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Discrepancy inequality of LeVeque

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1 Introduction

In [4] LeVeque obtained an upper bound of the discrepancy in terms of the exponential sums different from [1].

In [2], using summation method, we defined the discrepancy (called V-discrepancy) of a weighted uniformly distributed sequence having a continuous distribution function. In [2: Theorem 8], we mentioned the LeVeque's type upper bound without proof.

In this paper, we shall show the detailed proof. Our methods mainly owe to [3: Theorem 1.4, 2.4 and 2.5 of Chapter 2] and [4].

2 Definitions, Notations and Assertion

Let V_N be a regular summation method of the sequence $g(1), g(2), \cdots, g(N)$. Let μ be a Borel probability measure on $X = [0, 1]$, $F(x) = \mu([0, x))$ be a continuous function and $\{x\}$ mean the fractional part of x.

Definition 1. *Let $(g(n))$ be a sequence and $V_N(g(n))$ be a regular summation method of $g(1), \cdots, g(N)$. If $\lim_{N\to\infty} V_N(g(n)) = \sigma$, then $(g(n))$ is said to be V-summable to σ.*

Definition 2. *The sequence $(g(n))$ is said to be (V, μ)-u.d. mod 1 if for all intervals $J \in X$, we have*

$$\lim_{N\to\infty} V_N(C_J(g(n))) = \int_X C_J d\mu,$$

where C_J denotes the characteristic function of J.

Definition 3. *Let $(g(n))$ be a sequence of real numbers and $J = [\alpha, \beta) \subseteq [0, 1]$. The number*

$$D_N = \sup_J \left| V_N(C_J(g(n))) - \int_X C_J d\mu \right|,$$

is called the (V, μ)-discrepancy of $g(n))$.

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Setting $V_N(\bullet) = \frac{1}{s(N)} \sum_{n=1}^{N} p(n) \bullet$, we have the ordinary weighted uniform distribution i.e. $(p(n), \mu) - u.d.$.

By Definition 3, we obtain the following

Assertion 1. *The sequence $(g(n))$ is (V, μ)-u.d. mod 1 if and only if*

$$\lim_{N \to \infty} D_N = 0.$$

3 Lemma and Theorems

Theorem 1. *If we set, for $g(n) \in [0, 1]$, $n = 1, 2, \cdots$,*

$$\Delta_N(y) = V_N(C_{[0,y)}(g(n))) - F(y),$$

then

$$\int_0^1 \Delta_N(y)^2 dy = (V_N(g(n) - G))^2 + \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| V_N(e^{-2\pi i h g(n)}) - \int_0^1 e^{-2\pi i h y} dF(y) \right|^2,$$

where $G = \int_0^1 y dF(y)$.

Proof. We remark that $\Delta_N(y)$ is a piecewise continuous function in $[0, 1]$ with finitely many discontinuities at $y = g(1), \cdots, g(N)$. Moreover, we have $\Delta_N(0) = \Delta_N(1)$.

We expand $\Delta_N(y)$ into a Fourie series $\sum_{h=-\infty}^{\infty} a_h e^{2\pi i h x}$ which will represent $\Delta_N(y)$ apart from finitely many points. So we have

$$a_h = \int_0^1 \Delta_N(y) e^{-2\pi i h y} dy,$$

and

$$\begin{aligned} a_0 = \int_0^1 \Delta_N(y) dy &= V_N\left(\int_0^1 C_{[0,y)}(g(n))\right) - \int_0^1 F(y) dy \\ &= V_N(1 - g(n)) - (1 - G) = -V_N(g(n) - G), \end{aligned}$$

where $G = \int_0^1 y dF(y)$.

For $h \neq 0$, we obtain

$$a_h = \int_0^1 \Delta_N(y) e^{-2\pi i h y} = V_N\left(\int_0^1 C_{[0,y)}(g(n)) e^{-2\pi i h y} dy\right) - \int_0^1 F(y) e^{-2\pi i h y} dy$$

$$= V_N\left(\int_{g(n)}^{1} e^{-2\pi ihy}dy\right) - \left(\left[\frac{F(y)}{-2\pi ih}e^{-2\pi ihy}\right]_0^1 + \int_0^1 \frac{e^{-2\pi ihy}}{2\pi ih}dF(y)\right)$$

$$= \frac{1}{2\pi ih}\left(V_N(e^{-2\pi hg(n)}) - \int_0^1 e^{-2\pi ihy}dF(y)\right).$$

By Parseval's identity, we have

$$\int_0^1 \Delta_N^2(y) = a_0^2 + 2\sum_{h=1}^{\infty}|a_h|^2,$$

and the desired result follows immediately.

