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Remarks on Totally Symmetric Self-Complementary Plane Partitions

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1. Introduction

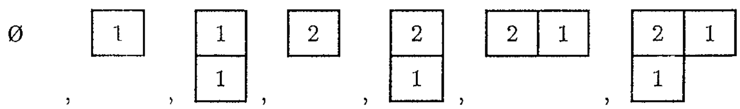
In [Sta3], R. P. Stanley classified the problem of enumerating plane partitions under various symmetries into ten cases. In this paper we consider a refinement of one of those cases, i.e. the case of totally symmetric self-complementary plane partitions (abbreviatedd as TSSCPP's). To enumerate TSSCPP's we introduce a set $\mathscr{C}_n$ of plane partitions which will be shown to have the same cardinality with the set of TSSCPP's. We mainly study this set $\mathscr{C}_n$ and obtain a certain generating function by means of a pfaffian.

For a positive integer n, we set

$$A_n := \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{1.2}$$

We introduce the set $\mathscr{C}_n$ of row-strict plane partititions whose entries in the i-th row do not exceed $n-i$. Then we will obtain a bijection between $\mathscr{C}_n$ and the set of totally symmetric self-complementary plane partitions in Section 3.

Example 1.1. $\mathscr{C}_3$ is composed of the following seven elements.



Theorem 1.1. *There is an explicit bijection (see Corollay 3.1) between the set of the totally symmetric $(2n, 2n, 2n)$-self-complementary plane partitions and $\mathscr{C}_n$.*

This $\mathscr{C}_n$ is the main object which we study in this paper. To study $\mathscr{C}_n$ we redefine the function U_k which is introduced in [MRR3] as the function of $\mathscr{C}_n$ into $\mathbb{N}$ by using the bijection between TSSCPP and $\mathscr{C}_n$ established in Theorem 1.4. This function U_k is expected to have the same distribution as that of 1's in the top row of alternating sign matrices (see [MRR3]). Namely we can expect that the number

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of elements γ in $\mathscr{C}_n$ such that $U_k(\gamma)=r$ would be equal to the number of alternating sign matrices which have 1 in the $(r+1)$-st position of the first row. If we put $\bar{U}_k(\gamma)=n-1-U_k(\gamma)$ for each $1\le k\le n$ and $\gamma\in\mathscr{C}_n$, then $\bar{U}_k(\gamma)$ has a simple interpretation as follows.

Let γ be an element of $\mathscr{C}_n$. If a non-zero entry in the i-th row of γ equals $n-i$, we call this entry a maximal part of γ. Maximal parts can occur only in the first column because γ is row-strict. We will prove that the $\bar{U}_k(\gamma)$ is the sum of the number of parts which are equal to k and the number of maximal parts which are less than k. Namely the following proposition will be proved in Sectin 4.

Proposition 1.1. *For each $\gamma\in\mathscr{C}_n$ and $k\in[1,n]$, we have*

$$\bar{U}_k(\gamma)=\#\{(i,j)\in[1,n]^2\,|\,\gamma_{ij}=k\}+\#\{1\le i\le k-1\,|\,\gamma_{n-i,1}=i\}.$$

From Conjecture 3 in [Sta3] we can expect the following.

Conjecture 1.1 [MRR3]. *For $0\le r\le n-1$ and $1\le k\le n$, we set*

$$\mathscr{C}_{n,r}^k=\{\gamma\in\mathscr{C}_n\,|\,\bar{U}_k(\gamma)=r\}.$$

Then the cardinality of $\mathscr{C}_{n,r}^k$ would be given by

$$\#\mathscr{C}_{n,r}^k=\frac{\binom{n+r-1}{n-1}\binom{2n-r-2}{n-1}A_{n-1}}{\binom{2n-2}{n-1}}. \tag{1.3}$$

Definition 1.1. Let A be an n by m rectangular matrix $(1\le n\le m)$:

$$A:=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m}\\ a_{21} & a_{22} & \cdots & a_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nm}\end{pmatrix}.$$

Then we put

$$\mathrm{d}_n(A):=\sum_{1\le j_1<j_2<\dots<j_n\le m}\begin{vmatrix} a_{1j_1} & a_{1j_2} & \cdots & a_{1j_n}\\ a_{2j_1} & a_{2j_2} & \cdots & a_{2j_n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{nj_1} & a_{nj_2} & \cdots & a_{nj_n}\end{vmatrix}, \tag{1.4}$$

i.e. $\mathrm{d}_n(A)$ is the sum of all $n\times n$ minors of A.

Let us give a formula which expresses the multivariable generating function of $\mathscr{C}_n$ weighted by $\bar{U}_k$ and in terms of d_n.

Theorem 1.2. *Let n and k be positive integer such that $1\le k\le n$. We define the n by $(2n-1)$ rectangular matrix $P_n(t,x)$ by*

$$P_n(t, x) := (p_{i,j}^{(n)}(t, x))_{\substack{i=0,\ldots,n-1 \\ j=0,\ldots,2n-2}}$$

where

$$p_{ij}^{(n)}(t, x) = e_{j-i}^{(i)}(t_1x_1, t_2x_2, \cdots, t_{i-1}x_{i-1}, (\prod_{\nu=i}^{n} t_\nu)x_i). \tag{1.5}$$

Then

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^\gamma = \mathrm{d}_n(P_n(t, x)). \tag{1.6}$$

Here $e_r^{(n)}(x_1, x_2, \cdots, x_n)$ is the elementary symmetric function of degree r with n variables $x_1, x_2, \cdots, x_n$ and $t^{\bar{U}(\gamma)}$ stands for $\prod_{k=1}^{n} t_k^{\bar{U}_k(\gamma)}$. If γ contains m_1 1's, m_2 2's, m_3 3's, $\cdots$, then we write x^γ for $x_1^{m_1}x_2^{m_2}x_3^{m_3}\cdots$.

By using a formula given in [Ok], we express the sum of minors $\mathrm{d}_n(P_n)$ by a pfaffian.

Theorem 1.3. *Let n be a positive integer.*

(1) *If n even then we have*

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^\gamma = \mathrm{Pf}_n(f_n(i, j; t, x))_{0 \le i,j \le n-1}.$$

(2) *If n is odd then we have*

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^\gamma = \mathrm{Pf}_{n-1}(f_n(i, j; t, x))_{1 \le i,j \le n-1}.$$

In the above expressions the entry $f_n(i, j; t, x)$ is given as follows.

$$\begin{aligned} &f_n(i, j; t, x) \\ &= \prod_{\nu=1}^{n} t_\nu \prod_{\nu=1}^{i} x_\nu \sum_{r \ge 2i-j} e_r^{(i+j)}(t_1^{-1}x_1^{-1}, \cdots, t_{i-1}^{-1}x_{i-1}^{-1}, (\prod_{\nu=i}^{n} t_\nu^{-1})x_i^{-1}, t_1x_1, \cdots, t_{j-1}x_{j-1}, (\prod_{\nu=j}^{n} t_\nu)x_j) \\ &\quad - \prod_{\nu=1}^{n} t_\nu \prod_{\nu=1}^{j} x_\nu \sum_{r \ge 2j-i} e_r^{(i+j)}(t_1x_1, \cdots, t_{i-1}x_{i-1}, (\prod_{\nu=i}^{n} t_\nu)x_i, t_1^{-1}x_1^{-1}, \cdots, t_{j-1}^{-1}x_{j-1}^{-1}, (\prod_{\nu=j}^{n} t_\nu^{-1})x_j^{-1}). \end{aligned} \tag{1.7}$$

Here Pf_n stands for the pfaffian of an $n \times n$ skew-symmetric matrix.

Now we consider a special case: i.e. we substitute $x_i = 1$ for $1 \le i \le n$ and $t_i = 1$ for $1 \le 1 \le n$ except for a fixed k. In this case we can simplify the entries of the pfaffan and obtain the following corollary.

Corollary 1.1.

(1) *If n is even then we have*

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}_k(\gamma)} = (1 + t)\,\mathrm{Pf}_n(h_{i,j}(t))_{2 \le i,j \le n-1}.$$

(2) *If is odd then we have*

$$\sum_{\gamma\in\mathscr{C}_n} t^{\bar{U}_k(\gamma)} = \mathrm{Pf}_{n-1}(h_{i,j}(t))_{1\le i,j\le n-1}$$

Here the entry $h_{ij}(t)$ *is given as follows: For* $i=j=1$ *we set* $h_{ij}(t)=0$ *and for* (i,j) *such that* $i+j\ge 3$ *we set*

$$\begin{aligned} h_{ij}(t) &= \frac{3(j-i)(3i-2)(3j-2)}{(i+j)(i+j-1)(i+j-2)}\binom{i+j}{2i-j}(1+t^2) \\ &+ \frac{3(j-i)\{9(5i^3j-8i^2j^2+5ij^3)-21(x^3+y^3)+4(5i^2-8ij+5j^2)+7(i+j)-6\}}{(i+j)(i+j-1)(i+j-2)(2i-j+1)(2j-i+1)} \\ &\times \binom{i+j}{2i-j}t. \end{aligned} \tag{1.8}$$

If we put $t=1$ in this formula and compute the entries of the pfaffian, then we obtain the following corollary.

