Fix $\mathscr{A}$ *and its divisions* (U, C) *and* (A, B). *Fix another* $\mathscr{A}'$ *and its divisions* (U', C') *and* (A', B'). *The map in Definition* 4.9 *from admissible matrices* $a \in \mathscr{M}(A', B', A, B)$ *to pairs* (π, σ), *where* π *is* (A, B)*-partially strict tableau,* σ *is* (A', B')*-partially strict tableau and* π *and* σ *have the same shape, is a bijction.*

The following proposition is also easy to see from definitions.

Proposition 4.4

Let $a \in \mathscr{M}(A', B', A, B)$. If $p(a)$ correspond to a by the map in Definition 4.10, *then the inverse biword $p(a)^{-1}$ correspond to a^t. Here a^t denote the conjugate matix of a.*

From Proposition 4.4 we obtain the following theorem.

Theorem 4.2

Fix $\mathscr{A}$ and its divisions (U, C) and (A, B). Assume that (π, σ) correspond to a by the bijection in Definition 4.9, *where $a \in \mathscr{M}(A, B, A, B)$, and π and σ are (A, B)-partially strict tableau having the same shape. Then (σ, π) corresponds to a^t by the same bijection.*

Example 4.12

Let a be as Example 4.7. Then

$$a^t = \begin{pmatrix} \mathit{2} & 0 & \mathit{0} & 1 \\ \mathit{0} & 1 & \mathit{3} & 0 \\ 0 & \mathit{1} & 1 & \mathit{1} \\ 0 & \mathit{2} & 0 & \mathit{2} \end{pmatrix}$$

and

$$l(a) = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & 4 & 4 & 4 & 4 \\ \mathbf{1} & \mathbf{1} & {}^\circ 4 & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & {}^\circ\mathbf{3} & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ 4 & {}^\circ 4 & {}^\circ 4 & 2 & 2 \end{pmatrix}.$$

It's easy to make sure that

$$\pi = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & 4 & 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{2} & 4 \\ \cline{1-3} \mathbf{1} & {}^\circ 3 \\ \cline{1-2} {}^\circ\mathbf{2} & 4 \\ \cline{1-2} {}^\circ\mathbf{2} \\ \cline{1-1} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|c|} \hline \mathbf{1} & 2 & 2 & 2 & {}^\circ\mathbf{3} & {}^\circ 4 \\ \hline \mathbf{1} & {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-3} 2 & {}^\circ\mathbf{3} \\ \cline{1-2} {}^\circ\mathbf{3} & {}^\circ 4 \\ \cline{1-2} {}^\circ 4 \\ \cline{1-1} \end{array}$$

Definition 4.11

Fix $\mathscr{A}$ and its divisions (U, C) and (A, B). Let $a = (a_{ij})_{i,j \in \mathscr{A}} \in \mathscr{M}(A, B, A, B)$ be an admissible symmetric matrix. We define $\mathrm{tr}_{(A,B)}\, a$ by

$$\mathrm{tr}_{(A,B)}\, a = \sum_{i \in A} a_{ii} + \sum_{i \in B} \mathrm{odd}\,\{a_{ii}\}$$

where $\mathrm{odd}\,\{x\} = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}.$

Corollary 4.1

Fix $\mathscr{A}$ and its divisions (U, C) and (A, B). The map in Definition 4.9 *givs a bijection from admissible symmetric matrices $a \in \mathscr{M}(A, B, A, B)$ onto (A, B)-partially strict tableaux π. In this bijection we have*

$$\mathrm{tr}_{(A,B)}\, a = \mathrm{odd}\,(\lambda)$$

where λ is the shape of π and odd (λ) *stands for the number of odd length columns in λ.*

Example 4.13

Let $A = \{1, {}^\circ 3\}$ and $B = \{{}^\circ\mathbf{2}, \mathbf{4}\}$. Let a be an admissible symmetric matrix given by

$$a = \begin{pmatrix} \mathit{1} & 0 & \mathit{2} & 0 \\ 0 & \mathit{2} & 1 & \mathit{0} \\ \mathit{2} & 1 & \mathit{0} & 1 \\ 0 & \mathit{0} & 1 & \mathit{1} \end{pmatrix}.$$