We have the following result (see [2]).

Lemma 1 [2: Theorem 5]. *Let F be a continuous distribution function. If $(g(n))$ is (V, μ)-u.d., then we have*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|V_N(e^{2\pi ihg(n)})|$$
$$+ \frac{4}{m+1}\int_0^1 (F(y) - y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy.$$

The following is an analogue of LeVeque's Inequality (cf. [2: Chap. 2 Rh. 2.4]).

Theorem 2 [2: Theorem 6]. *Under the $F(y) = \mu([0, y)) = y$, we have*

$$D_N \leq \left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}|V_N(e^{2\pi ihg(n)})|^2\right)^{\frac{1}{3}}.$$

Proof. We put

$$S_N = V_N(g(n) - G), \quad G = \int_0^1 ydF(y)$$

and

$$T_N(y) = V_N(C_{[0,y)}(g(n))) - F(y) + S_N \qquad \text{for } 0 \leq y \leq 1.$$

The function is monotonely decreasing except finite positive jumps at $g(n)$, $n = 0, 1, \cdots N$. Since $T_N(0) = S_N = T_N(1)$, we can extend $T_N(y)$ to R with a period of 1.

Let α and β be numbers in $[0, 1]$, with $T_N(\alpha) > 0$ and $T_N(\beta) < 0$. Such numbers exist because

$$\int_0^1 T_N(y)dy = \int_0^1 V_N(C_{[0,y)}(g(n)))dy - \int_0^1 F(y)dy + S_N$$
$$= V_N\left(\int_0^1 C_{[0,y)}(g(n))dy\right) - \int_0^1 F(y)dy + S_N$$

$$= V_N(1-g(n)) - \left(1 - \int_0^1 ydF(y)\right) + S_N = V_N(G - g(n)) + S_N = 0.$$

In the interval $[\alpha, \alpha + T_N(\alpha)]$, the graph of $T_N(y)$ will not lie below the curve segment joining the points $(\alpha, T_N(\alpha))$ and $(\alpha + T_N(\alpha), 0)$.

By the periodicity of T_N, there exists $\beta_1 \in [\alpha, \alpha + 1]$ will $T_N(\beta_1) = T_N(\beta)$.

In the interval $[\beta_1 + T_N(\beta_1), \beta_1]$, the graph of T_N will not lie above the curve segment joining $(\beta_1 + T_N(\beta_1), 0)$ and $(\beta_1, T_N(\beta_1)$; therefore the graph of $|T_N|$ *will not lie below the curve segment joining* $(\beta_1 + T_N(\beta_1), 0)$ and $(\beta_1 - T_N(\beta_1))$. Moreover, the intervals $[\alpha, \alpha + T_N(\alpha)]$ and $[\beta_1 + T_N(\beta_1), \beta_1]$ can have at most one point in common, because of the properties that the graph of T_N satisfies there. Thus

$$\int_0^1 T_N^2(y)dy = \int_\alpha^{\alpha+1} T_N^2(y)dy \geq \int_\alpha^{\alpha+T_N(\alpha)} T_N^2(y)dy + \int_{\beta_1+T_N(\beta_1)}^{\beta_1} T_N^2(y)dy$$
$$\geq \int_{-T_N(\alpha)}^0 (-y)^2 dy + \int_0^{-T_N(\beta_1)} y^2 dy = \frac{1}{3} T_N(\alpha)^3 + \frac{1}{3}(-T_N(\beta_1))^3.$$

For non-negative real numbers r and s, we put $t = \frac{1}{2}(r+s)$ and $u = \frac{1}{2}(r-s)$. Then

$$r^3 + s^3 = (t+u)^3 + (t-u)^3 = 2t^3 + 6tu^2 \geq 2t^3 = \frac{1}{4}(r+s)^3.$$

Applying this inequality with $r = T_N(\alpha)$ and $s = -T_N(\beta)$, we have

$$\frac{1}{12}(T_N(\alpha) - T_N(\beta))^3 \leq \int_0^1 T_N^2(y)dy.$$

So

$$\frac{1}{12}(V_N(C_{[\beta,\alpha)}(g(n)) - (F(\alpha) - F(\beta)))^3 \leq \int_0^1 T_N(y)^2 dy \qquad \text{for all } \alpha, \beta.$$

