J. Fac. Educ. Tottori Univ.
(Nat, Sci.), **42** (1993) 1–15

A generalized Adams spectral sequence based on a BP-associated spectrum

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(Received April 20, 1992)

§ 1. Introduction

In the stable homotopy theory, one of the main problems is to compute homotopy groups of a spectrum. We usually compute them after localizing by a prime number p, say, homotopy groups of p-localized spectrum. One method to compute them is the one using an E-Adams spectral sequence for a ring spectrum E, which is first introduced by F. Adams taking the Eilenberg-MacLane spectrum for the ring spectrum E. Then S. P. Novikov developed the theory to the Brown-Peterson spectrum BP and succeeded to compute the first line of its E_2-term, which is formerly known as the J-image for an odd prime. Here the Brown-Peterson spectrum BP is a ring spectrum and has the coefficient ring $\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$. After that, H. Miller, D. Ravenel and S. Wilson showed the computability of the BP-Adams spectral sequence. Their theory is not only showing the computability but also giving the deep insight in the stable homotopy theory. One of them is related to Bousfield's localization theory with respect to a spectrum. In view of these, D. Ravenel introduces the chromatic filtration in the stable homotopy category. This is closely related to the n-th Johnson-Wilson spectrum $E(n)$. This spectrum is characterized by the coefficient ring $E(0)_* = \boldsymbol{Q}$, and $E(n)_* = \boldsymbol{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$. These theories are developed under the hypothesis that $E(n)$ is a ring spectrum, which is stated in [6, Cor. 2. 16]. Unfortunately, T. Ohkawa found an error in their proofs that $E(n)$ is a ring spectrum. So far, we have no alternative proof for it, though Ohkawa proved that $v_n^{-1}BP$ is a ring spectrum (see [10] for the proof). Here we notice that $v_n^{-1}BP$ gives the same Bousfield class as $E(n)$, which means that there's no distinction between $v_n^{-1}BP$-localized spectra and $E(n)$-localized ones. Thus there seems everything goes well if we use $v_n^{-1}BP$ instead of $E(n)$. But when we compute the E_2-term of the generalized Adams spectral sequence computing homotopy groups of $L_{E(n)}X$, it is sometimes easier to use $E(n)$ than to do $v_n^{-1}BP$. Here we define $E(n)$-Adams spectral sequence although $E(n)$ is not a ring spectrum.

We call a spectrum E *BP-associated spectrum* if it satisfies $E_*(X) = E_*(S^0) \otimes_{BP_*} BP_*(X)$ for all spectra X. The typical example of BP-associated spectrum is $E(n)$ by Landweber's exact functor theorem ([4]). In this paper we give a generalized Adams spectral sequence based on BP-associated spectrum E and show that it converges to the homotopy groups of E-local spectrum for $E = E(n)$. That is,

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THEOREM. *For any prime p, we have the $E(n)$-Adams spectral sequence converging to the homotopy groups of $E(n)$-localized spectrum $L_{E(n)}X$ with* E_2-term

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(X)).$$

As is seen in the definition, we can define R-associated spectrum E for a ring spectrum R, and give a E-Adams spectral sequence. But it does not seem much interesting and so we give here only for $R = BP$.

In the next section we define BP-associated spectrum, and in §3, we introduce E-Adams spectral sequence, and then discuss the E-nilpotency in §4 and E-completion in §5. In the last section we prove the main theorem by using the results of M. Hopkins and D. Ravenel.

§2. *BP*-associated spectrum

Let BP denotes the Brown-Peterson spectrum at the prime p. Then the homotopy group $\pi_*(BP \wedge -) = BP_*(-)$ is well known to be a homology theory over the category of CW-spectra. The coefficient group $BP_* = BP_*(S^0)$ is a polynomial algebra $\mathbf{Z}_{(p)}[v_1, v_2, \cdots]$ over Hazewinkel's generators v_n and it acts on $BP_*(X)$ from the left by $vx = (\mu \wedge X)(v \wedge x)$ for $v \in BP_*$ and $x \in BP_*(X)$, where $\mu: BP \wedge BP \to BP$ denotes the multiplication of the ring spectrum BP. Since BP is a ring spectrum, we obtain a Hopf algebroid $(BP_*, BP_*(BP))$ by a well known fashion (*cf.* [1], [7]).

