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Realization of a subalgebra of a generalized Steenrod algebra

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§1. Introduction

For a ring spectrum E, $(A, \Gamma) = (E_*, E_*(E))$ is a Hopf algebroid if Γ is flat over A (*cf.* [8], [1]). In this case, the homology groups $E_*(X)$ of a spectrum X have a Γ-comodule structure in the natural way (*cf.* [1]). What can be said for the converse statement? In other words, for a given Γ-comodule M, is there a spectrum X such that $E_*(X) = M$? This is called a problem of realizability of a comodule. Originally, this problem was stated in the language of cohomologies. That is to say, it asks if there is any spectrum X such that $H^*(X; \boldsymbol{Z}/p) = \mathscr{B}$ for a given subalgebra $\mathscr{B}$ of the Steenrod algebra $\mathscr{A} = H^*(H\boldsymbol{Z}/p; \boldsymbol{Z}/p)$ at each prime number p, where $H\boldsymbol{Z}/p$ denotes the mod p Eilenberg-MacLane spectrum. Since $H^*(H\boldsymbol{Z}/p; \boldsymbol{Z}/p) = [H\boldsymbol{Z}/p, H\boldsymbol{Z}/p]^*$ as homotopy sets, we can ask what is going on if we generalize this by replacing $H\boldsymbol{Z}/p$ with a ring spectrum E. Furthermore, this is rewritten in the language of homologies, which is treated here.

One of the ways to solve this is a way to use the Adams-Bousfield resolution, which is explained as follows:

For a spectrum X, we can construct an Adams-Bousfield resolution

$$pt \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

such that $X_n \to X_{n+1} \to E \wedge \bar{E}^n \wedge X$ for each n is a cofiber sequence (up to suspension), where $\bar{E}$ denotes the cofiber of the unit map $i: S^0 \to E$ of the ring spectrum E. Then we see that $E_*(X) = E_*(X^\wedge)$, where $X^\wedge = \lim_n X_n$. Our idea to find a solution for the problem is construct this kind of resolution without X. By this, we mean that we construct the resolution out of a spectrum EM such that $\pi_*(EM) = M$ for the given comodule M. This M corresponds to $E_*(X)$ (*i.e.* $E \wedge X = EM$), if X exists. In fact, we study whether or not we can construct a cofibration $X_n \to X_{n+1} \to EM \wedge \bar{E}^n$ for each n. If we construct them for all n, then we see that $M^\wedge = \lim_n X_n$ satisfies $E_*(M^\wedge) = \pi_*(EM) = M$. That is, $EM = E \wedge M^\wedge$.

Under this idea, several authors have succeeded to construct spectra. Set first $E = H\boldsymbol{Z}/p$, the mod p Eilenberg-MacLane spectrum. Then the dual of the Steenrod algebra $\mathscr{A}_* = E_*(E)$ for odd p is known to be the tensor product of the polynomial algebra $\mathscr{P}_* = F_p[\xi_1, \xi_2, \cdots]$ with $|\xi_n| = 2p^n - 2$, the dual of the algebra $\mathscr{P}$ of the reduced power operations, and the exterior algebra $\Lambda(\tau_0, \tau_1, \cdots)$ with $|\tau_n| = 2p^n - 1$. For $p = 2$, $\mathscr{A}_* = E_*(E) = F_2[\xi_1, \xi_2, \cdots]$ with $|\xi_n| = 2^n - 1$, where the dual of the

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algebra $\mathscr{P}$ of the square operations (mod 2 reduced power operations) are embedded as $\mathscr{P}_* = F_2[\xi_1^2, \xi_2^2, \cdots]$. Then define the spectra BP and $V(n)$ $(n \geq 0)$ by $E_*(BP) = \mathscr{P}_*$ and $E_*(V(n)) = \Lambda(\tau_0, \cdots, \tau_n)$. Now take the $\mathscr{A}_*$-module M to be $\mathscr{P}_*$ or $\Lambda(\tau_0, \tau_1, \tau_2, \tau_3)$. Using the method explained above, E. Brown and F. Peterson showed the existence of the spectrum BP in [3], which is known to be the Brown-Peterson spectrum, and H. Toda showed in [11] the existence of the spectrum $V(3)$ for $p > 5$, which is called the Toda-Smith spectrum. ($V(n)$ for $n < 3$ had been constructed by then for a prime number $p > 2n$ in different methods by H. Toda and L. Smith.)

