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On the discrepancy of (V, μ)-uniformly distributed sequence

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1 Abstract

[GK] and [G] generalized the discrepancy of weighted uniformly distributed sequence using the regular summation method. We obtained the upper and lower bounds estimation of the generalized discrepancy (called V-discrepancy) of a weighted uniformly distributed sequence having a continuous distribution function.

For the computation of the discrepancy D_N^* of ordinary uniformly distributed sequence, there exists a simple algorithm ([N; Chap. 2 Th. 1.4). [NT] obtained the algorithm which gives the discrepancy of the weighted uniformly distributed sequence.

In this paper, we have the similar algorithm for the discrepancy of the (V, μ)-uniformly distributed sequence ([G], [GK]), and the relation between the discrepancy of the weighted uniformly distributed sequence and that of the ordinary uniformly distributed sequence. Our methods mainly owe to [N1][N2][NK] and [NT].

2 Definitions and Notations

Let V_N be a regular summation method of the sequence $g(1), g(2), \cdots, g(N)$. Let μ be a Borel probability measure on an arbitrary compact metric space X.

We assume that $X = [0, 1]$ and $F(x) = \mu([0, x))$ is a continuous function.

DEFINITION 1. *Let $(g(n))$ be a sequence and $V_N(g(n))$ be a summation of $g(1), \cdots, g(N)$. If $\lim_{N\to\infty} V_N(g(n)) = \sigma$, then $(g(n))$ is said to be V-summable to σ.*

DEFINITION 2 ([GK]). *The sequence $(g(n))$ is said to be (V, μ)-u.d. if for all intervals $J \in X$, we have*

$$\lim_{N\to\infty} V_N(C_J(g(n))) = \int_X C_J d\mu,$$

where C_J denotes the characteristic function of J.

DEFINITION 3 ([GK]). *Let $(g(n))$ be a sequence of real numbers and $J = [\alpha, \beta) \subseteq [0, 1]$. The number*

$$D_N = \sup_J \left| V_N(C_J(g(n))) - \int_X C_J d\mu \right|,$$

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and

$$D_N^* = \sup_{0 \le \alpha < 1} \left| V_N(C_{[0,\alpha)}(g(n))) - \int_X C_{[0,\alpha)} d\mu \right|,$$

are called the (V, μ)*-discrepancy.*

Setting $V_N(\bullet) = \frac{1}{s(N)} \sum_{n=1}^{N} p(n) \bullet$, we have the ordinary weighted uniform distribution i.e. $(p(n), \mu) - u.d.$.

DEFINITION 4. *For* $0 \le \alpha < \beta \le 1$, *let* $A([\alpha, \beta): g(n))$ *be the number of terms* $g(n)$, $1 \le n \le N$, *for which the fractional part of* $g(n) \in [\alpha, \beta)$. *We set*

$$D_N^{(p)} = \sup_{\alpha,\beta} \left| \frac{1}{S(N)} \sum_{n=1}^{N} p(n) A([\alpha, \beta): g(n)) - (\beta - \alpha) \right|,$$

$$D_N = \sup_{\alpha,\beta} \left| \frac{1}{N} \sum_{n=1}^{N} A([\alpha, \beta): g(n)) - (\beta - \alpha) \right|,$$

$$D_N^{(p,\mu)} = \sup_{\alpha,\beta} \left| \frac{1}{S(N)} \sum_{n=1}^{N} p(n) A([\alpha, \beta): g(n)) - (F(\beta) - F(\alpha)) \right|,$$

$$D_N^{(1,\mu)} = \sup_{\alpha,\beta} \left| \frac{1}{N} \sum_{n=1}^{N} A([\alpha, \beta): g(n)) - (F(\beta) - F(\alpha)) \right|.$$

3 Theorems

By the Definition 3, we obtain the following Theorem 1 immediately.

THEOREM 1. *The sequence* $(g(n))$ *is* (V, μ)*-u.d. mod* 1 *if and only if*

$$\lim_{N \to \infty} D_N = 0.$$

REMARK. $D_N^* \le D_N \le 2D_N^*$.

[NT] obtained the following: for $0 \le x_1 \le \cdots \le x_N$,

$$D_N^*(P, x_n) = \max_{1 \le n \le N} \max \left(\left| x_n - \frac{1}{P(N)} \sum_{j=1}^{n} p_j \right|, \left| x_n - \frac{1}{P(N)} \sum_{j=1}^{n-1} p_j \right| \right)$$

$$= \max_{1 \le n \le N} \left(\frac{p_n}{2P(N)} + \left| x_n - \frac{1}{P(N)} \sum_{j=1}^{n-1} p_j - \frac{p_n}{2P(N)} \right| \right) \ge \frac{\max(p_1, \cdots, p_N)}{2P(N)}.$$

The following theorem is a generalization of the result of [NT].