Then

$$l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & \mathbf{4} & \mathbf{4} \\ {}^\circ 3 & {}^\circ 3 & 1 & {}^\circ 3 & {}^\circ\mathbf{2} & {}^\circ\mathbf{2} & \mathbf{4} & 1 & 1 & {}^\circ 2 & \mathbf{4} & {}^\circ 3 \end{pmatrix}.$$

and

$$\pi = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & {}^\circ\mathbf{2} & 4 \\ \hline {}^\circ\mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 \\ \cline{1-4} {}^\circ\mathbf{2} & \mathbf{4} \\ \cline{1-2} {}^\circ 3 \\ \cline{1-1} \end{array}$$

Corollary 4.2

Fix $\mathscr{A}$ and its division (A, B).

$$\sum_{\lambda} HS_{\lambda}^{(A,B)}(x)\, t^{\mathrm{odd}(\lambda)} = \prod_{\substack{(i,j)\in A\times A\cup B\times B \\ i<j}} \frac{1}{1 - x_i x_j} \prod_{(i,j)\in A\times B} (1 + x_i x_j) \prod_{i\in A} \frac{1}{1 - t x_i} \prod_{i\in B} \frac{1 + t x_i}{1 - x_i^2}$$

In particular,

$$\sum_{\lambda' \,\mathrm{even}} HS_{\lambda}^{(A,B)}(x) = \prod_{\substack{(i,j)\in A\times A\cup B\times B \\ i<j}} \frac{1}{1 - x_i x_j} \prod_{(i,j)\in A\times B} (1 + x_i x_j) \prod_{i\in B} \frac{1}{1 - x_i^2}$$

Now we investigate the skew case. Let $\mathrm{PST}_{(A,B)}(\lambda/\mu)$ denote the set of (A, B)-partially strict skew tableaux which have skew shape λ/μ.

Theorem 4.3

Fix $\mathscr{A}$ and its divisions (U, C) and (A, B). Fix another $\mathscr{A}'$ and its divisions (U', C') and (A', B'). Let α and β be fixed partitions. Then the map

$$(a, \tau, \kappa) \leftrightarrow (\pi, \sigma)$$

defined below is a bijection between admissible matrices $a \in \mathscr{M}(A, B, A', B')$ with $\tau \in \mathrm{PST}_{(A,B)}(\alpha/\mu)$ and $\kappa \in \mathrm{PST}_{(A',B')}(\beta/\mu)$, on the one hand, and $\pi \in \mathrm{PST}_{(A,B)}(\lambda/\beta)$ and $\sigma \in \mathrm{PST}_{(A',B')}(\lambda/\alpha)$, on the other, such that $\tau \bigcup \check{l}(a) = \pi$ and $\kappa \bigcup \hat{l}(a) = \sigma$.

Proof.

Let n be the largest letter of $\kappa \bigcup \hat{l}(a)$. We construct (π_r, σ_r), for $r = 0, 1, \ldots, n$, as follows. Start with $(\pi_0, \sigma_0) = (\tau, \emptyset_\alpha)$. Form π_r from π_{r-1} as follows.

Case 1: $r \in A'_u \cup B'_c$

At first we insert all the circled letters of $\check{l}(a)$ paired with r's in $\hat{l}(a)$, where these circled letters are arranged in decreasing order. Next we internally insert all the letters of π_{r-1} corresponding to r's in σ_{r-1}. If $r \in A'_u$, the insertion proceed left to right, and if $r \in B'_c$, the insertion proceed top to bottom. Finally we insert all the uncircled letters of $\check{l}(a)$ paired with r's in $\hat{l}(a)$, where these uncircled letters are arranged in increasing order.

Case 2: $r \in A'_c \cup B'_u$

At first we insert all the uncircled letters of $\check{l}(a)$ paired with r's in $\hat{l}(a)$, where these uncircled letters are arranged in decreasing order. Next we internally insert all the letters of π_{r-1} corresponding to r's in σ_{r-1}. If $r \in A'_c$, the insertion proceed left to right, and if $r \in B'_u$, the insertion proceed top to bottom. Finally we insert all the circled letters of $\check{l}(a)$ paired with r's in $\hat{l}(a)$, where these uncircled letters are arranged in increasing order.