Instead of HZ/p, take the Brown-Peterson spectrum BP (resp. Johnson-Wilson spectrum $E(n)$ for $n \geq 0$ ([6])), where the coefficient ring BP_* (resp. $E(n)_*$) is the polynomial algebra $Z_{(p)}[v_1, v_2, \cdots]$ (resp. $Z_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}]$) over the generators v_k with $|v_k| = 2p^k - 2$. Then we found some condition for the existence of the spectrum X with $BP_*(X) = v_n^{-1}BP_*/(p^{i_0}, v_1^{i_1}, \cdots, v_{n-1}^{i_{n-1}})$ (resp. $E(m)_*(X) = E(m)_*/(p^{i_0}, v_1^{i_1}, \cdots, v_{n-1}^{i_{n-1}})$) in [10] (resp. [9]). In this case, if such a resolution exists, then $BPM = BP \wedge M^\wedge$, which is what we want to study here. To do this, for a ring spectrum E with $E_*(E)$ E_*-flat, we give the general result:

THEOREM. *Let E, F and G be spectra of such that E and F are ring spectra and that $E_*(F)$ and $E_*(G)$ are Hopf algebroids over F_* with $E_*(G)$ F_*-free and $G \subsetneq F$ inducing the map of Hopf algebroids $E_*(G) \to E_*(F)$. Furthermore, assume that there exists a map $E \to F$ which induces the map $E_*(E) \to E_*(F)$ of Hopf algebroids. Then there exists a spectrum X such that*

$$X \wedge G = F.$$

The proof is given in §3 by setting $F = E$. For general result, it is almost identical to the case $F = E$ but a little more complicated, and so we omit here.

Let $p = 2$ and $D(A_1)$ denote the cofiber of the essential map $\Sigma^5 M_\eta \wedge M_\nu \to M_\eta \wedge M_\nu$, and $E(2)/2 = E(2) \wedge M_2$. Here M_α for $\alpha \in \pi_t(S^0)$ denotes the cofiber of the map $f\colon S^t \to S^0$ which represents α. Here $\eta \in \pi_1(S^0) = Z/2$ and $\nu \in \pi_3(S^0) = Z/8$ are the generators. Note that $E(2)_* = \pi_*(E(2)) = Z/2[v_1, v_2, v_2^{-1}]$. As a corollary,

COROLLARY. *There exists a spectrum X at the prime 2 such that*

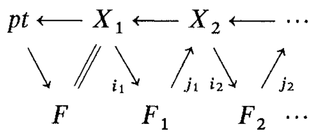
$$X \wedge D(A_1)/2 = E(2)/2.$$

By the same manner given in [5], the connected cover of X gives a counter example to the result of [4] which claims that $\mathscr{A}//\mathscr{A}_2$ is not realizable as a cohomology of a spectrum. Here $\mathscr{A}_2$ denotes the subalgebra of the Steenrod algebra generated by Sq^{2^i} with $i = 0, 1, 2$.

§2. Adams-Bousfield resolution and geometric resolution

Let E denote a ring spectrum such that $E_*(E)$ is flat over E_* and M be a $E_*(E)$-comodule. Consider the diagram

(2.1)



of spectra and maps such that every triangle is a cofibering, and that the compositions $i_{k+1}j_k (k \in \boldsymbol{Z})$ yield the long exact sequence

$$E_*(F) \xrightarrow{i_{1*}} E_*(F_1) \xrightarrow{i_{2*}j_{1*}} E_*(F_2) \longrightarrow \tag{2.2}$$

with $\operatorname{Ker} i_{1*} = M$. The maps $k_i: X_{i+1} \to X_i$ induce the map $\kappa_n: \lim X_i \to X_n$ by the canonical projection. Let F_n denote the kernel of κ_n. Then we have a filtration

$$\lim X_i = F_0 \supset F_1 \supset \cdots .$$

Now suppose that

$$\bigcap_k F_k = 0. \tag{2.3}$$

Then we have

THEOREM 2.4.

$$E_*(\lim X_n) = M.$$

PROOF. Apply the functor $E_*(-)$ to the diagram (2.1), and we obtain an exact couple. This yields the spectral sequence

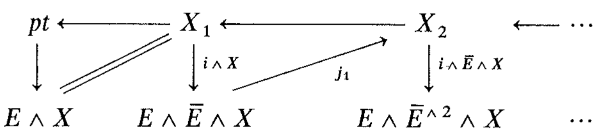
$$E_1^s = E_*(F_s) \Longrightarrow E_*(\lim X_n),$$

which converges by the assumption (2.3). Furthermore, d_1 is given by $i_k j_{k-1}$ ($j_0 = id$). Thus, E_2-term is the cohomology of the complex (E_1^*, d_1) which is the one given in (2.2). Since it is exact, the homology turns out to be

$$E_2^s = \begin{cases} M & s = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus this spectral sequence collapses and we have the result. q.e.d.