Corollary 1.2.

(1) *If* n *is even,*

$$\#\mathscr{C}_n = 2\,\mathrm{Pf}_n(b_{i,j})_{2\le i,j\le n-1}.$$

(2) *If* n *is odd,*

$$\#\mathscr{C}_n = \mathrm{Pf}_{n-1}(b_{i,j})_{1\le i,j\le n-1}.$$

Here the entry $b_{i,j}$ *is given by*

$$b_{i,j} = \frac{3(j-i)(3i+1)(3j+1)}{(i+j)(2i-j+1)(2j-i+1)}\binom{i+j}{2i-j} \tag{1.9}$$

Note that this formula is equivalent to the formula obtained in [Ste].

This paper is organized as follows. In Section 2 we give elementary definitions on plane partitions. So the reader who knows these objects may skip this section and only refer to it afterwards when necessary. In Section 3 we prove Theorem 1.4 by constructing a bijection concretely. In Section 4 we redefine the function U_k, investigate the relation between $\mathscr{C}_n$ and the shifted plane partition defined in [MRR3] and prove Proposition 1.1. In Section 5 we employ the lattice path method and give the generating functions of $\mathscr{C}_n$ in the form of the sum of minors. Theorem 1.2 and its corollaries will also be proved in the section. In Section 6 we transform a sum of minors into a pfaffian and simplify the entries of the pfaffian as far as possible. Theorem 1.3 and its corollaries will also be obtained.

Recently J. R. Stembridge [Ste] obtained a formula on the number of totally symmetric self-comlementary plane partitions and G. E. Andrews evaluated the pfaffian and proved the conjecture on the number of totally symmetric self-complementary plane partitions in [Sta2]. But this work is independent of thier results and most parts of this paper are written as my master thesis in January of 1988. It was delayed to translate them into English and publish them, and it was the auther's fault. The auther expresses special thanks to H. Kimura and I. Terada for checking English.

§2 Preparation

In this section we introduce some notation and terminology concerning plane partitions. Some terminology describing various symmetries of plane partitions will be defined at the end of this section.

We use the following notation. We denote the set of positive integers by **P**, the set of nonnegative integers by **N** and the set of integers by **Z**. We write

$$[i, j] := \{x \in \mathbf{Z} \mid i \leq x \leq j\}.$$

We denote by $\# X$ or Card X the number of elements of the finite set X, and by $\binom{n}{r}$ the binomial coefficient.

The *Gaussian binomial coefficient* is by definition

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{[n]_q!}{[r]_q!\,[n-r]_q!} \tag{2.1}$$

where $[i]_q! = [i]_q[i-1]_q \cdots [1]_q$ and $[i]_q = \dfrac{q^i - 1}{q - 1}$. (We agree to define also that $\begin{bmatrix} n \\ r \end{bmatrix}_q$ is zero if $r < 0$ or $r > n$.)

We use the notation in the book [Mc] for partitions. For example a *partition* is a weakly decreasing sequence of nonnegative integers $\lambda := (\lambda_1, \lambda_2, \lambda_3, \cdots)$ with finitely many nonzero entries. We denote the *length* of a partition by $l(\lambda)$ and the *conjugate partition* of λ by λ'. The *Young diagram* (or *Ferrers graph*) of the partition λ is a subset of $\mathbf{P}^2$ defined by

$$D(\lambda) := \{(i, j) \in \mathbf{P}^2 \mid j \leq \lambda_i\} \tag{2.2}$$

From now on we identify a partition λ with its Young diagram and denote $D(\lambda)$ simply by λ.

Next we define the plane partitions in accordance with [Mc] and [St1].

Definition 2.2. Let λ be a partition. *A plane partition* $\pi := (\pi_{ij})_{(i,j)\in\lambda}$ is a filling of the Young diagram λ which satisfies the following conditions.

(*i*) $\pi_{ij} \in \mathbf{Z}$ if $(i, j) \in \lambda$,

(*ii*) $\pi_{ij} \geq \pi_{i+1\,j}$ if $(i, j), (i+1, j) \in \lambda$, (*iii*) $$\pi_{ij} \geq \pi_{i\,j+1} \quad \text{if } (i, j), (i, j+1) \in \lambda.$$

The entries π_{ij} ($(i, j) \in \lambda$) are called the *parts* of π, and $|\pi| := \sum_{(i,j)\in\lambda} \pi_{ij}$ the *weight* of π. In particular we denote by $\emptyset$ the plane partition which has no entry (i.e. $\lambda = \emptyset$).

In most parts of this paper we only consider the plane partitions whose parts are positive integers. So from now on we assume that the plane partitions have only positive parts and we regard that points outside λ are filled with zero unless otherwise mentioned. The subset of $\mathbf{P}^3$ defined by

$$F(\pi) := \{(i, j, k) \in \mathbf{P}^3 \mid (i, j) \in \lambda,\ k \leq \pi_{ij}\} \tag{2.3}$$

is called the *Ferrers graph* of π. A Ferrers graph F of a plane partition always satisfies the following conditions:

(*i*) If $(x', y', z') \in F$, then any $(x, y, z) \in \mathbf{P}^3$ satisfying $x \leq x'$, $y \leq y'$ and $z \leq z'$ belongs to F,

(*ii*) Card $F < \infty$.

Note that any subset of $\mathbf{P}^3$ which satisfies the above conditions defines a Ferrers graph of a plane partition. So we will henceforce identify a plane partition with its Ferrers graph and denote it by the same symbol. Now let π be a plane partition. The partition defined by

$$bs(\pi) := \{(x, y) \in \mathbf{P}^2 \mid (x, y, 1) \in \pi\}$$

is usually called the *shape* of π, but in this paper we sometimes call it the *bottom shape* of π in order to distinguish it from the side shape defined below. The partition defined by

$$ss(\pi) := \{(x, z) \in \mathbf{P}^2 \mid (x, 1, z) \in \pi\}$$

is by definition the *side shape* of π.

Next we recall some difinitions concerning the symmetries of plane partitions which will be needed later. (For the details see [St3].)

Definition 2.3. Let $\pi := (\pi_{ij})_{(i,j)\in\mathbf{P}^2}$ be a plane partition.

1) We say π is *column-strict* if $\pi_{ij} > \pi_{i+1\,j}$ for all $(i, j) \in \mathbf{P}^2$ such that $\pi_{ij} \neq 0$.

2) We say π is *row-strict* if $\pi_{ij} > \pi_{ij+1}$ for all $(i, j) \in \mathbf{P}^2$ such that $\pi_{ij} \neq 0$.

3) We say π is *symmetric if* π satisfies $\forall (x, y, z) \in \mathbf{P}^3 : (x, y, z) \in \pi \Leftrightarrow (y, x, z). \in \pi$.

4) We say π is *cyclically symmtric* if π satisfies $\forall (x, y, z) \in \mathbf{P}^3 : (x, y, z) \in \pi \Leftrightarrow (z, x, y) \in \pi$.

5) We say π is *totally symmtric*

if π is both symmetric and cyclically symmetric.

6) For $l, m, n \in \mathbf{P}$ set $X_{lmn} := [1, l] \times [1, m] \times [1, n]$.
We say π is *(l, m, n)-self-complementary*
if π satisfies the following condition:
π is contained in X_{lmn}, and
for all $(x, y, z) \in X_{lmn}$, $(x, y, z) \in \pi$ and only if
$(l+1-x, m+1-y, n+1-z) \notin \pi$.

The condition of π being a (l, m, n)-seld-complementary is equivalent to saying that π is contained in the box X_{lmn} and π and $X_{lmn} - \pi$ are symmetrical to each other with respect to the center $\left(\frac{l}{2}, \frac{m}{2}, \frac{n}{2}\right)$ of the box X_{lmn}.

§3 Certain Classes of Plane Partitions and Bijections Among Them

The aim of this section is to construct a bijection between the set of totally symmetric self-complementary plane partitions and $\mathscr{C}_n$.

Let $n \in \mathbf{P}$. We denote by $\mathscr{R}_n$ the set of all cyclically symmetric $(2n, 2n, 2n)$-self-complementary plane partitions. We denote by $\mathscr{S}_n$ the set of all totally symmetric $(2n, 2n, 2n)$-self-complementary plane partitions.

Definition 3.4. Let $\mathscr{E}_n$ be the set of plane partitions ε which satisfy:
(i) $\varepsilon \subset [1, n]^3$,
(ii) for all $(x, y) \in \mathbf{P}^2$ we have

$$(x, y, 1) \in \varepsilon \Rightarrow (n+1-y, 1, n+1-x) \notin \varepsilon. \tag{3.1}$$

Example 3.2.

$$\mathscr{E}_1 = \{\emptyset\},$$
$$\mathscr{E}_2 = \{\emptyset, 1, 2, 1\ \ 1\}$$

Theorem 3.4. *Let Φ: $\mathscr{R}_n \to \mathscr{E}_n$ be the map which associates $\rho \in \mathscr{R}_n$ with $\varepsilon \in \mathscr{E}_n$ defined by the condition:*

$$\forall (x, y, z) \in \mathbf{P}^3: (x, y, z) \in \varepsilon \Leftrightarrow (x, y+n, z+n) \in \rho. \tag{3.2}$$

Then Φ is a bijection from $\mathscr{R}_n$ to $\mathscr{E}_n$.

