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A note on the chromatic convergence theorem

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§1. Introduction

Let $\mathcal{C} = \mathcal{C}_{(p)}$ denote the homotopy category of p-local spectra for a prime number p and $E(n)$, the Johnson-Wilson spectrum whose coefficient ring is $E(n)_* = Z_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$ for each n. Then we have the Bousfield localization functor $L_n: \mathcal{C} \to \mathcal{C}$ with respect to $E(n)_*$. Since $L_n L_m = L_m$ if $m \le n$, the natural transformation $\eta_m: id \to L_m$ yields another one $L_n \to L_m$. In particular, we have the sequence

$$L_0 \longleftarrow L_1 \longleftarrow \cdots \longleftarrow L_n \longleftarrow L_{n+1} \longleftarrow \cdots \longleftarrow id,$$

which is called the *chromatic tower.* Thus we can consider the homotopy inverse limit $\underleftarrow{\lim}\, L_n X$ on a spectrum X. Then Mike Hopkins and Doug Ravenel show the following chromatic convergence theorem:

THEOREM ([8, Th. 7.5.7]). *For a p-local finite CW-complex X, the chromatic tower converges in the sense that*

$$X \simeq \underleftarrow{\lim}\, L_n X.$$

In this paper, we will prove this by another method using results of [2].

§2. The Adams-Novikov spectral sequence

Let E be a ring spectrum. Then we have the cofiber sequence

$$S^0 \xrightarrow{\iota} E \longrightarrow \overline{E} \xrightarrow{k} S^1$$

for the unit map ι. Applying $- \wedge \overline{E}^s$ to it, we have another cofiber sequence

$$\overline{E}^s \xrightarrow{\iota} E \wedge \overline{E}^s \longrightarrow \overline{E}^{s+1} \xrightarrow{k} \Sigma^1 \overline{E}^s.$$

This gives rise to the exact couple

$$\pi_*(\overline{E}^s \wedge X) \xrightarrow{\iota_*} E_*(\overline{E}^s \wedge X) \longrightarrow \pi_*(\overline{E}^{s+1} \wedge X) \xrightarrow{k_*} \pi_{*-1}(\overline{E}^s \wedge X)$$

for a spectrum X. The E-*Adams spectral sequence for computing the homotopy groups*

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$\pi_*(X)$ is now defined to be the spectral sequence associated to the above exact couple.

Note that the pair $(E_*, E_*(E))$ is the Hopf algebroid with the canonical structure maps. Besides, if $E_*(E)$ is flat over E_*, the category of $E_*(E)$-comodules has enough injectives. So we can do homological algebra in the category. Let F be an E-module spectrum with structure map ν: $E\wedge F \to F$. Then we can define the map κ: $[X, F]_* \to \mathrm{Hom}_{E_*}(E_*(X), F_*)$ by $\kappa(x) = \nu_*(E\wedge x)_*$. Furthermore, suppose that $E_*(E)$ is flat over E_* and κ induces an isomorphism

$$\kappa : [Y, F]_* \cong \mathrm{Hom}_{E_*}(E_*(Y), F_*)$$

for a spectrum Y with $E_*(Y)$ is free over E_* and an E_*-module spectrum F. Then the E_2-term of the E-Adams spectral sequence is given by

$$E_2^{s,t}(X) = \mathrm{Ext}_{E_*(E)}(E_*, E_*(X)),$$

since the category of $E_*(E)$-comodules has enough injectives, and we have the change of rings theorem

$$\mathrm{Hom}_{E_*}(E_*(Y), F_*) \cong \mathrm{Hom}_{E_*(E)}(E_*(Y), E_*(F)).$$

As examples, we have the Brown-Peterson spectrum BP and the Johnson-Wilson spectrum $E(n)$. These spectra represent the homology theories $BP_*(-)$ and $E(n)_*(-)$ with coefficient rings

$$BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \cdots] \text{ and } E(n)_* = \boldsymbol{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}].$$

Here v_n is the Hazewinkel generators with $|v_n| = 2(p^n - 1)$. We call these spectral sequences the *Adams-Novikov spectral sequences.*

In this section we recall [2] its results. Following [2], we use the notation:

$$(A, \Gamma) = (BP_*, BP_*(BP)) \text{ and}$$
$$(A(n), \Gamma(n)) = (E(n)_*, (E(n)_*(E(n))),$$

in which $E(n)_*(E(n)) = E(n)_* \otimes_A \Gamma \otimes_A E(n)_*$.