In either case placing r's in the appropriate cells of σ_{r-1} result in σ_r. It is not hard to see that the cells where r's are placed are horizontal or vertical strip in σ_r. At last we put $(\pi_n, \sigma_n) = (\pi, \sigma)$. ■

Example 4.14

Let $A = \{1, {}^\circ 2\}$, $B = \{{}^\circ\mathbf{3}, \mathbf{4}\}$, $A' = \{1, {}^\circ 3\}$, $B = \{\mathbf{2}, {}^\circ\mathbf{4}\}$, $\alpha = (221)$ and $\beta = (43)$. Let $a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ so that the matrix word of a is $l(a) = \begin{pmatrix} 1 & \mathbf{2} & \mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ 3 & {}^\circ\mathbf{4} \\ {}^\circ 2 & \mathbf{4} & {}^\circ 2 & 1 & 1 & {}^\circ 2 & {}^\circ\mathbf{3} \end{pmatrix}$. Let

$$\tau = \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \mathbf{4} \\ \hline {}^\circ\mathbf{3} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \kappa = \begin{array}{|c|c|c|c|}\hline & 1 & \mathbf{2} & {}^\circ\mathbf{4} \\ \hline 1 & {}^\circ 3 & {}^\circ\mathbf{4} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & & & 1 & 1 & {}^\circ\mathbf{3} & {}^\circ\mathbf{4} \\ \hline & & & 1 & {}^\circ 2 & \multicolumn{3}{c}{} \\ \cline{1-5} 1 & {}^\circ 2 & \mathbf{4} & \multicolumn{5}{c}{} \\ \cline{1-3} {}^\circ 2 & {}^\circ\mathbf{3} & \multicolumn{6}{c}{} \\ \cline{1-2} \end{array} \qquad \sigma = \begin{array}{|c|c|c|c|c|c|c|c|}\hline & & 1 & 1 & \mathbf{2} & {}^\circ 3 & {}^\circ 3 & {}^\circ\mathbf{4} \\ \hline & & \mathbf{2} & {}^\circ 3 & {}^\circ 3 & \multicolumn{3}{c}{} \\ \cline{1-5} & 1 & {}^\circ\mathbf{4} & \multicolumn{5}{c}{} \\ \cline{1-3} \mathbf{2} & {}^\circ\mathbf{4} & \multicolumn{6}{c}{} \\ \cline{1-2} \end{array}$$

Corollary 4.3

Fix $\mathscr{A}$ and its division (A, B). Fix another $\mathscr{A}'$ and its division (A', B'). Let α and β be fixed partitions. Then

$$\sum_{\lambda} HS^{(A,B)}_{\lambda/\beta}(x)\, HS^{(A',B')}_{\lambda/\alpha}(y)$$

$$= \sum_{\mu} HS^{(A,B)}_{\alpha/\mu}(x)\, HS^{(A',B')}_{\beta/\mu}(y) \prod_{(i,j)\in A\times A' \cup B\times B'} \frac{1}{1 - x_i y_j} \prod_{(i,j)\in A\times B' \cup B\times A'} (1 + x_i y_j)$$

Theorem 4.4

Let $\mathscr{A} = \mathscr{A}'$, $(A, B) = (A', B')$, $(U, C) = (U', C')$ *and* $\alpha = \beta$ *in Theorem* 4.3. *If* (a, τ, κ) *correspond to* (π, σ) *by the bijection in Theorem* 4.3 *then* (a^t, κ, τ) *correspond to* (σ, π) *by the same bijection.*

Theorem 4.5

Fix $\mathscr{A}$ *and its divisions* (U, C) *and* (A, B). *Let* α *be a fixed partition. Then the mapping in Theorem* 4.3 *restricts to a bijection*

$$(a, \tau) \leftrightarrow \pi$$

where $a \in \mathscr{M}(A, B, A, B)$ *is a symmetric matrix,* $\tau \in \mathrm{PST}_{(A,B)}(\alpha/\mu)$, $\pi \in \mathrm{PST}_{(A,B)}(\lambda/\mu)$, *and* $\check{l}(a) \bigcup \tau = \pi$. *In this bijection we always have*

$$\mathrm{tr}_{(A,B)}\, a + \mathrm{odd}\,(\mu) = \mathrm{odd}\,(\lambda)$$

Example 4.15

Let $A = \{1, {}^\circ 2\}$, $B = \{{}^\circ\mathbf{3}, \mathbf{4}\}$ and $\alpha = (221)$. Let $a = \begin{pmatrix} \mathit{0} & 1 & \mathit{1} & 1 \\ 1 & \mathit{2} & 0 & \mathit{0} \\ \mathit{1} & 0 & \mathit{0} & 0 \\ 1 & \mathit{0} & 0 & \mathit{0} \end{pmatrix}$ so that the matrix word of a is $l(a) = \begin{pmatrix} 1 & 1 & 1 & {}^\circ 2 & {}^\circ 2 & {}^\circ 2 & {}^\circ\mathbf{3} & \mathbf{4} \\ {}^\circ\mathbf{3} & {}^\circ 2 & \mathbf{4} & {}^\circ 2 & {}^\circ 2 & 1 & 1 & 1 \end{pmatrix}$. Let

$$\tau = \begin{array}{|c|c|} \hline & \\ \hline & \mathbf{4} \\ \hline {}^\circ\mathbf{3} \\ \cline{1-1} \end{array}$$