Proof. In order to show that Φ is well defined, we have to verify that $\varepsilon \in \mathscr{E}_n$. But it is clear that ε defined above is actually a plane partition and is contained in $[1, n]^3$, so we have only to show that

$$\forall (x, y) \in \mathbf{P}^2: (x, y, 1) \in \varepsilon \Leftrightarrow (n+1-y, 1, n+1-x) \notin \varepsilon. \tag{3.3}$$

Using the properties of ρ and the definition of ε, this immediately follows by an easy calculation. Next we construct the inverse map Ψ: $\mathscr{E}_n \to \mathscr{R}_n$. We define the domains

D_{pqr} (where $p, q, r \in \{1, 2\}$) as follows:

$$D_{pqr} := [(p-1)n+1, pn] \times [(q-1)n+1, qn] \times [(r-1)n+1, rn] \tag{3.4}$$

Given $\varepsilon \in \mathscr{E}_n$, we define $\rho \in \mathscr{R}_n$ by specifying the points of ρ in each domain D_{pqr} as follows:

(1) $\forall (x, y+n, z+n) \in D_{122} : (x, y+n, z+n) \in \rho \Leftrightarrow (x, y, z) \in \varepsilon$,

(2) $\forall (x+n, y, z+n) \in D_{212} : (x+n, y, z+n) \in \rho \Leftrightarrow (y, z, x) \in \varepsilon$,

(3) $\forall (x+n, y+n, z) \in D_{221} : (x+n, y+n, z) \in \rho \Leftrightarrow (z, x, y) \in \varepsilon$,

(4) $\forall (x+n, y, z) \in D_{211} : (x+n, y, z) \in \rho \Leftrightarrow (n+1-x, n+1-y, n+1-z) \notin \varepsilon$,

(5) $\forall (x, y+n, z) \in D_{121} : (x, y+n, z) \in \rho \Leftrightarrow (n+1-y, n+1-z, n+1-x) \notin \varepsilon$,

(6) $\forall (x, y, z+n) \in D_{112} : (x, y, z+n) \in \rho \Leftrightarrow (n+1-z, n+1-x, n+1-y) \notin \varepsilon$,

(7) $\forall (x, y, z) \in D_{111} : (x, y, z) \in \rho$,

(8) $\forall (x+n, y+n, z+n) \in D_{222} : (x+n, y+n, z+n) \notin \rho$,

(where x, y and z belong to $[1, n]$ in the above notation.)

First we have to show that $\rho \in \mathscr{R}_n$. It is enough to show that ρ becomes a plane partition since it is clear from the definition of ρ that ρ is cyclically symmetric and (2n, 2n, 2n)-self-complementary. Suppose that $P' = (x', y', z')$ belongs to ρ and let $P = (x, y, z) \in [1, 2n]^3$ be a point satisfying $x \leq x'$, $y \leq y'$, $z \leq z'$. Then we have to show that $P \in \rho$. This is clear if P and P' lie in the same domain. Further in the case that $P \in D_{111}$ or $P' \in D_{222}$, this is trivial. So the remaining cases to be considered are as follows.

$$\begin{aligned} &P \in D_{pqr} : \text{(where one of } p, q, r \text{ is 2 and the rest are 1)} \\ &P' \in D_{stu} : \text{(where one of } s, t, u \text{ is 1 and the rest are 2)} \\ &p \leq s,\ q \leq t,\ r \leq u \end{aligned} \tag{3.5}$$

For example we consider the case where

$$\begin{aligned} &P = (x, y, z+n) \in D_{112},\ P' = (x', y'+n, z'+n) \in D_{122} : \\ &x \leq x', \text{ and } z \leq z'. \end{aligned} \tag{3.6}$$

Since the claim was shown in the case where two points lie in the same domain, we can assume that $x = x'$ and $z = z'$. Then an easy calculation as follows leads to the coclusion.

$$\begin{aligned} &P' = (x, y'+n, z+n) \in \rho \Leftrightarrow (x, y', z) \in \varepsilon \\ &\Rightarrow (x, 1, z) \in \varepsilon \Rightarrow (n+1-z, n+1-x, 1) \notin \varepsilon \\ &\Rightarrow (n+1-z, n+1-x, n+1-y) \notin \varepsilon \Leftrightarrow P = (x, y, z+n) \notin \rho \end{aligned}$$

So in this case the claim was shown. We can prove the other cases similarly, and this shows that ρ is a plane partition. The remaining task is to show that Φ and

Ψ are the inverse mappings of each other. It is clear from the definition that $\Phi \circ \Psi = id_{\mathscr{E}_n}$, so we have to show that Φ is injective. This is equivalent to show that, for any $\varepsilon \in \mathscr{E}_n$, $\rho = \Psi(\varepsilon) \in \mathscr{R}_n$ is the only one which satisfies $\Phi(\rho) = \varepsilon$. But this is an easy consequence of the fact that ρ is cyclically symmetric and (2n, 2n, 2n)-self-complementary. □

Definition 3.5. For $n \in \mathbf{P}$, we put

$$\mathscr{C}_n : \{\gamma \mid \gamma \text{ is a row-strict plane partition satisfying } (x, y, z) \in \gamma \Rightarrow x + z \leq n\}, \tag{3.7}$$

$$\begin{aligned} \mathscr{D}_n := \{\delta \mid \delta &\text{ is a plane partition satisfying the conditions:} \\ &\forall (x, y, z) \in \mathbf{P}^3 : (x, y, z) \in \delta \Leftrightarrow (x, z, y) \in \delta \\ &\forall (x, y, z) \in \mathbf{P}^3 : (x, y, z) \in \delta \Leftrightarrow x + y \leq n\}. \end{aligned} \tag{3.8}$$

Example 3.3. $\mathscr{D}_3$ is composed of the following seven elements.

$$\emptyset, \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \quad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c|}{1} \\ \hline 1 & \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}, \quad \begin{array}{|c|c}\hline 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & \\ \cline{1-1}\end{array}$$

Proposition 3.2. *Let $\Phi_2 : \mathscr{D}_n \to \mathscr{C}_n$ be the map which associates $\delta \in \mathscr{D}_n$ with $\gamma \in \mathscr{C}_n$ defined by*

$$\forall (x, y, z) \in \mathbf{P}^3 : (x, y, z) \in \gamma \Leftrightarrow (x, y, y + z - 1) \in \delta. \tag{3.9}$$

Then Φ_2 is a bijection.

Proof. The plane partitions in $\mathscr{D}_n$ are symmetric in the directions of y and z (i.e. $(x, y, z) \in \delta$ if and only if $(x, z, y) \in \delta$.), whereas each row of the plane partitions in $\mathscr{C}_n$ is a strict partition. There is a well known one to one correspondence between symmetric partitions and strict partitions so that it is easy to verify the above construction gives a one to one correspondence between $\mathscr{C}_n$ and $\mathscr{D}_n$. □

Theorem 3.5. *The restriction of the mapping $\Phi : \mathscr{R}_n \to \mathscr{E}_n$ in Theorem 3.1 to $\mathscr{S}_n$ gives one to one correspondence between $\mathscr{S}_n$ and $\mathscr{D}_n$.*

Proof. Let $\rho \in \mathscr{R}_n$, $\varepsilon \in \mathscr{E}_n$ and $\varepsilon = \Phi(\rho)$. Then

$$\begin{aligned} &\rho \text{ is symmetric} \\ \Leftrightarrow &[(x, y + n, z + n) \in \rho \Leftrightarrow (x, z + n, y + n) \in \rho] \\ \Leftrightarrow &[(x, y, z) \in \varepsilon \Leftrightarrow (x, z, y) \in \varepsilon]. \end{aligned}$$

Finally we have to show that, under this condition, the condition

$$(i) \qquad (x, y, 1) \in \varepsilon \Rightarrow (n + 1 - y, 1, n + 1 - x) \notin \varepsilon$$

is equivalent to the condition

(*ii*) $$(x, y, z) \in \varepsilon \Rightarrow x + y \leq n.$$

This is an easy calculation. □

Corollary 3.3. *The mapping* $\Phi_2 \circ \Phi|_{\mathscr{S}_n}$ *is a bijection from* $\mathscr{S}_n$ *to* $\mathscr{C}_n$. *To sum up, this bijetion of* $\mathscr{S}_n$ *onto* $\mathscr{C}_n$ *is given by the following rule. Given* $\delta \in \mathscr{S}_n$, *we construct* $\gamma \in \mathscr{C}_n$ *as follows*:

$$\forall (x, y, z) \in \mathbf{P}^3 : (x, y, z) \in \gamma \Leftrightarrow (x, y + n, y + z + n - 1) \in \sigma \tag{3.10}$$

§4 Weighting the elements of $\mathscr{C}_n$

In this section we investigate the set $\mathscr{C}_n$ First we introduce the terminology defined in [MRR3], then investigate the relation between $\mathscr{C}_n$ and $\mathscr{B}_n$ and redifine the function U_k. We will prove Proposition 4.1 and introduce two conjectures from [MRR3] at the end of this section.