As an example, we have the Adams-Bousfield resolution which is given as follows: Let $\bar{E}$ denote the cofiber of the unit map $i: S^0 \to E$, and we have the cofibering $X = S^0 \wedge X \xrightarrow{i \wedge X} E \wedge X \to \bar{E} \wedge X$. Then we have the diagram:



Since E is a ring spectrum, we have the long exact sequence corresponding to

(2.2). Furthermore it is shown in [2] that the diagram above yields the filtration $F_0 \supset F_1 \supset \cdots$ satisfying (2.3). Thus this shows the converging spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*(E)}(E_*, E_*(E)) \Longrightarrow \pi_{t-s}(X_E^{\wedge}),$$

where $X_E^{\wedge} = \lim X_n$. This spectral sequence is called the generalized *Adams spectral sequence* based on E. In particular, if we take $E = BP$, we call it the *Adams-Novikov spectral sequence.*

§3. Smash decomposition of a ring spectrum

Let E be a ring spectrum with $E_*(E)$ being flat over E_*. Suppose that F is a spectrum such that $E_*(F)$ is a Hopf algebroid over E_* as well as a free left E_*-module.

LEMMA 3.1. *Suppose that $E_*(F)$ is a free left E_*-module. Then*

$$E \wedge F \simeq \vee E.$$

PROOF. Put $E_*(F) = E_*\{g_\lambda \mid \lambda \in \Lambda\}$, a free E_*-module over the generators $\{g_\lambda\}$. Suppose that a map $f_\lambda\colon S^0 \to E \wedge F$ represents the generator g_λ. Here note that everything is considered up to suspension. Then we have a map $f_\lambda\colon E \to E \wedge S^0 \xrightarrow{E\wedge f_\alpha} E \wedge E \wedge F \xrightarrow{\mu\wedge F} E \wedge F$ for the multiplication $\mu\colon E \wedge E \to E$, which induces the E_*-module map $\varphi_\lambda\colon E_* \to E_*(F)$ such that $\varphi_\lambda(1) = g_\lambda$. Thus we obtain a map

$$\vee f_\lambda\colon \bigvee_{\lambda\in\Lambda} E \longrightarrow E \wedge F,$$

which induces an isomorphism on homotopy $\pi_*(-)$. Now use the J.H.C. Whitehead theorem to get the desired homotopy equivalence. q.e.d.

LEMMA 3.2. *Suppose that $E_*(F)$ is a free right E_*-module. Then there exists a map $\varphi\colon E \wedge F \wedge E \to E \wedge F$, which represents the right action. That is to say, the right action xy for $x \in E_*(F)$ and $y \in E_*$ is given by the composition $S^0 \xrightarrow{x\wedge y} E \wedge F \wedge E \xrightarrow{\varphi} E \wedge F$.*

PROOF. Let $\{\overline{g_\lambda}\}$ denote the free generators of the right E_*-module $E_*(F)$. Then in the same way as above, we obtain an equivalence

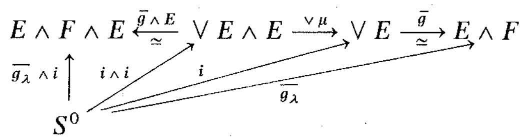
$$\bar{g} = \vee \bar{g}_\lambda\colon \bigvee_\lambda E \simeq E \wedge F.$$

Now define the map φ by the composition

$$E \wedge F \wedge E \xrightarrow{\bar{g}^{-1}\wedge E} \bigvee_\lambda E \wedge E \xrightarrow{\vee\mu} \bigvee_\lambda E \xrightarrow{\bar{g}} E \wedge F.$$

Then for a generator $\overline{g_\lambda}$ and an element $y \in E_*$, $\overline{g_\lambda}\, y$ is represented by the composition $\varphi(\overline{g_\lambda} \wedge y)$. In fact, $\varphi(\overline{g_\lambda} \wedge 1) = \overline{g_\lambda}$ as in the following commutative diagram, since

$1 \in E_*$ is represented by the unit $i: S^0 \to E$.



q.e.d.