THEOREM 2. (see [N1], [N2], [NT]) *Let* $0 \le g(1) \le g(2) \le \cdots \le g(N) \le 1$ *be* N *numbers in* $[0, 1]$. *Then the discrepancy* D_N^* *is given by*

$$D_N^* = \max_{1 \le i \le N} \max (|V_N(C_J(g(n))) - F(g(i))|, |V_N(C_J(g(n))) - F(g(i+1))|).$$

PROOF. We set $g(0) = 0$, $g(N+1) = 1$, $J = [0, \alpha)$ and $F(\alpha) = \int_X c_J d\mu$ for convinience. For the distinct values of the numbers $g(i)$, $0 \le i \le N+1$, we have

$$D_N^* = \max_{0 \le i \le N, g(i) < g(i+1)} \sup_{g(i) < \alpha \le g(i+1)} |V_N(C_J(g(n)) - F(\alpha)|.$$

Whenever $g(i) < g(i+1)$, the function of α, $|V_N(C_J(g(n)) - F(\alpha)|$ attains its maximum in $[g(i), g(i+1)]$ at one of the end point of the interval. Therefore

$$D_N^* = \max_{0 \le i \le N, g(i) < g(i+1)} \max (|V_N(C_J(g(n)) - F(g(i))|, |V_N(C_J(g(n)) - F(g(i+1))|). \quad (1)$$

Now we show that we may drop the restriction $g(i) < g(i+1)$ in the first maximum. Suppose we have $g(i) < g(i+1) = \cdots = g(i+s) < g(i+s+1)$ with some $s \ge 2$. The indices not admitted in the first maximum in (1) are the integers $i+j$ with $1 \le j \le s-1$. We prove that for $1 \le j \le r-1$,

$$|V_N(C_J(g(n))) - F(g(i+j))| \quad \text{and} \quad |V_N(C_J(g(n))) - F(g(i+j+1))|,$$

where $\widetilde{V_N}$ is the summation for $g(0), g(1), \cdots, g(i+j)$, which are excluded in (1), are dominated by numbers appear in (1).

For $1 \le j \le s-1$, we get by the same reasoning

$$\begin{aligned} |\widetilde{V_N}(C_J(g(n))) - F(g(i+j))| &= |\widetilde{V_N}(C_J(g(n))) - F(g(i+1))| \\ &< \max (|V_N^1(C_J(g(n))) - F(g(i+1))|, |V_N^2(C_J(g(n))) - F(g(i+1))|), \\ &= \max (|V_N^1(C_J(g(n))) - F(g(i+1))|, |V_N^2(C_J(g(n))) - F(g(i+s))|), \end{aligned}$$

where V_N^1 is the summation of $g(0), \cdots, g(i)$ and V_N^2 is the summation of $g(0), \cdots, g(i), \cdots, g(i+s)$ and both numbers in the last maximum occure in (1).

Thus, considering

$$|\widetilde{V_N}(C_J(g(n))) - F(g(i+j+1))|, \qquad 1 \le j \le s-1,$$

we have the desired result.

We have the following theorem from definiton 4.

THEOREM 3. *We have*

$$D_N^{(p)} \le \frac{1}{S(N)} P_N,$$

$$D_N^{(p,\mu)} \le \frac{1}{S(N)} P_N^{(1,\mu)} \le$$

$$\leq \frac{1}{S(N)}\{P_N + (N_p(N) + \sum_{j=1}^{N-1} j\,|p(j) - p(j-1)|) \sup_{\alpha,\beta}((\beta-\alpha) - (F(\beta) - F(\alpha)))\},$$

$$D_N \leq \frac{1}{N} Q_N^{(p)} \quad \textit{and} \quad D_N^{(1,\mu)} \leq \frac{1}{N} Q_N^{(p,\mu)},$$

where

$$P_N = D_N N p(N) + \sum_{j=1}^{N-1} j\,|p(j) - p(j+1)|\,D_j,$$

$$P_N^{(1,\mu)} = D_N^{(1,\mu)} N p(N) + \sum_{j=1}^{N-1} j\,|p(j) - p(j+1)|\,D_j^{(1,\mu)},$$

$$D_n^{(1,\mu)} \leq D_n + \sup_{\alpha,\beta}((\beta-\alpha) - (F(\beta) - F(\alpha))),$$

$$Q_N^{(p)} = D_N^{(p)} \frac{S(N)}{p(N)} + \sum_{n=1}^{N-1} S(n) \left| \frac{1}{p(n)} - \frac{1}{p(n-1)} \right| D_n^{(p)},$$

$$Q_N^{(p,\mu)} = D_N^{(p,\mu)} \frac{S(N)}{p(N)} + \sum_{n=1}^{N-1} S(n) \left| \frac{1}{p(n)} - \frac{1}{p(n-1)} \right| D_n^{(p,\mu)}.$$