Then we have

$$\pi = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline & & {}^\circ 2 & {}^\circ\mathbf{3} \\ \hline & 1 & {}^\circ 2 & \mathbf{4} \\ \hline {}^\circ 2 & {}^\circ\mathbf{3} \\ \cline{1-2} \end{array}$$

Corollary 4.4

Fix $\mathscr{A}$ *and its division* (A, B). *Let* α *be a fixed partition.*

$$\sum_{\lambda} HS^{(A,B)}_{\lambda/\alpha}(x)\, t^{\mathrm{odd}(\lambda)}$$

$$= \sum_{\mu} HS^{(A,B)}_{\alpha/\mu}(x)\, t^{\mathrm{odd}(\mu)} \prod_{\substack{(i,j) \in A \times A \cup B \times B \\ i<j}} \frac{1}{1 - x_i x_j} \prod_{(i,j) \in A \times B} (1 + x_i x_j) \prod_{i \in A} \frac{1}{1 - t x_i} \prod_{i \in B} \frac{1 + t x_i}{1 - x_i^2}$$

In particular,

$$\sum_{\lambda' \text{ even}} HS^{(A,B)}_{\lambda/\alpha}(x) = \sum_{\mu' \text{ even}} HS^{(A,B)}_{\alpha/\mu}(x) \prod_{\substack{(i,j)\in A\times A\cup B\times B \\ i<j}} \frac{1}{1-x_ix_j} \prod_{(i,j)\in A\times B} (1+x_ix_j) \prod_{i\in B} \frac{1}{1-x_i^2}$$

Acknowledgement

The author wish to thank Prof. Okamoto, Kimura, Koike, Terada, and Roby.

Reference

[BR] A. Berele and A. Regev, Hook Young Diagrams with Applications to Combinatorics and to Representations of Lie Superalgabras, *Adv. Math.* **64**, (1987) 118–175

[Fo1] Sergey V. Fomin, Finite partially ordered sets and Young tableaux, *Soviet Math. Dokl.* **19**, (1978) 1510–1514

[Fo2] Sergey V. Fomin, Generalized Robinson-Sdhensted-Knuth Correspondence, *preprint*

[Gr] Curtis Greene, An Extension of Schensted's Theorem, *Adv. in Math.* **14**, (1974) 254–265

[GK] Curtis Greene and Daniel J. Kleitman, The Structure of Sperner k-Families, *J. Combi. Theory Ser. A* **20** (1976), 41–68

[Ha] Mark D. Haiman, On Mixed Insertion, Symmetry, and Shifted Young Tableux, *J. Combi. Theory Ser. A* **50** (1989), 196–225

[Mc] I. G. Macdonald, *Symmetric Functions and Hall Polynomials Clarendon Press,* Oxford 1979

[Ok] Soichi Okada, Partially Strict Shifted Plane Partitions, *J. Combi. Theory Ser. A* **53** (1990), 143–156

[Re] Jeffrey B. Remmel, The Combinatorics of (k, l)-Hook Schur Functions *Contemporary Math. Vol,* **34**, (1984), 253–287

[Ro] Thomas W. Roby, Applications and Extensions of Fomin's Generalization of the Robinson-Shensted Correspondence to Differential posets, *Ph. D. thesis, MIT,* 1991

[Sa] Bruce E. Sagan, *The Symmetric Group Representations, Combinatorial Algorithms, and Symmetric Functions,* Wadsworth & Brooks/Cole

[SS] Bruce E. Sagan and Richard P. Stanley, Robinson-Schensted Algorithm for Skew Tableax, *J. Combi. Theory Ser. A* **55** (1990), 161–193

[Sc] M. P. Schützenberger, La correspondence de Robinson, in *Combinatoire et Représentation du Groupe Symetrique,* D. Foata ed., Lecture Notes in Math., Vol. **579**, Springer-Verlag, New York, NY/(1977), 59–135

[To] G. P. Thomas, On A Construction of Schüzenberger, *Descrete Math.* **17** (1977), 107–118