Consider the homology theories $h_*(-) = E_*(F \wedge -)$ and $k_*(-) = E_*(F) \otimes_{E_*} E_*(-)$, and the natural transformation $\psi: k_* \to h_*$ defined by $\psi_X(x \otimes y) = \varphi(x \wedge y)$ for the map φ in Lemma 3.2, where $\psi_X: k_*(X) \to h_*(X)$. Then, ψ_{S^0} turns out to be an isomorphism, and so is ψ_X for any spectrum X. In particular, we have

(3.3) $$E_*(F \wedge E) = E_*(F) \otimes_{E_*} E_*(E), \quad E_*(F \wedge F) = E_*(F) \otimes_{E_*} E_*(F).$$

Now consider a cobar resolution

$$E_* \longrightarrow E_*(F) \longrightarrow E_*(F)^{\otimes 2} \longrightarrow E_*(F)^{\otimes 3} \longrightarrow \cdots.$$

Assume that $E_*(F) \to E_*(E)$ is an inclusion as comodules. Put $A = E_*(E) \square_{E_*(F)} E_*$. Applying $A \otimes_{E_*} -$ to the resolution, we have a resolution

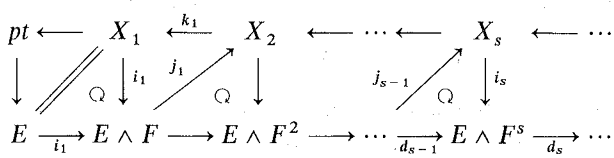
$$A \longrightarrow E_*(F) \longrightarrow E_*(E) \otimes_{E_*} E_*(F) \longrightarrow E_*(E) \otimes_{E_*} E_*(F)^{\otimes 2} \longrightarrow \cdots.$$

Note that $[E \wedge F^k, E \wedge F^{k'}] \cong \mathrm{Hom}_{E_*(E)}(E_*(E \wedge F^k), E_*(E \wedge F^{k'}))$. Therefore, by (3.3), we have a sequence

$$E \longrightarrow E \wedge F \longrightarrow E \wedge F \wedge F \longrightarrow \cdots,$$

whose E_*-homology is the above resolution. We call this sequence of spectra a *geometric resolution.*

PROPOSITION 3.4. *Suppose that* $\mathrm{Ext}_{E_*(F)}^{n+1, n-1}(E_*, E_*) = 0$ *for* $n < s$. *Then we have the following exact couple*:



in which the bottom sequence is the geometric resolution.

PROOF. We construct this by the induction on s. For $s = 1$, we just put $X_1 = E$. Suppose that we have the exact couple up to s. Apply $G^n(-) = [-, E \wedge F^{s+1}]_{-n}$ to the exact couple and obtain the algebraic exact couple which gives rise to the spectral sequence

$$E_1^t = G^*(E \wedge F^t) \Longrightarrow G^*(X_s).$$

By this, we obtain

$$G^0(X_s) = \mathrm{Ker}\, d_{s-1}^* \oplus E_{s-1}(F^{s+1}),$$

in which $\mathrm{Ker}\, d_{s-1} = \mathrm{Im}\, j_{s-1}^*$. Take $d_s i_s \in G^0(X_s)$. Then $j_{s-1}^*(d_s i_s) = d_s d_{s-1} = 0$, and so $d_s i_s = o_s k$ for some $o_s \in E_{s-1}(F^{s+1})$. Here $k = k_1 k_2 \cdots k_{s-1}$. Since k_* is monomorphic, $d_{s-1}(o_s k) = d_{s+1} d_s i_s = 0$ implies $o_s \in \mathrm{Ext}_{E_*(F)}^{s+1,s-1}(E_*, E_*) = 0$. Thus $o_s \in \mathrm{Im}\, d_s$. Put $o_s = d_s o_s'$. Define $i_s' = i_s - o_s' k$. Then $i_s' j_{s-1} = d_{s-1}$, and $d_s i_s' = d_s i_s - o_s k = 0$. Now define X_{s+1} to be a cofiber of i_s', and we have the case for $s+1$. q.e.d.

THEOREM 3.5. *If* $\mathrm{Ext}_{E_*(F)}^{n+1,n-1}(E_*, E_*) = 0$ *for* $n \geq 2$, *then we have a spectrum* X *such that*

$$E = F \wedge X.$$

PROOF. Put $X = \lim X_s$ and consider the spectral sequence obtained by applying $F_*(-)$ to the exact couple. Since $F_*(E \wedge Y) = E_*(F) \otimes_{E_*} E_*(Y)$, the E_1-term yields the cobar resolution over E_*. Therefore, we have

$$F_*(X) = E_*.$$

q.e.d.

Note that there exist maps $i: F \subsetneq E$ (by the assumption) and $k: X \to X_1 = E$ which yields the homotopy equivalence $\mu(i \wedge k): F \wedge X \to E$.