Let E_* be a BP_*-algebra such that the action of BP_* satisfies

$$vx = xv \quad \text{for} \quad x \in E_* \quad \text{and} \quad v \in BP_*$$

and the functor

$$E_*(-) = E_* \otimes_{BP_*} BP_*(-)$$

is a homology theory. Then we have a spectrum E representing the homology theory $E_*(-)$. In other words,

$$E_*(X) = \pi_*(E \wedge X)$$

for a spectrum X. We call the spectrum E *BP-associated spectrum.* This spectrum is constructed by Brown's representation theorem as follows: By the properties of S-duality, for a finite spectra X, we have

$$\begin{aligned} E^*(X) = E_{-*}(DX) &= E_* \otimes_{BP_*} BP_{-*}(DX) \\ &= E^* \otimes_{BP^*} BP^*(X), \end{aligned}$$

where DX denotes the S-dual of X. Note that every spectrum X has a direct system $\{X_\alpha\}$ consisting of finite spectra X_α such that $X = \mathrm{colim}\ X_\alpha$. We have Milnor's short exact sequence

$$0 \longrightarrow \lim{}^1 E^*(X_\alpha) \longrightarrow E^*(X) \longrightarrow \lim E^*(X_\alpha) \longrightarrow 0.$$

Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \lim^1 E^*(X_\alpha) & \longrightarrow & E^*(X) & \longrightarrow & \lim E^*(X_\alpha) & \longrightarrow 0 \\ & \| & & \downarrow & & \| & \\ 0 \longrightarrow & \lim^1 E^* \otimes_{BP^*} BP^*(X_\alpha) & \longrightarrow & E^* \otimes_{BP^*} BP^*(X) & \longrightarrow & \lim E^* \otimes_{BP^*} BP^*(X_\alpha) & \longrightarrow 0, \end{array}$$

and obtain an isomorphism

$$E^*(X) = E^* \otimes_{BP^*} BP^*(X) \qquad \text{for any } X$$

by the Five Lemma. Now we apply Brown's representation theorem to obtain the spectrum E such that

$$E^*(X) = [X, E]_*.$$

Here, the natural isomorphism $\varphi\colon [X, E]_* \to E^*(X) = E^* \otimes_{BP^*} BP^*(X)$ is defined by $\varphi(x) = x^*(u)$ for a unit $u \in E^*(E)$ that represents the homotopy class of the identity map $1_E\colon E \to E$. By this isomorphism, we have an element κ of $BP^*(E)$ such that $u = 1 \otimes \kappa$. Then we have

$$\varphi(x) = 1 \otimes x^*(\kappa) = 1 \otimes \kappa x. \tag{2.1}$$

Take $X = S^0$ and we have a map $i_E\colon S^0 \to E$ that represents the unit 1 of the algebra $E^* = E^*(S^0)$. That is,

$$\varphi(i_E) = 1 \otimes i$$

for the unit map $i\colon S^0 \to BP$ of the ring spectrum BP. Furthermore we have a map $\iota\colon BP \to E$ such that ι corresponds to the unit $1 \otimes 1 \in E^*(BP) = E^* \otimes_{BP^*} BP^*(BP)$ by the isomorphism φ, where 1 on the left is the unit of E^* as above and 1 on the right is the unit that represents the identity map $1\colon BP \to BP$. Then

$$\begin{aligned} 1 \otimes 1 = \varphi(\iota) &= 1 \otimes \iota^*(\kappa) \\ &= 1 \otimes \kappa\iota, \end{aligned} \tag{2.2}$$

and we have

LEMMA 2.3. *Let $i\colon S^0 \to BP$ be the unit map of the ring spectrum BP. Then we have*

$$\iota i = i_E.$$

PROOF. It suffices to show that $i^*\colon E^*(BP) \to E^*(S^0) = E^*$ maps ι to i_E. The map i^* is induced from another $i^*\colon BP^*(BP) \to BP^*(S^0) = BP^*$, which sends the unit 1, the homotopy class of the identity, to the unit 1, the homotopy class of i. So we compute $\varphi(i^*(\iota)) = (1 \otimes i^*)(\varphi(\iota))$ by the naturality of φ, which equals to $(1 \otimes i^*)(1 \otimes 1) = 1 \otimes i^*(1) = 1 \otimes i = \varphi(i_E)$. q.e.d.

LEMMA 2.4. $\iota\kappa = 1 \in [E, E]_*$.

PROOF. Since φ is an isomorphism, it is sufficient to show that $\varphi(\iota\kappa) = \varphi(1) = 1 \otimes \kappa$. Thus we compute

$$\begin{aligned}\varphi(\iota\kappa) &= 1 \otimes (\iota\kappa)^*(\kappa) = 1 \otimes \kappa\iota\kappa \\ &= 1 \otimes \kappa^*(\kappa\iota) = (1 \otimes \kappa^*)(1 \otimes \kappa\iota) \\ &= (1 \otimes \kappa^*)(1 \otimes 1) \text{ by (2.2)} \\ &= 1 \otimes \kappa.\end{aligned}$$

q.e.d.