A strict partition is by definition a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$. With a strict partition λ we can associate a shifted diagram $SD(\lambda)$ which is defined by

$$SD(\lambda) := \{(i, j) \in \mathbf{P}^2 \mid 1 \leq i \leq j \leq \lambda_i\}, \tag{4.1}$$

Definition 4.6. Let λ be a strict partition. *A shifted plane partition* π of the shape $SD(\lambda)$ is a triangular array

$$\pi := (\pi_{ij})_{(i,j) \in SD(\lambda)}$$

which satisfies the following conditions

(*i*)	$\pi_{ij} \in \mathbf{Z}$	if $(i, j) \in SD(\lambda)$
(*ii*)	$\pi_{ij} \geq \pi_{i+1,j}$	if $(i, j), (i + 1, j) \in SD(\lambda)$
(*iii*)	$\pi_{ij} \geq \pi_{i,j+1}$	if $(i, j), (i, j + 1) \in SD(\lambda)$

From now on we only consider the shifted plane partitions whose parts are positive integers and regard that all parts outside the shape $SD(\lambda)$ are filled with zero. We can define a Ferrers graph $F(\pi)$ of a shifted plane partition π in the same way as that of an ordinary plane partition:

$$F(\pi) := \{(i, j, k) \in \mathbf{P}^3 \mid i \leq j \text{ and } k \leq \pi_{ij}\} \tag{4.2}$$

Definition 4.7. We denote the strict partition $(n - 1, n - 2, \cdots, 1)$ by $\lambda^{(n)}$. Let $\mathscr{B}_n$ be the set of shifted plane partitions

$$\beta := (\beta_{ij})_{(i,j) \in SD(\lambda^{(n)})}$$

which satisfy

$$n - i \leq \beta_{ij} \leq n \qquad \text{for all } (i, j) \in SD(\lambda^{(n)}). \tag{4.3}$$

Example 4.4. $\mathscr{B}_3$ has the following seven elements.

$$\begin{array}{|c|c|}\hline 3&3\\ \hline \multicolumn{1}{c|}{}&3\\ \cline{2-2}\end{array},\quad \begin{array}{|c|c|}\hline 3&3\\ \hline \multicolumn{1}{c|}{}&2\\ \cline{2-2}\end{array},\quad \begin{array}{|c|c|}\hline 3&3\\ \hline \multicolumn{1}{c|}{}&1\\ \cline{2-2}\end{array},\quad \begin{array}{|c|c|}\hline 3&2\\ \hline \multicolumn{1}{c|}{}&2\\ \cline{2-2}\end{array},\quad \begin{array}{|c|c|}\hline 3&2\\ \hline \multicolumn{1}{c|}{}&1\\ \cline{2-2}\end{array},\quad \begin{array}{|c|c|}\hline 2&2\\ \hline \multicolumn{1}{c|}{}&2\\ \cline{2-2}\end{array},\quad \begin{array}{|c|c|}\hline 2&2\\ \hline \multicolumn{1}{c|}{}&1\\ \cline{2-2}\end{array}$$

In [MRR3] the set $\mathscr{B}_n$ was defined, and the following theorem was proved.

Theorem 4.6. *For $n \in \mathbf{P}$, the mapping $\mathscr{S}_n \to \mathscr{B}_n$, $\sigma = (\sigma_{ij})_{(i,j)\in\mathbf{P}^2} \mapsto (\beta_{ij})_{(i,j)\in SD(\lambda^{(n)})}$ defined by*

$$\beta_{ij} = \sigma_{i+1,j+1} - n \qquad (1 \le i \le j \le n-1) \tag{4.4}$$

is a bijection.

(see [MRR3], p. 280 Theorem 1.)

Interpreting in terms of the Ferrers graph, we can say that the mapping $\mathscr{S}_n \to \mathscr{B}_n$ takes $\sigma \in \mathscr{S}_n$ to $\beta \in \mathscr{B}_n$ defined by

$$\begin{aligned} &\forall (x, y, z) \in \mathbf{P}^3 \quad \text{such that} \quad 1 \le x \le y \le n-1: \\ &\qquad (x, y, z) \in \beta \Leftrightarrow (x+1, y+1, z+n) \in \sigma \end{aligned} \tag{4.5}$$

Combining Theorem 4.4 and Corollary 3.12 of Section 3, we obtain the following corollary.

Corollary 4.4. *The map $\mathscr{B}_n \to \mathscr{C}_n$, $\beta \mapsto \gamma$ defined as follows is a bijection. For each $\beta \in \mathscr{B}_n$, $\gamma \in \mathscr{C}_n$ is determined by*

$$\begin{aligned} &\forall (x, y, z) \in [1, n]^3 \quad \textit{such that} \quad x + z \le n: \\ &\qquad (x, y, z) \in \gamma \Leftrightarrow (n+1-x-z, n-x, n+1-y) \notin \beta \end{aligned} \tag{4.6}$$

The inverse mapping $\mathscr{C}_n \to \mathscr{B}_n$ is given by the following rule. With each $\gamma \in \mathscr{C}_n$, we associate $\beta \in \mathscr{B}_n$ determined by

$$\begin{aligned} &\forall (x, y, z) \in [1, n]^3 \quad \textit{such that} \quad 1 \le x \le y \le n-1: \\ &\qquad (x, y, z) \in \beta \Leftrightarrow (n+1-z, n-y, 1-x+y) \notin \gamma \end{aligned} \tag{4.7}$$

Mills, Robbins and Rumsey defined the following function U_k in [MRR3].

Definition 4.8. For $\beta \in \mathscr{B}_n$ and $k \in [1, n]$, let

$$U_k(\beta) := \sum_{t=1}^{n-k} (\beta_{t,t+k-1} - \beta_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \{\beta_{t,n-1} > n-t\} \tag{4.8}$$

Here we set $\beta_{t,n} = n - t$ for all $t \in [1, n-1]$ by convention. Also $\{\cdots\}$ has value 1 when the statement "$\cdots$" is true and 0 otherwise.

We also use the function

$$\bar{U}_k(\beta) := n - 1 - U_k(\beta) \tag{4.9}$$

for $\beta \in \mathscr{B}_n$. Identifying $\mathscr{B}_n$ and $\mathscr{C}_n$ by the bijection defined in Corollary 4.5, we define $U_k(\gamma)$ and $\bar{U}_k(\gamma)$ for $\gamma \in \mathscr{C}_n$.

Definition 4.9. Let $\gamma = (\gamma_{ij})_{(i,j)\in \mathbf{P}^2} \in \mathscr{C}_n$. We call the parts γ_{ij} which satisfy $\gamma_{ij} = n - i$ *the maximal parts of* γ. The maximal parts, if they exist, appear only in the first column.

For example the maximal parts of an element of $\mathscr{C}_5$

4	3	1
3	2	1
1	1	
1		

are the boldfaced entries.

Next we show that the $\bar{U}_k(\gamma)$ is the sum of the number of parts which are equal to k and the number of maximal parts which are equal to $1, 2, \cdots, k-1$ in γ.

Proposition 4.3. *For each* $\gamma \in \mathscr{C}_n$ *and* $k \in [1, n]$, *we have*

$$\bar{U}_k(\gamma) = \#\{(i, j) \in [1, n]^2 \mid \gamma_{ij} = k\} + \#\{1 \le i \le k-1 \mid \gamma_{n-i,1} = i\}. \tag{4.10}$$

Proof. Recall that the definition of U_k is

$$U_k(\beta) := \sum_{t=1}^{n-k} (\beta_{t,t+k-1} - \beta_{t,t+k}) + \sum_{t=n-k+1}^{n-1} \{\beta_{t,n-1} > n - t\}.$$

If $n - k + 1 \le t \le n - 1$,

$$\beta_{t,n-1} > n - t \Leftrightarrow (t, n-1, n+1-t) \in \beta \Leftrightarrow (t, 1, n-t) \notin \gamma \Leftrightarrow \gamma_{t,1} < n - t.$$

It follows that

$$\begin{aligned} \sum_{t=n-k+1}^{n-1} \{\beta_{t,n-1} > n - t\} &= \#\{t \in [1, k-1] \mid \gamma_{n-t,1} < t\} \\ &= k - 1 - \#\{t \in [1, k-1] \mid \gamma_{n-t,1} = t\}. \end{aligned} \tag{4.11}$$

On the other hand if $t \in [1, n-k]$

$$\begin{aligned} \beta_{t,t+k-1} &= \#\{m \in [1, n] \mid (t, t+k-1, m) \in \beta\} \\ &= \#\{m \in [1, n] \mid (n+1-m, n-k+1-t, k) \notin \gamma\} \\ &= \#\{m \in [1, n] \mid \gamma_{m,n-k+1-t} < k\} \end{aligned} \tag{4.12}$$

and

$$\beta_{t,t+k} = \begin{cases} \#\{m \in [1, n] \mid \gamma_{m,n-k-t} \le k\} & \text{if } t < n-k \\ k & \text{if } t = n-k \end{cases}. \tag{4.13}$$

From (4.12) and (4.13) we conclude

$$\begin{aligned}\sum_{t=1}^{n-k} (\beta_{t,t+k-1} - \beta_{t,t+k}) &= \#\{(i,j) \in [1, n] \times [1, n-k] \mid \gamma_{i,j} < k\} \\ &\quad - \#\{(i,j) \in [1, n] \times [1, n-k-1] \mid \gamma_{i,j} \le k\} - k.\end{aligned} \tag{4.14}$$

It is easy to see that

$$\#\{(i,j) \mid i \in [1, n],\ j = n-k \text{ and } \gamma_{i,j} \le k\} = n. \tag{4.15}$$

From (4.14) and (4.15)

$$\sum_{t=1}^{n-k} (\beta_{t,t+k-1} - \beta_{t,t+k}) = n - k - \#\{(i,j) \in [1, n] \times [1, n-k] \mid \gamma_{i,j} = k\}. \tag{4.16}$$

(4.11) and (4.16) immediately imply the proposition. □

We introduce the notion of alternating sign matrices defined by Mills, Robbins and Rumsey.