REMARK. This will hold true for E, F and G such that E and F are ring spectra, and $E_*(F)$ and $E_*(G)$ are Hopf algebroids with $G \subsetneq F$ inducing the map of Hopf algebroids. In the later, we consider the case for $E = E(2)$, $F = E(2)/(2)$ and $G = D(A_1)$ at the prime 2.

§4. Application

As is remarked in the previous section, we have consider for $E(2)$, $E(2)/(2)$ and $D(A_1) \wedge M_2$ at the prime 2. Here $D(A_1)$ is the cofiber of the essential map

$$h_{20}: \Sigma^5 M_\eta \wedge M_\nu \longrightarrow M_\eta \wedge M_\nu,$$

where M_α denotes the mapping cone of the elements $\alpha \in \pi_*(S^0)$. The existence of h_{20} is shown in [5]. Then our start line is

(4.1) *There is a map* $D(A_1) \subsetneq E(2)$, *that induces* $E(2)_*(D(A_1)) \to E(2)_*(E(2))$ *the map of coalgebras.*

PROOF. Note that $E(2) = v_2^{-1} BP\langle 2\rangle$ and so $E(2)/2 = v_2^{-1} BP\langle 2\rangle \wedge M_2$. Consider the cofiber sequence $S^1 \xrightarrow{\eta} S^0 \to M_\eta$. Then the unit map $i: S^0 \to BP\langle 2\rangle$ is extended to $\tilde{i}: M_\eta \to BP\langle 2\rangle$, since the composition $\eta i = 0: S^1 \to BP\langle 2\rangle$, for odd t. We also have the cofiber sequence $\Sigma^3 M_\eta \xrightarrow{\nu} M_\eta \to M_\eta \wedge M_\nu$, in which $\Sigma^3 M_\eta = S^3 \cup e^5$. Therefore, we have $[\Sigma^3 M_\eta, BP\langle 2\rangle]_0 = 0$ and so the map $\tilde{i}$ is extended to $M_\eta \wedge M_\nu \to BP\langle 2\rangle$.

In the same way, we have $D(A_1) \to BP\langle 2\rangle$, and compose with $BP\langle 2\rangle \subsetneq E(2)$. By this construction, we see that the induced map $E(2)_*(D(A_1)) \to E(2)_*(E(2))$ is an inclusion, which is map of comodules by the universal coefficient isomorphism

$$[D(A_1) \wedge M_2, E(2)/(2)]_* \cong \mathrm{Hom}^*_{E(2)_*(E(2))}(E(2)_*(D(A_1) \wedge M_2), E(2)_*(E(2)/(2))).$$

The coalgebra structure of $E(2)_*(D(A_1))$ is now read off from this inclusion. q.e.d.

COROLLARY 4.2. *There exists a spectrum EO_2 at the prime 2 such that*

$$EO_2 \wedge D(A_1)/2 = E(2)/2.$$

This EO_2 gives a counter example to the result of [4] which states that $\mathscr{A}//\mathscr{A}_2$ is not representable. The existence of EO_2 is stated in [5] but their proof seems to have some subtle gaps. Here they use the notation EO_2 by the analogy of BO. In fact, BO is given as a fixed point set of the $\boldsymbol{Z}/2$-action on BU, and EO_2 is defined as a homotopy fixed point set of an action on E_2, which is a completion of $E(2)$. Then using the spectral sequence, we obtain

$$E_2(EO_2 \wedge M_v \wedge A_1) = K(2)_*[h_{20}]. \tag{4.3}$$

In fact,

$$\begin{aligned} E_2(EO_2 \wedge M_v \wedge A_1) &= \mathrm{Ext}_{E(2)_*(E(2))}(E(2)_*, E(2)_*(EO_2 \wedge M_v \wedge A_1)) \\ &= \mathrm{Ext}_{E(2)_*(E(2))}(E(2)_*, E(2)_*(E(2)) \square_C E(2)_*(M_v \wedge A_1)) \\ &= \mathrm{Ext}_C(E(2)_*, E(2)_*(M_v \wedge A_1)) \\ &= \mathrm{Ext}_D(K(2)_*, K(2)_*) \\ &= K(2)_*[h_{20}], \end{aligned}$$

where $C = E(2)_*[t_1, t_2]/(t_1^4, t_2^2)$ and $D = K(2)_*[t_2]/(t_2^2)$ for the Morava K-theory $K(2)_* = F_2[v_2, v_2^{-1}]$.

In the same way as above, we also have W which relates to EO_2 at the prime 3:

COROLLARY 4.4. *There exists a spectrum W at the prime 3 such that*

$$W \wedge X = E(2),$$

for $X = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$.

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