Let (B, Σ) denote one of the Hopf algebroids Γ and $\Gamma(n)$'s. Consider an ideal I_n generated by p, $v_1, \cdots, v_{n-1}$. Then it is an invariant prime ideal. Now we define Σ-comodules B_i^j and LB_i^j inductively out of B. First put $B_i^0 = B/I_i$. Then it is a Σ-comodule with structure map induced from the right unit η_R: $B \to \Sigma$. We denote it also by η_R. Now suppose that B_i^j is defined and put $LB_i^j = v_{i+j}^{-1} B_i^j$. Then the canonical inclusion $B_i^j \to LB_i^j$ is a map of comodules (see [5]) and induces its cokernel B_i^{j+1} a comodule structure since Σ is flat over B. Thus we have completed the induction and B_i^j's are defined.

THEOREM 2.1. *For $i+j \le n$, we denote $t = n-i-j$. Then there exists an exact sequence*

$$\cdots \xrightarrow{\eta} \mathrm{Ext}^s_\Gamma(A, A^j_i) \longrightarrow \mathrm{Ext}^s_{\Gamma(n)}(A(n), A(n)^j_i) \longrightarrow \mathrm{Ext}^{s-t}_\Gamma(A, A^{n-i+1}_i) \xrightarrow{\eta} \mathrm{Ext}^{s+1}_\Gamma(A, A^j_i) \longrightarrow \cdots,$$

in $\mathrm{Ext}^{s-t}_\Gamma = 0$ *for* $s - t < 0$.

To prove this we also showed in [2] the following

THEOREM 2.2. *Let I_n denote the invariant prime ideal of $A = BP_*$.*

1) ([6]) *Let s and j be positive integers such that $s > n^2$ and $j < n$. Then we have an isomorphism*

$$\mathrm{Ext}^s_\Gamma(A, A^{n-j+1}_j) \cong \mathrm{Ext}^{s+n-j+1}_\Gamma(A, A^0_j).$$

2) *If $0 \le s \le n - j$ and $0 \le j \le n$, then*

$$\mathrm{Ext}^s_\Gamma(A, A/I_j) \cong \mathrm{Ext}^s_{\Gamma(n)}(A(n), A(n)/I_j).$$

3) ([5, Th. 2.10]) $\mathrm{Ext}^s_\Gamma(A, v_n^{-1}A/I_n) \cong \mathrm{Ext}^s_{\Gamma(n)}(A(n), A(n) \otimes_A v_n^{-1}A/I_n)$.

§3. Thick subcategory

In this section we will give another proof of the theorem by using the results given in [2]. Recall [3] first the definition of a thick subcategory. Let $\mathcal{FH}_{(p)}$ denote the category of p-local finite spectra and homotopy classes of maps. The subcategory $\mathcal{C}$ of $\mathcal{FH}_{(p)}$ is said to be *thick* if it satisfies the following:

(i) If two of three spectra in a cofiber sequence

$$X \xrightarrow{f} Y \longrightarrow C_f$$

are in $\mathcal{C}$, then so is the third; and

(ii) A retract of a spectrum in $\mathcal{C}$ is in $\mathcal{C}$.

Consider the Morava K-theory $K(n)$, which is characterized by its homotopy groups $\pi_*(K(n)) = Z/p[v_n, v_n^{-1}]$. Denote $\mathcal{F}_{p,n}$ the full subcategory of $\mathcal{FH}_{(p)}$ consisting of the spectra X with $K(n)_*(X) \neq 0$ and $K(n-1)_*(X) = 0$. Then we have the thick subcategory theorem:

THEOREM 3.1 (Hopkins). *Let $\mathcal{F}$ be a thick subcategory of $\mathcal{FH}_{(p)}$. Then $\mathcal{F}$ is either all of $\mathcal{FH}_{(p)}$, the trivial category or $\mathcal{F}_{p,n}$ for some n.*

THEOREM 3.2. *Let $\mathcal{F} = \{X \in \mathcal{FH}_{(p)} \mid X = \varprojlim L_n X\}$. Then $\mathcal{F}$ is a thick subcategory.*

PROOF. By the definition of the product of spectra, we see that products preserve cofiber sequences. Therefore the homotopy inverse limit preserves cofiber sequences. Thus the five lemma shows the condition (i).

Let $i: A \to X$ be an inclution and $r: X \to A$ a retraction. Then $ri = 1$. Since both of $\varprojlim$ and L_n are functors, we have maps

$$\varprojlim L_n(i) : \varprojlim L_n(A) \longrightarrow \varprojlim L_n(X) \quad \text{and} \quad \varprojlim L_n(r) : \varprojlim L_n(X) \longrightarrow \varprojlim L_n(A)$$

such that $\varprojlim L_n(r) \varprojlim L_n(i) = 1$. Consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ r\ } & A \\ & \searrow^{r'} & j \downarrow \uparrow ri' \\ & & \varprojlim L_n A, \end{array}$$

in which $i' = \varprojlim L_n(i)$ and $r' = \varprojlim L_n(r)$, and $i'j = i$ and $jr = r'$. Therefore we compute

$$ri'j = ri = 1 \quad \text{and} \quad jri' = r'i' = 1,$$

and so we have the homotopy equivalent

$$A = \varprojlim L_n A,$$

which shows $A \in \mathcal{F}$. q.e.d.