Hence we obtain

$$\frac{1}{12} D_N^3 \leq \int_0^1 T_N^2(y)dy. \tag{1}$$

Now we compute the right-hand side of (1).

$$\int_0^1 T_N^2(y)dy = \int_0^1 (\Delta_N(y) + S_N)^2 dy = \int_0^1 \Delta_N(y)^2 dy + 2S_N \int_0^1 \Delta_N(y)dy + S_N^2$$
$$= \int_0^1 \Delta_N(y)^2 dy - 2S_N^2 + S_N^2 = \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} |V_N(e^{2\pi i h g(n)})|^2, \tag{2}$$

because of Theorem 1.

By (1) and (2), we obtain

$$D_N \leq \left(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} | V_N(e^{2\pi i h g(n)})|^2 \right)^{\frac{1}{3}}.$$

This completes the proof.

Corollary 1. *If* $(g(n))$, $g(n) \in [0, 1]$ *is a* (V, μ)*-u.d., then*

$$D_N \leq \left(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} | V_N(e^{2\pi i h F(g(n))})|^2 \right)^{\frac{1}{3}},$$

where $F(y) = \mu([0, y))$ *is continuous on* $0 \leq y \leq 1$.

Proof. Since $F(g(n))$ is a $(V, 1)$-u.d., we obviously obtain by applying the Theorem 2.

Theorem 3. *Under the condition such that* $F'(y) < \infty$ *exists for* $0 \leq y \leq 1$,

$$D_N \leq \left(\| F' \| \frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| V_N \left(e^{2\pi i h y g(n)} - \int_0^1 e^{2\pi i h y} dF(y) \right) \right|^2 \right)^{\frac{1}{3}},$$

where $\| F' \|$ *is a supremum norm of* F'.

Proof. We put

$$S_N = V_N(g(n) - G), \quad G = \int_0^1 y dF(y)$$

and

$$T_N(y) = V_N(C_{[0,y)}(g(n)) - F(y) + S_N) \qquad \text{for } 0 \leq y \leq 1.$$

The function is a monotone decreasing except finite positive jump at $g(n)$, $n = 0, 1, \cdots N$. Since $T_N(0) = S_N = T_N(1)$, we can extend $T_N(y)$ to R with a period of 1.

Let α and β be numbers from $[0, 1]$, with $T_N(\alpha) > 0$ and $T_N(\beta) < 0$. Such numbers exists because of $\int_0^1 T_N(y)dy = 0$, as in the proof of Theorem 2.

In the interval $[\alpha, \alpha + y_0]$ such that $F(y_0 + \alpha) = T_N(\alpha) + F(\alpha)$, we have

$$T_N(y) \geq T_N(\alpha) - F(y) + F(\alpha) > 0.$$

In the interval $[\beta + y_1, \beta]$ such that $F(\beta + y_1) = T_N(\beta) + F(\beta)$, we have

$$T_N(y) \leq T_N(\beta) - F(y) + F(\beta) < 0.$$

As the similar way of Theorem 2, we obtain

$$\int_0^1 T_N^2(y)dy \geq \int_{\alpha}^{\alpha + y_0} T_N^2(y)dy + \int_{\beta + y_1}^{\beta} T_N^2(y)dy$$

$$\geq \int_{\alpha}^{\alpha+y_0} (T_N(\alpha) - F(y) + F(\alpha))^2 dy + \int_{\beta+y_1}^{\beta} (T_N(\beta) - F(y) + F(\beta))^2 dy$$
$$= \int_{-T_N(\alpha)}^{0} t^2 \frac{dt}{F'(y)} + \int_{0}^{-T_N(\beta_1)} t^2 \frac{dt}{F'(y)}.$$

Since $F'(y) \leq \|F'\| = M$ by assumption, we have

$$\int_0^1 T_N^2(y)dy \geq \frac{1}{M}\left(\frac{1}{3} T_N(\alpha)^3 + \frac{1}{3}(-T_N(\beta_1))^3\right).$$