Definition 4.10. An alternating sign matrix is a square matrix which satisfies

(*i*) all entries are 1, − 1 or 0,

(*ii*) every row and column has sum 1,

(*iii*) in every row and column the nonzero entries alternate in sign.

Let $\mathscr{A}_n$ be the set of n by n alternating sign matrices.

Example 4.5.

$$\mathscr{A}_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

We refer to some conjectures from [MRR3] in order to show some significance of the functions U_k.

Conjecture 4.2. ([*MRR*3]) *Let* $0 \le r \le n-1$ *and* $1 \le k \le n$. *Then*

$$\#\{\alpha = (\alpha_{ij})_{1 \le i,j \le n} \in \mathscr{A}_n \mid \alpha_{1r+1} = 1\} = \#\{\beta \in \mathscr{B}_n \mid U_k(\beta) = r\}. \tag{4.17}$$

Conjecture 4.3 ([*MRR*3]) *Let* $n \ge 2$ *and r, s be integers such that* $0 \le r, s < n$. *Then*

$$\#\{\alpha=(\alpha_{ij})_{1\le i,j\le n}\in\mathscr{A}_n \mid \alpha_{1r+1}=1,\ \alpha_{nn-s}=1\}=\#\{\beta\in\mathscr{B}_n \mid U_1(\beta)=r,\ U_2(\beta)=s\}. \tag{4.18}$$

By Conjecture 3 in [Sta3] we expect the following conjecture.

Conjecture 4.4. *Let $0\le r\le n-1$ and $1\le k\le n$. Set*

$$\mathscr{C}_{nr}^{k}=\{\gamma\in\mathscr{C}_n \mid \bar{U}_k(\gamma)=r\}.$$

Then $\#\mathscr{C}_{nr}^{k}$ would be given by

$$\#\mathscr{C}_{nr}^{k}=\frac{\binom{n+r-1}{n-1}\binom{2n-r-2}{n-1}A_{n-1}}{\binom{2n-2}{n-1}} \tag{4.19}$$

Conjecture 4.5. *Let $n\ge 2$ and r, s be integers such that $0\le r, s<n$. Then*

$$\begin{aligned}&\#\{\gamma\in\mathscr{C}_n \mid \bar{U}_1(\gamma)=r,\ \bar{U}_2(\gamma)=s\}.\\ &=\#\{\alpha=(\alpha_{ij})_{1\le i,j\le n}\in\mathscr{A}_n \mid \alpha_{1r+1}=1,\ \alpha_{nn-s}=1\}\end{aligned} \tag{4.20}$$

§5 Generating Functions

In this section we will give the generating function of $\mathscr{C}_n$ which is weighted by the multiplicities of parts and the function $\bar{U}_k$. The main theorem of this section is Theorem 5.1. We will obtain several corollaries of this theorem.

In the first place we summarize the lattice path method by recalling some terminology and stating the results of Gessel-Viennot [GV] as Lemma 5.1 and Lemma 5.2. Let D be an acyclic digraph in which every edge is assigned an element of a fixed commutative ring R. This element is called the weight of the edge. In our application the ring R will be the ring of polynomials in several variables. A *k-vertex* is by definition a k-tuple of vertices of D for a fixed integer k. If $\mathbf{u}=(u_1, u_2,\cdots,u_k)$ and $\mathbf{v}=(v_1, v_2,\cdots,v_k)$ are k-vertices of D, a *k-path* from $\mathbf{u}$ to $\mathbf{v}$ is a k-tuple $\mathbf{A}=(A_1, A_2,\cdots,A_k)$ of paths such that each A_i is a path from u_i to v_i. The k-path is said to be *disjoint* if the paths A_i are vertex disjoint. We define the weight of a path A to be the product of the weights of its edges and denote it by $wt(A)$. Similarly the weight of the k-path $\mathbf{A}$ is defined to be the product of each path and denoted by $wt(\mathbf{A})$. If u and v are two vertices, we write the set of paths from u to v as $\mathscr{P}(u, v)$ and if $\mathbf{u}$ and $\mathbf{v}$ are k-vertices, we write the set of k-paths from $\mathbf{u}$ to $\mathbf{v}$ as $\mathscr{P}(\mathbf{u}, \mathbf{v})$. And we denote the set of disjoint paths from $\mathbf{u}$ to $\mathbf{v}$ by $\mathscr{N}(\mathbf{u}, \mathbf{v})$. We write

$$P(u, v)=\sum_{A\in\mathscr{P}(u,v)} wt(A) \tag{5.1}$$

$$P(\mathbf{u}, \mathbf{v})=\sum_{\mathbf{A}\in\mathscr{P}(\mathbf{u},\mathbf{v})} wt(\mathbf{A}) \tag{5.2}$$

$$N(\mathbf{u}, \mathbf{v}) = \sum_{\mathbf{A} \in \mathcal{N}(\mathbf{u}, \mathbf{v})} wt(\mathbf{A}) \tag{5.3}$$

Let S_k be the symmetric group on $\{1, 2, \cdots, k\}$. If $\mathbf{v}$ is a k-path and $\pi \in S_k$, then let $\pi(\mathbf{v}) = (v_{\pi(1)}, v_{\pi(2)}, \cdots, v_{\pi(k)})$.

Lemma 5.1. ([*GV*] *Theorem* 1)

$$\sum_{\pi \in S_k} (\operatorname{sgn}\pi) N(\mathbf{u}, \pi(\mathbf{v})) = \det(P(u_i, v_j))_{i,j=1,2,\dots,k} \tag{5.4}$$

Let us say that a pair $(\mathbf{u}, \mathbf{v})$ of k-vertices is *nonpermutable* if $N(\mathbf{u}, \pi(\mathbf{v}))$ is empty unless π is the identity element. Then we have the following corollary.

Lemma 5.2. ([*GV*] *Corollary* 2) *If* $(\mathbf{u}, \mathbf{v})$ *is nonpermutable, then*

$$N(\mathbf{u}, \mathbf{v}) = \det(P(u_i, v_j))_{i,j=1,2,\dots,k}. \tag{5.5}$$

We need several kinds of symmetric functions to describe the generating functions of plane paritions. Let us prepare some notation here. We use countable many variables $x = (x_i)_{i \in \mathbf{Z}}$. For $n, m, r \in \mathbf{Z}$ such that $n \geq m$, we write *the r-th elementary symmetric function* in $n - m$ variables $x_{m+1}, x_{m+2}, \cdots x_n$ as $e_r^{(n/m)}(x)$. Its precise definition is as follows.
In the case $n > m$ we define $e_r^{(n/m)}(x)$ by

$$e_r^{(n/m)}(x) := \begin{cases} \sum_{m+1 \leq i_1 < i_2 < \dots < i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r} & \text{if } n > m \text{ and } r > 0. \\ \delta_{r,0} & \text{if } n = m. \end{cases} \tag{5.6}$$

(We use the convention that $e_r^{(n/m)}(x) = 1$ if $r = 0$ and $e_r^{(n/m)}(x) = 0$ if $r < 0$.) If $n = m$, we put $e_r^{(n/m)}(x) = \delta_{r,0}$. If m $= 0$, we abbreviate $e_r^{(n/0)}(x)$ to $e_r^{(n)}(x)$. The generating function of $e_r^{(n/m)}(x)$ is given by

$$\sum_{r \in \mathbf{Z}} e_r^{(n/m)}(x) t^r = \prod_{i=m+1}^{n} (1 + x_i t). \tag{5.7}$$

We prepare some notation to describe our theorem.