PROOF. We set

$$x_n = \frac{1}{n} \sum_{j=1}^{n} A([\alpha, \beta) : g(j)) - (\beta - \alpha),$$

then we have $|x_n| \leq D_n$ for all n.
Since $A([\alpha, \beta) : g(n)) - (\beta - \alpha) = N x_N - (N-1) x_{N-1}$, we obtain by Abel's method,

$$\frac{1}{S(N)} \sum_{n=1}^{N} p(n) A([\alpha, \beta) : g(n)) - (\beta - \alpha) = \frac{1}{S(N)} \sum_{n=1}^{N} p(n)(n x_n - (n-1) x_{n-1})$$

$$\leq \frac{1}{S(N)}(D_N N p(N) + (N-1) D_{N-1} |p(N-1) - p(N)|$$

$$+ (N-2) D_{N-2} |p(N-2) - p(N-1)| + \cdots + 1 \cdot D_1 |p(1) - p(2)|) = \frac{1}{S(N)} P_N.$$

By the same way, we obtain

$$\frac{1}{S(N)} \sum_{n=1}^{N} p(n) A([\alpha, \beta) : g(n)) - (F(\beta) - F(\alpha))$$

$$\leq \frac{1}{S(N)}(D_N^{(1,\mu)} N p(N) + (N-1) D_{N-1}^{(1,\mu)} |p(N-1) - p(N)| + \cdots) = \frac{1}{S(N)} P_N^{(1,\mu)}.$$

Therefore

$$D_N^{(p,\mu)} \le \frac{1}{S(N)} P_N^{(1,\mu)}.$$

Moreover we have

$$D_N^{(1,\mu)} \le D_N + \sup_{\alpha,\beta} ((\beta - \alpha) - (F(\beta) - F(\alpha))),$$

because

$$\frac{1}{N} \sum_{n=1}^{N} A([\alpha, \beta): g(n)) - (F(\beta) - F(\alpha))$$

$$= \frac{1}{N} \sum_{n=1}^{N} A([\alpha, \beta): g(n)) - (\beta - \alpha) + (\beta - \alpha) - (F(\beta) - F(\alpha))$$

$$\le D_N + \sup_{\alpha,\beta} ((\beta - \alpha) - (F(\beta) - F(\alpha))).$$

Thus

$$D_N^{(p,\mu)} \le \frac{1}{S(N)} P_N^{(1,\mu)}$$

$$\le \frac{1}{S(N)} (P_N + (Np(N) + \sum_{j=1}^{N-1} j\,|p(j) - p(j-1)|) \cdot \sup_{\alpha,\beta} ((\beta - \alpha) - (F(\beta) - F(\alpha))).$$

We set

$$y_n = \frac{1}{s(n)} \sum_{j=1}^{n} p(j) A([\alpha, \beta): g(j)) - (\beta - \alpha).$$

Then $|y_n| \le D_n^{(p)}$ and

$$A([\alpha, \beta): g(n)) - (\beta - \alpha) = \frac{1}{p(n)} (S(n) y_n - S(n-1) y_{n-1}).$$

So by Abel's method

$$\frac{1}{N} \sum_{n=1}^{N} A([\alpha, \beta): g(n)) - (\beta - \alpha) = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{p(n)} (S(n) y_n - S(n-1) y_{n-1})$$

$$= \frac{1}{N} \left(\frac{S(N)}{p(N)} y_N + S(N-1) \left(\frac{1}{p(N-1)} - \frac{1}{p(N)} \right) y_{N-1} \right.$$

$$\left. + S(N-2) \left(\frac{1}{p(N-2)} - \frac{1}{p(N-1)} \right) y_{N-2} + \cdots + S(1) \left(\frac{1}{p(1)} - \frac{1}{p(2)} \right) y_1 \right) \le \frac{1}{N} Q_N^{(p)}.$$

Thus

$$D_N \leq \frac{1}{N} Q_N^{(p)}.$$

As the same way,

$$D_N^{(1,\mu)} \leq \frac{1}{N} Q_N^{(p,\mu)}.$$

Therefore we obtain theorem 3.

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