§4. The proof of the chromatic convergence theorem

By virtue of the thick subcategory theorem, it is sufficient to show that some spectrum of $\mathcal{F}_{p,n}$ satisfies $X = \varprojlim L_n X$. Consider an invariant regular sequence of the form:

$$J_j : p, v_1^{e_j}, \cdots, v_{j-1}^{e_{j-1}}.$$

Then Theorem 2.2 2) yields inductively an isomorphism

$$\lambda_* : \mathrm{Ext}_{\Gamma}^s(A, A/J_j) \cong \mathrm{Ext}_{\Gamma(n)}^s(A(n), A(n)/J_j), \tag{4.1}$$

for $0 \leq s \leq n-j$ and $0 \leq j \leq n$. It is known by [4] (see also [7]) that there exists a spectrum XJ_j whose BP_*-homology is BP_*/J_j for each j, and XJ_j belongs to $\mathcal{F}_{p,j}$.

LEMMA 4.2. *Let x and y be elements of the E_2-terms $\mathrm{Ext}_{\Gamma}^s(A, A/J_j)$ and $\mathrm{Ext}_{\Gamma(n)}^s(A(n)/J_j)$ of the Adams-Novikov spectral sequences for $\pi_*(XJ_j)$ and $\pi_*(L_n XJ_j)$, respectively, such that $\lambda(x) = y$. Further suppose that*

$$|x| \leq v(n-j).$$

Then if y is a permanent cycle, then so is x. Here $v(s)$ is an integer $p(p-1)s$ if s is even, and $p(p-1)(s-1)+2(p-1)$ if s is odd.

PROOF. Let $d_r: E_r^{s,t} \to E_r^{s+r,t+r-1}$ be the differential of the Adams-Novikov spectral sequence. Then we deduce, from the hypothesis on y, that $d_r(x)=0$ for $r \leq n-j-s$ inductively by the isomorphism (4.1) and the naturality of the differential. Furthermore, $Ext_\Gamma^{u,t}(A,A/J_j)=0$ if $t < v(u)$ by the vanishing line theorem of the spectral sequence. Therefore $d_r(x)=0$ for $r > n-j-s$. q.e.d.

Now we are ready to prove the chromatic convergence theorem.

PROOF OF THE THEOREM. Putting n to infinity, the isomorphisms (4.1) give rise to an isomorphism of the E_2-terms

$$\lambda_*: \mathrm{Ext}_\Gamma^s(A, A/J_j) \cong \varprojlim \mathrm{Ext}_{\Gamma(n)}^s(A(n), A(n)/J_j),$$

which shows the desired isomorphism of the homotopy groups

$$\lambda_*: \pi_*(XJ_j) \cong \varprojlim \pi_*(L_n XJ_j)$$

by the naturality of the differentials of the Adams-Novikov spectral sequence. Here $\Gamma(n)$ and $A(n)$ denote $E(n)_*(E(n))$ and $E(n)_*$, respectively. In fact, suppose that $\lambda_*(\xi)=0$ in $\pi_*(L_n XJ_j)$ and ξ is detected by $x \in \mathrm{Ext}_\Gamma^s(A,A/J_j)$. Then $\lambda_*(x)$ dies or is killed. If it dies,

$$d_r(\lambda_*(x)) = \lambda_*(d_r(x)) \neq 0,$$

and so $d_r(x) \neq 0$, which shows that x dies. If it is killed, then there exists an element y such that $d_r(y)=\lambda_*(x)$. Since λ_* is an epimorphism, we have an element w such that $\lambda_*(w)=y$. Therefore the naturality shows that $d_r(y)=x$ and x is killed. Thus we have seen that λ_*: $\pi_*(XJ_j) \to \varprojlim \pi_*(L_n XJ_j)$ is a monomorphism.

Now turn to show that λ_* is an epimorphism. Let $l_k: \varprojlim \pi_*(L_n XJ_j) \to \pi_*(L_k XJ_j)$ be the projection. For any element $x \in \varprojlim \pi_*(L_n XJ_j)$, there is an integer k such that $l_k(x) \neq 0$. Put filt $l_k(x)=m$, and take n so that $n > m+j+1$ and $|x| \leq v(n)$. Then $l_n(x)$ is represented by a permanent cycle x_n and there exists an element y in $\mathrm{Ext}_\Gamma^m(A, A/J_j)$ such that $\lambda_*(y)=x_n$ by the isomorphism (4.1). Now apply Lemma 4.2 to see that y is a permanent cycle. The composition $\pi_*(XJ_j) \to \pi_*(L_n XJ_j) \to \pi_*(\varprojlim L_n XJ_j)$ sends y to x by definition, which shows the map λ_* is epic. q.e.d.

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