Definition 5.11. By the generating function of $\mathscr{C}_n$ weighted by $\bar{U}_k$, we mean $\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^{\gamma}$.
If $\pi = (\pi_{ij})_{(i,j) \in \lambda}$ is a plane partition of the shape λ, we write x^{π} for the monomial

$$\prod_{(i,j) \in \lambda} x_{\pi_{ij}}. \tag{5.8}$$

For $\gamma \in \mathscr{C}_n$, we write $\bar{U}(\gamma) = (\bar{U}_1(\gamma), \bar{U}_2(\gamma), \cdots, \bar{U}_n(\gamma))$ and

$$t^{\bar{U}(\gamma)} = t_1^{\bar{U}_1(\gamma)} t_2^{\bar{U}_2(\gamma)} \dots t_n^{\bar{U}_n(\gamma)} \tag{5.9}$$

For example, for the following element γ in $\mathscr{C}_5$

$$\gamma = \begin{array}{|c|c|c|c|} \hline \mathbf{4} & 3 & 2 & 1 \\ \hline 2 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} \mathbf{2} & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

we have $t^{\bar{U}(\gamma)}x^{\gamma} = t_1^3 t_2^4 t_3^3 t_4^3 t_5^3 x_1^3 x_2^3 x_3 x_4$.

Now we give the generating function of $\mathscr{C}_n$ expressed as the sum of minors of a rectangular matrix whose entries are certain elemntary symmeric functions. We can apply Lemma 1 in two different ways to obtain the generating function of $\mathscr{C}_n$. We use only one of them here.

Theorem 5.7. *Let* $n \in \mathbf{P}$ *and* $k \in [1, n]$. *We define the* n *by* $(2n-1)$ *rectangular matrix* $P_n(t, x)$ *by*

$$P_n(t, x) := (p_{i,j}^{(n)}(t, x))_{\substack{i=0,\dots,n-1 \\ j=0,\dots,2n-2}}$$

where

$$p_{ij}^{(n)}(t, x) = e_{j-i}^{(i)}(t_1 x_1, t_2 x_2, \cdots, t_{i-1} x_{i-1}, (\prod_{\nu=i}^{n} t_\nu) x_i). \tag{5.10}$$

Then

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^{\gamma} = \mathrm{d}_n(P_n(t, x)). \tag{5.11}$$

Before proving the theorem, we first show an example, and then we state some corollaries which are immediately deduced from Theorem 5.4.

Example 5.6. When $n = 3$ and $k = 1$,

$$P_3(t, x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & t_1 t_2 t_3 x_1 & 0 & 0 \\ 0 & 0 & 1 & t_1 x_1 + t_2 t_3 x_2 & t_1 t_2 t_3 x_1 x_2 \end{pmatrix}$$

and

$$\begin{aligned} \mathrm{d}_3(P_3(t, x)) = 1 &+ t_1 x_1 + t_2 t_3 x_2 + t_1 t_2 t_3 x_1 x_2 + t_1^2 t_2 t_3 x_1^2 \\ & t_1 t_2^2 t_3^2 x_1 x_2 + t_1^2 t_2^2 t_3^2 x_1^2 x_2^2. \end{aligned}$$

Whereas the elements of $\mathscr{C}_3$ are

$\gamma \in \mathscr{C}_3$	$\bar{U}_1(\gamma)$	$\bar{U}_2(\gamma)$	$\bar{U}_3(\gamma)$	$t^{\bar{U}(\gamma)}x^\gamma$
ϕ,	0,	0,	0,	1
1,	1,	0,	0,	$t_1 x_1$
1 1'	2,	1,	1,	$t_1^2 t_2 t_3 x_1^2$
2,	0,	1,	1,	$t_2 t_3 x_2$
2 1'	1,	2,	2,	$t_1 t_2^2 t_3^2 x_1 x_2$
2 1,	1,	1,	1,	$t_1 t_2 t_3 x_1 x_2$
2 1 1 ,	2,	2,	2,	$t_1^2 t_2^2 \quad t_3^2 x_1^2 x_2^2$.

Putting $t_i = 1$ $(1 \leq i \leq n)$ in Theorem 5.1, we obtain

Corollary 5.5. *For $n \in \mathbf{P}$, let $L_n(x)$ be the n by $(2n-1)$ rectangular matrix defined by*

$$L_n(x) := (e_{j-i}^{(i)}(x))_{\substack{i=0,\ldots,n-1 \\ j=0,\ldots,2n-2}}. \tag{5.12}$$

Then

$$\sum_{\gamma \in \mathscr{C}_n} x^\gamma = \mathrm{d}_n(L_n(x)). \tag{5.13}$$

Putting $x_i = q^i$ $(1 \leq i \leq n)$ in Corollary 5.1, we obtain

Corollary 5.6. *For $n \in \mathbf{P}$, let $M_n(q)$ be the n by $2n-1$ rectangular matrix defined by*

$$M_n(q) := \left(q^{\frac{(j-i)(j-i-1)}{2}} \begin{bmatrix} i \\ j-i \end{bmatrix}_q \right)_{\substack{i=0,\ldots,n-1 \\ j=0,\ldots,2n-2}}. \tag{5.14}$$

Then

$$\sum_{\gamma \in \mathscr{C}_n} q^{|\gamma|} = \mathrm{d}_n(M_n(q)) \tag{5.15}$$

Putting q = 1 in Corollary 5.2, we obtain

Corollary 5.7. *For $n \in \mathbf{P}$, let $N_n(x)$ be the n by $(2n-1)$ rectangular matrix defined by*

$$N_n(t, x) := \left(\binom{i}{j-i} \right)_{\substack{i=0,\ldots,n-1 \\ j=0,\ldots,2n-2}}. \tag{5.16}$$

Then

$$\# \mathscr{S}_n = \# \mathscr{C}_n = \mathrm{d}_n(N_n). \tag{5.17}$$

If we put $x_i = 1$ $(1 \le i \le n)$ in Theorem 2, we obtain

Corollary 5.8. *For $n \in \mathbf{P}$, let $Q_n(t)$ be the n by $(2n-1)$ rectangular matrix defined by*

$$Q_n(t) := \Big(e^{(i)}_{j-i}(t_1, t_2, \cdots, t_{i-1}, \prod_{\nu=i}^{n} t_\nu)\Big)_{\substack{i=0,\dots,n-1\\ j=0,\dots,2n-2}}. \tag{5.18}$$

Then

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} = \mathrm{d}_n(Q_n(t)). \tag{5.19}$$

Proof of Theorem 5.1. We consider the digraph in which the vertices are lattice points in the plane and the edges go from (i, j) to $(i, j-1)$ and $(i+1, j-1)$. The edges which go from (i, j) to $(i, j+1)$ are called the vertical steps and those which go from (i, j) to $(i+1, j-1)$ the horizontal steps, although they are not exactly horizontal. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition whose length is exatly k. And let

$$\mathscr{C}^{(n)}_\lambda := \{\gamma \in \mathscr{C}_n \mid bs(\gamma) = \lambda\}. \tag{5.20}$$

We define the k-vertices $\mathbf{u}$ and $\mathbf{v}$ by

$$u_i = (1-i, n-i), \qquad v_i = (\lambda_i - i + 1, 0). \tag{5.21}$$

Then we can define a bijective correspondence between the $\gamma \in \mathscr{C}^{(n)}_\lambda$ and the disjoint k-paths $\mathbf{A}$ from $\mathbf{u}$ to $\mathbf{v}$ as folows. The i-th row of the plane partition determines the i-th path A_i of $\mathbf{A}$. A_i contains the horizontal step from (l, h) to $(l+1, h-1)$ in i-th path if and only if $\gamma_{i,l+i}$ is equal to h. Vertical steps are appended appropriately so that the i-th path A_i becomes a path from u_i to v_i.
Recall that

$$\bar{U}_k(\gamma) = \#\{(i, j) \in [1, n]^2 \mid \gamma_{ij} = k\} + \#\{1 \le i \le k-1 \mid \gamma_{n-i,1} = i\}. \tag{5.22}$$

To realize this weight we define the weights of the edges in the above digraph as follows. The weights of vertical steps are 1. If the horizontal step from (l, h) to $(l+1, h-1)$ is in the form

$$l = 1 - i, \qquad h = n - i$$

then the weight of the step is $t_h t_{h+1} \cdots t_n x_h$. Otherwise, the weight of the step is $t_h x_h$. Let $u_i = (a_i, b_i)$ and $v_i = (c_i, d_i)$. In this digraph the pair $(\mathbf{u}, \mathbf{v})$ is nonpermutable if $a_{i+1} \le a_i$, $b_{i+1} \le b_i + a_i - a_{i+1}$, $c_{i+1} \le c_i$ and $d_{i+1} \le d_i + c_i - c_{i+1}$. Since the above pair satisfies this condition, we can apply Lemma 2 and obtain

$$N(\mathbf{u}, \mathbf{v}) = \det(P(u_i, v_j))_{i,j=1,2,\dots,k}. \tag{5.23}$$

In this identity we can express $P(u_i, v_j)$ using the elementary symmetric function:

$$P(u_i, v_j) = e^{(n-i)}_{\lambda_j - j + i}(t_1 x_1, t_2 x_2, \cdots, t_{n-i-1} x_{n-i-1}, (\prod_{\nu=n-i}^{n} t_\nu) x_{n-i}). \tag{5.24}$$