As the similar argument of Theorem 2, we have

$$\frac{1}{12M} D_N^3 \leq \int_0^1 T_N^2(y)dy. \tag{3}$$

Now we compute the right-hand side of (3).

$$\int_0^1 T_N^2(y)dy = \int_0^1 (\Delta_N(y) + S_N)^2 dy = \int_0^1 \Delta_N(y)^2 dy - 2S_N^2 + S_N^2$$
$$= \frac{1}{2\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| V_N\left(e^{-2\pi i h g(n)} - \int_0^1 e^{-2\pi i h y} dF(y)\right)\right|^2. \tag{4}$$

By (3) and (4), we obtain

$$D_N \leq \left(\|F'\| \frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \left| V_N\left(e^{2\pi i h g(n)} - \int_0^1 e^{2\pi i h y} dF(y)\right)\right|^2\right)^{\frac{1}{3}}.$$

This completes the proof.

Theorem A. [5: *p.* 226] *If $g'(t)$ is monotone and $|g'(t)| \leq 1 - \delta$ for some $\delta > 0$ in (a, b) and $p(t)$ is monotonically decreasing and differentiable, then*

$$\left|\sum_{a<n\leq b} p(n)e^{2\pi i g(n)} - \int_a^b p(t)e^{2\pi i g(t)}dt\right| \leq A_\delta \max_{a\leq t\leq b} p(t).$$

Theorem 4. *Let $0 < K(t) = \dfrac{g'(t)}{p(t)}$ be monotone and differentiable $\uparrow$ or $\downarrow$ $K > 0$. Let $g'(t)$ be monotonically decreasing and $|g'(t)| \leq 1 - \delta$ and $p(t)$ be monotonicallly decreasing to zero and differentiable. Then we have*

$$\limsup_{N\to\infty} |\sum_{n=1}^{N} p(n)e^{2\pi i g(n)}| = \infty.$$

Proof. Assume that $\limsup_{N\to\infty} |\sum_{n=1}^{N} p(n)e^{2\pi i g(n)}| = A < \infty$. There exists a sequence $(N_k)_{k=1}^{\infty}$ such that for sufficient large k,

$$\sum_{n=1}^{N_k-1} p(n)e^{2\pi i g(n)} = Ae^{i\theta_k} + o(1).$$

For any sufficiently large integer N, there exists a k such that $N_k \leq N < N_{k+1}$. Applying Theorem A, we have

$$\left|\sum_{n=a}^{N} p(n)e^{2\pi i g(n)} - \int_a^N p(t)e^{2\pi i g(t)}dt\right| \leq A_\delta \max_{a \leq t \leq N} p(t).$$

Thus, by using the estimation in the proof of [2: lemma 4],

$$\begin{aligned}\sum_{n=1}^{N} p(n)e^{2\pi i g(n)} &= \sum_{n=1}^{N_k-1} p(n)e^{2\pi i g(n)} + \sum_{n=N_k}^{N} p(n)e^{2\pi i g(n)} \\ &= \sum_{n=1}^{N_k-1} p(n)e^{2\pi i g(n)} + \int_{N_k}^{N} p(t)e^{2\pi i g(t)}dt + o(1) \\ &= Ae^{i\theta_k} + \frac{1}{2\pi i K}(e^{2\pi i g(N)} - e^{2\pi i g(N_k)}) + o(1),\end{aligned}$$

and

$$\sum_{n=1}^{N} p(n)e^{2\pi i g(n)} = \sum_{n=1}^{N_{k+1}} - \sum_{N+1}^{N_{k+1}} = Ae^{i\theta_{k+1}} + \frac{1}{2\pi i K}(e^{2\pi i g(N_{k+1})} - e^{2\pi i g(N)}) + o(1),$$

Therefore

$$0 = A(e^{i\theta_k} - e^{i\theta_{k+1}}) + \frac{1}{2\pi i K}(e^{2\pi i g(N)} - e^{2\pi i g(N_k)} - e^{2\pi i g(N_{k+1})}) + o(1).$$

This leads a contradiction. Thus we have $\limsup_{N\to\infty} |\sum_{n=1}^{N} p(n)e^{2\pi i g(n)}| = \infty$.

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