Up to this point we assumed that $1 \leq i, j \leq k$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. But we may identify the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ such that $\lambda_{k+1} = 0, \cdots, \lambda_n = 0$, and from (5.24) we obtain

$$\det(P(u_i, v_j))_{i,j=1,2,\ldots,k} = \det(P(u_i, v_j))_{i,j=1,2,\ldots,n}. \tag{5.25}$$

So we sum up this generating function with λ ranging over all partitions and obtain

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^{\gamma} = \sum_{\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_n)} \det(P(u_i, v_j))_{i,j=1,2,\ldots,n}, \tag{5.26}$$

where λ ranges under the condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0. \tag{5.27}$$

Now we replace λ_j by

$$\mu_{n-j} = n - j + \lambda_j \qquad (1 \leq j \leq n). \tag{5.28}$$

Then μ ranges under the condition

$$0 \leq \mu_0 < \mu_1 < \cdots < \mu_n \leq 2n - 2. \tag{5.29}$$

Substituting $\lambda_j - j + i = \mu_{n-j} + i - n$ into (5.24), we obtain

$$P(u_i, v_j) = e^{(n-i)}_{\mu_{n-j}+i-n}(t_1 x_1, t_2 x_2, \cdots, t_{n-i-1} x_{n-i-1}, (\prod_{\nu=n-i}^{n} t_\nu) x_{n-i}). \tag{5.30}$$

From (5.30) we have

$$P(u_{n-i}, v_{n-j}) = e^{(i)}_{\mu_j - i}(t_1 x_1, t_2 x_2, \cdots, t_{i-1} x_{i-1}, (\prod_{\nu=i}^{n} t_\nu) x_i). \tag{5.31}$$

Substituting (5.31) into (5.26), we obtain (5.11). □

§6 A Formula with Pfaffian

In this section we express $d_n(P_n(t, x))$ of (5.11) obtaind in Theorem 5.1 in the form of a pfaffian by using [O] Theorem 3 and then simplify the entries of the pfaffian as far as possible. Theorem 1.3 and Corollary 1.2 is the main result of this section. Let $Z := (z_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ be an n by m matrix whose entries are the variables z_{ij}.

Let $(a_1, a_2, \cdots, a_r)$ and $(b_1, b_2, \cdots, b_r)$ be integers such that

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_r \leq n, \qquad 1 \leq b_1 \leq b_2 \leq \cdots \leq b_r \leq m.$$

We write

$$d(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots, b_r) = \det(z_{a_i b_j})_{1 \leq i,j \leq r} \tag{6.1}$$

and

$$\mathrm{d}(a_1, a_2, \cdots, a_r) = \sum_{1 \le b_1 \le b_2 \le \ldots \le b_r \le m} \mathrm{d}(a_1, a_2, \cdots, a_r; b_1, b_2, \cdots, b_r). \tag{6.2}$$

Lemma 6.3. [*O, Theorem* 3] *Let* $1 \le a_1 \le a_2 \le \cdots \le a_r \le n$.
(1) *If* r *is even, then we have*

$$\mathrm{d}(a_1, a_2, \cdots, a_r) = \mathrm{Pf}_r(\mathrm{d}(a_i, a_j))_{1 \le i,j \le r}$$

(2) *If* r *is odd, then we have*

$$\mathrm{d}(a_1, a_2, \cdots, a_r) = \mathrm{Pf}_{r+1}(x_{ij})_{1 \le i,j \le r+1}$$

where

$$x_{ij} := \begin{cases} 0 & (i = j = 1) \\ \mathrm{d}(a_{j-1}) & (i = 1, 2 \le j \le r+1) \\ -\mathrm{d}(a_{i-1}) & (j = 1, 2 \le i \le r+1) \\ \mathrm{d}(a_{i-1}, a_{j-1}) & (2 \le i, j \le r+1) \end{cases}$$

Here we denote the pfaffian of degree r *by* Pf_r.

Definition 6.12. For $n \in \mathbf{P}$ and $i, j \in \mathbf{N}$, we put

$$f_n(i, j; t, x) := \sum_{0 \le k < l} \begin{vmatrix} p_{ik}^{(n)}(t, x) & p_{il}^{(n)}(t, x) \\ p_{jk}^{(n)}(t, x) & p_{jl}^{(n)}(t, x) \end{vmatrix}, \tag{6.3}$$

where $p_{ij}^{(n)}(t, x)$ is as in Theorem 5.1.

If n is even, by virtue of Lemma 6.1 (1) we can express the sum of determinants in Theorem 5.4 by a pfaffian as in the following theorem. If n is odd, we remove the first row from the rectangular array of Theorem 5.4 and then apply Lemma 6.1 (1).

Theorem 6.8. *Let* $n \in \mathbf{P}$.

(1) *If* n *is even,* $\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^{\gamma} = \mathrm{Pf}_n(f_n(i, j; t, x))_{0 \le i,j \le n-1}$.

(2) *If* n *is odd,* $\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}(\gamma)} x^{\gamma} = \mathrm{Pf}_{n-1}(f_n(i, j; t, x))_{1 \le i,j \le n-1}$.

S. Okada pointed out that Lemma 6.1, which transform the sum of minors into a pfaffian, is appliable to Theorem 5.1. From now on we try to simplify the pfaffian as far as possible. Set $y_i := t_i x_i$, $X_i := \prod_{\nu=1}^{i} x_\nu$, $T_i := \prod_{\nu=i}^{n} t_\nu$ and $T := T_1 = \prod_{\nu=1}^{n} t_\nu$.

Proposition 6.4. *For* $i, j \in \mathbf{N}$ *we have*

$$\begin{aligned} f_n(i, j; t, x) = {} & TX_i \sum_{r \ge 2i-j} e_r^{(i+j)}(y_1^{-1}, \cdots, y_{i-1}^{-1}, T_{i+1}^{-1} y_i^{-1}, y_1, \cdots, y_{j-1}, T_{j+1} y_j) \\ & + TX_j \sum_{r \ge 2j-i} e_r^{(i+j)}(y_1, \cdots, y_{i-1}, T_{i+1} y_i, y_1^{-1}, \cdots, y_{j-1}^{-1}, T_{j+1}^{-1} y_j^{-1}). \end{aligned} \tag{6.4}$$

Proof. Let S be the linear operator which associates the constant term a_0 with a Laurent series $\sum_{n\le m} a_n z^{-n}$ $(m\in\mathbf{Z})$. If $f(z)=\sum_{n\ge 0} a_n z^n$ is a polynomial, then we have

$$S\left(\frac{z^{-m}}{1-z^{-1}}f(z)\right)=\sum_{n\ge m} a_n. \tag{6.5}$$

We use this formula to transform $f_n(i,j;t,x)$. Let $f_n(i,j;t,x)$ be as in Definition 6.1, then we have

$$\begin{aligned} f_n(i,j;t,x)&=\sum_{k\ge 0}\sum_{l\ge k}\begin{vmatrix} p_{ik}^{(n)}(t,x) & p_{il}^{(n)}(t,x)\\ p_{jk}^{(n)}(t,x) & p_{jl}^{(n)}(t,x)\end{vmatrix}\\ &=\sum_{k\ge 0}\begin{vmatrix} p_{ik}^{(n)}(t,x) & \sum_{l\ge k} p_{jl}^{(n)}(t,x)\\ p_{jk}^{(n)}(t,x) & \sum_{l\ge k} p_{jl}^{(n)}(t,x)\end{vmatrix}. \end{aligned} \tag{6.6}$$

From (6.5), we have

$$\sum_{l\ge k} p_{il}^{(n)}(t,x)=S\left(\frac{z^{-k}}{1-z^{-1}}\sum_{l\ge 0} p_{il}^{(n)}(t,x)z^l\right) \tag{6.7}$$

Using the generating function of the elementary symmetric functions, we express the sum $\sum_{l=0}^{\infty} p_{il}^{(n)}(t,x)z^l$ as a product:

$$\begin{aligned} \sum_{l=0}^{\infty} p_{il}^{(n)}(t,x)z^l&=\sum_{l=0}^{\infty} e_{l-i}^{(i)}(y_1,\cdots,y_{i-1},T_{i+1}y_i)z^l\\ &=z^i\sum_{l=0}^{\infty} e_{l-i}^{(i)}(y_1,\cdots,y_{i-1},T_{i+1}y_i)z^{l-i}\\ &=z^i\left(\prod_{\nu=1}^{i-1}(1+y_\nu z)\right)(1+T_{i+1}y_i z). \end{aligned} \tag{6.8}$$

Combining (6.7) and (6.8), we express (6.6) as follows

$$\begin{aligned} &f_n(i,j;t,x)\\ &=\sum_{k\ge 0}\begin{vmatrix} p_{ik}^{(n)}(t,x) & S\left(\frac{z^{-k}}{1-z^{-1}}z^i\left(\prod_{\nu=1}^{i-1}(1+y_\nu z)\right)(1+T_{i+1}y_i z)\right)\\ p_{jk}^{(n)}(t,x) & S\left(\frac{z^{-k}}{1-z^{-1}}z^j\left(\prod_{\nu=1}^{j-1}(1+y_\nu z)\right)(1+T_{j+1}y_j z)\right)\end{vmatrix}\\ &=S\left(\frac{1}{1-z^{-1}}\begin{vmatrix} \sum_{k\ge 0} p_{ik}^{(n)}(t,x)z^{-k} & z^i\left(\prod_{\nu=1}^{i-1}(1+y_\nu z)\right)(1+T_{i+1}y_i z)\\ \sum_{k\ge 0} p_{jk}^{(n)}(t,x)z^{-k} & z^j\left(\prod_{\nu=1}^{j-1}(1+y_\nu z)\right)(1+T_{j+1}y_j z)\end{vmatrix}\right) \end{aligned}$$

$$= S\left(\frac{1}{1-z^{-1}} \begin{vmatrix} z^{-i}(\prod_{\nu=1}^{i-1}(1+y_\nu z^{-1}))(1+T_{i+1}y_i z^{-1}) & z^{i}(\prod_{\nu=1}^{i-1}(1+y_\nu z))(1+T_{i+1}y_i z) \\ z^{-j}(\prod_{\nu=1}^{j-1}(1+y_\nu z^{-1}))(1+T_{j+1}y_j z^{-1}) & z^{j}(\prod_{\nu=1}^{j-1}(1+y_\nu z))(1+T_{j+1}y_j z) \end{vmatrix} \right)$$

$$= S\left(\frac{z^{-2i+j}}{1-z^{-1}} TX_i(\prod_{\nu=1}^{i-1}(1+y_\nu^{-1}z))(1+T_{i+1}^{-1}y_i^{-1}z)(\prod_{\nu=1}^{j-1}(1+y_\nu z))(1+T_{j+1}y_j z) \right.$$
$$\left. - \frac{z^{-2j+i}}{1-z^{-1}} TX_j(\prod_{\nu=1}^{i-1}(1+y_\nu z))(1+T_{i+1}y_i z)(\prod_{\nu=1}^{j-1}(1+y_\nu^{-1}z))(1+T_{j+1}^{-1}y_j^{-1}z) \right).$$

This proves the lemma. □

From now on we deal with special cases since we are mainly concerned with the problem to count the number of TSSCPP and it is very complicated to simplify the entries of the pfaffian in the multivaliable case. Fix an integer k such that $0 \le k \le n-1$. We substitute $x_i = 1$ for $1 \le i \le n$, $t_i = 1$ for $1 \le i \le n$, $i \ne k$ and $t_k = t$ into $f_n(i, j; t, x)$. We denote by $g_{ij}(t)$ the function obtained by this substitution. Notice that $g_{ij}(t)$ is independent of k and written as follows.

If $i = j = 0$, then $g_{ij}(t) = 0$.
If $i = 0$ and $j > 0$, then $g_{ij}(t) = 2^{i-1}(t+1)$.
If $j = 0$ and $i > 0$, then $g_{ij}(t) = -2^{j-1}(t+1)$.
If $i, j > 0$, then

$$g_{ij}(t) = (t^2+1)\left\{ \sum_{r \ge 2i-j} \binom{i+j-2}{r-1} - \sum_{r \ge 2j-i} \binom{i+j-2}{r-1} \right\}$$
$$+ t\left\{ \sum_{r \ge 2i-j} \left\{ \binom{i+j-2}{r} + \binom{i+j-2}{r-2} \right\} - \sum_{r \ge 2j-i} \left\{ \binom{i+j-2}{r} + \binom{i+j-2}{r-2} \right\} \right\}. \tag{6.9}$$

If we use a recursion formula of binomial coefficients, we obtain the following lemma by easy calculation.

Lemma 6.4. *For $i, j \ge 1$ we have the recursive formula*

$$g_{ij}(t) = 2g_{ij-1}(t) + \left\{ \binom{i+j-2}{2i-j-1} + 2\binom{i+j-2}{2i-j} \right\}(1+t^2)$$
$$+ \left\{ \binom{i+j-2}{2i-j-2} + 2\binom{i+j-2}{2i-j-1} + \binom{i+j-2}{2i-j} + 2\binom{i+j-2}{2i-j+1} \right\} t. \tag{6.10}$$

If we utilize these recursive formulae and carry out some elementary transformations in rows and columns on the pfaffian in Theorem 6.4, we obtain the following theorem.

Theorem 6.9. *Fix* $0 \le k \le n-1$. *Then we have*

$$\sum_{\gamma \in \mathscr{C}_n} t^{\bar{U}_k(\gamma)} = \begin{cases} (1+t)\,\mathrm{Pf}_n(h_{ij}(t))_{2 \le i,j \le n-1} & \text{if } n \text{ is even,} \\ \mathrm{Pf}_n(h_{ij}(t))_{1 \le i,j \le n-1} & \text{if } n \text{ is odd,} \end{cases} \tag{6.11}$$

where the entries $h_{i,j}(t)$ *are as follows.*
If $i = j = 1$, *then* $h_{i,j}(t) = 0$.
If $i \ge 2$ *or* $j \ge 2$, *then*

$$\begin{aligned} h_{i,j}(t) = {} & \frac{3(j-i)(3i-2)(3j-2)}{(i+j-2)(2i-j)(2j-i)}\binom{i+j-2}{2i-j-1}(1+t^2) \\ & + \frac{(j-i)(3i-1)(3j-1)\{3(i^2-ij+j^2)-(i+j)-2\}}{(i+j-2)(2i-j+1)(2i-j)(2j-i+1)(2j-i)}\binom{i+j-2}{2i-j-1}t. \end{aligned} \tag{6.12}$$

Proof. First we consider the case where n is odd. The one variable generating function is given by $\mathrm{Pf}_{n-1}(g_{ij}(t))_{1 \le i,j \le n-1}$. We carry out the elementary transformations on $\det(g_{ij}(t))_{1 \le i,j \le n-1}$ which is the square of the pfaffian. In the determinant we subtract twice the $(n-2)$-th column from the $(n-1)$-th column, then subtract twice the $(n-3)$-th column from the $(n-2)$-th column. We continue these steps until we subtract twice the first column from the second column. By Lemma 6.5 the (i, j)-entry $(j \ge 2)$ of the resulting matix is given by

$$\begin{aligned} & \left\{\binom{i+j-2}{2i-j-1} + 2\binom{i+j-2}{2i-j}\right\}(1+t^2) \\ & + \left\{\binom{i+j-2}{2i-j-2} + 2\binom{i+j-2}{2i-j-1} + \binom{i+j-2}{2i-j} + 2\binom{i+j-2}{2i-j+1}\right\}t. \end{aligned} \tag{6.13}$$

And the i-th entry of the first column of the resulted matix is given by

$$\begin{cases} 0 & (i=1), \\ -(2+3t+2t^2) & (i=2), \\ -2^{i-1}(1+t^2) & (i \ge 3). \end{cases} \tag{6.14}$$

Next we perform the same elementary transformations with respect to rows. We subtract twice the $(n-2)$-th row from the $(n-1)$-th row, twice the $(n-3)$-th row from the $(n-2)$-th row and so on. The (i, j)-entry $(j \ge 2)$ of the resulting matix is given by

$$\left\{2\binom{i+j-3}{2i-j} + 3\binom{i+j-3}{2i-j-1} - 3\binom{i+j-3}{2i-j-2} - 2\binom{i+j-3}{2i-j-3}\right\}(1+t^2)$$

$$\left\{2\binom{i+j-3}{2i-j+1}+3\binom{i+j-3}{2i-j}-\binom{i+j-3}{2i-j-1}\right. \qquad (6.15)$$
$$\left.+\binom{i+j-3}{2i-j-2}-3\binom{i+j-3}{2i-j-3}-2\binom{i+j-3}{2i-j-1}\right\}t$$

Further the i-th entry of the first column of the resulting matix is given by

$$\begin{cases} 0 & (i=1) \\ -(2+3t+2t^2) & (i=2) \\ -2t & (i=3) \\ 0 & (i\geq 4) \end{cases} \qquad (6.16)$$

(6.16) coincides with (6.15) except when $i=j=1$. We calculate (6.15) and obtain (6.12). If n is even, we carry out the same elementary transformations on rows and columns. First we transform $n-1$, $n-2,\cdots,2$-th columns, then transform $n-1$, $n-2,\cdots,2$-th rows. We obtain the same entry as (6.15) except when $i=0$ or $j=0$. When $i=0$, we obtain the entries

$$\begin{cases} 0 & (j=0 \text{ or } j\geq 2) \\ t+1 & (j=1) \end{cases}. \qquad (6.17)$$

We exand the pfaffian with respect to the top row and obtain the theorem. □

If we put $t=1$ in the formula of Theorem 6.6, then we obtain the following corollary.

Corollary 6.9.

$$\#\mathscr{C}_n = \begin{cases} 2Pf_n(b_{ij})_{2\leq i,j\leq n-1} & \textit{if } n \textit{ is even.} \\ Pf_n(b_{ij})_{1\leq i,j\leq n-1} & \textit{if } n \textit{ is odd.} \end{cases}$$

where the entry $b_{i,j}$ *is given by*

$$b_{i,j}=\frac{3(j-i)(3i+1)(3j+1)}{(i+j)(2i-j+1)(2j-i+1)}\binom{i+j}{2i-j}. \qquad (6.18)$$

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