LEMMA 2.5. *Let E be a BP-associated spectrum. Then the induced map $\iota_*: BP^*(X) \to E^*(X) = E^* \otimes_{BP^*} BP^*(X)$ is given by*

$$\iota_*(x) = 1 \otimes x.$$

PROOF. By definition, we compute $\iota_*(x) = 1 \otimes (\iota x)^*(\kappa) = x^*(1 \otimes \iota^*(\kappa)) = x^*(1 \otimes 1) = 1 \otimes x^*(1)$.

LEMMA 2.6. *Let E be a BP-associated spectrum. Then we have a map $\nu: E \wedge BP \to E$ such that*

$$\nu(i_E \wedge BP) = \iota.$$

PROOF. In $E^*(S^0) = E^* \otimes_{BP^*} BP^*(S^0)$, we see that

$$1 \otimes \kappa i_E = 1 \otimes \kappa\iota i = i^*(1 \otimes \kappa\iota) = i^*(1 \otimes 1) = 1 \otimes i$$

by Lemmas 2.3 and 2.5. Thus we compute

$$\begin{aligned}\varphi(i^*(\iota)) &= \iota_*(i) = 1 \otimes i && \text{(by Lemma 2.5)} \\ &= 1 \otimes \kappa i_E && \text{(by the above equation)} \\ &= 1 \otimes \kappa\iota\kappa i_E && \text{(by Lemma 2.4)} \\ &= \varphi(\iota\kappa i_E) && \text{(by (2.1))} \\ &= \varphi(i_E^* \kappa^*(\iota)).\end{aligned}$$

Note that these induced maps are those of $BP^*(BP)$-algebras, and ι is a generator of $E^*(BP)$. These arguments give the commutative diagram

$$\begin{array}{ccc} E^*(E) & \xrightarrow{i_E^*} & E^* \\ \uparrow{\scriptstyle \kappa^*} & \nearrow{\scriptstyle i^*} & \\ E^*(BP). & & \end{array}$$

We obtain an isomorphism $BP^*(X \wedge BP) = BP^*(X) \otimes_{BP^*} BP^*(BP)$ by comparing homology theories. We further see that

$$E^*(X) \otimes_{BP^*} BP^*(BP) = E^* \otimes_{BP^*} BP^*(X) \otimes_{BP^*} BP^*(BP)$$

$$= E^* \otimes_{BP^*} BP^*(X \wedge BP) = E^*(X \wedge BP).$$

Thus tensoring $BP^*(BP)$ over BP^* to the above diagram from the right gives another commutative diagram

$$\begin{array}{ccc} E^*(E \wedge BP) & \xrightarrow{(i_E \wedge BP)^*} & E^*(BP) \\ \uparrow{\scriptstyle (\kappa \wedge BP)^*} & \nearrow{\scriptstyle (i \wedge BP)^*} & \\ E^*(BP \wedge BP). & & \end{array}$$

Since BP is a ring spectrum, we have an epimorphism

$$(i \wedge BP)^*\colon BP^*(BP \wedge BP) \longrightarrow BP^*(BP).$$

Now tensoring E^* to it from the left induces another epimorphism

$$(i \wedge BP)^*\colon E^*(BP \wedge BP) \longrightarrow E^*(BP),$$

since $E^*(X) = E^* \otimes_{BP^*} BP^*(X)$ for any X.

The above diagram gives $(i \wedge BP)^* = (i_E \wedge BP)^*(\kappa \wedge BP)^*$ and so we have an epimorphism $(i_E \wedge BP)^*\colon E^*(E \wedge BP) \to E^*(BP)$. Now we have an element $\nu \in E^*(E \wedge BP)$ such that $(i \wedge BP)^*(\nu) = \iota$ as desired. q.e.d.

LEMMA 2.7. *Let E be a BP-associated spectrum. For any spectrum X, the map $i_E \wedge X\colon X \to E \wedge X$ induces the epimorphism*

$$(i_E \wedge X)^*\colon [E \wedge X, E \wedge Y]_* \longrightarrow [X, E \wedge Y]_*.$$

PROOF. For any element $x \in [X, E \wedge Y]_*$, define $\tilde{x}$ by the composition $(\nu \wedge Y)(E \wedge (\kappa \wedge Y)x)$. Then

$$(i_E \wedge X)^*(\tilde{x}) = (\nu \wedge Y)(E \wedge (\kappa \wedge Y)x)(i_E \wedge X)$$
$$= (\nu \wedge Y)(i_E \wedge BP \wedge Y)(\kappa \wedge Y)x = (\iota \wedge Y)(\kappa \wedge Y)x = x.$$

q.e.d.

LEMMA 2.8. *For a BP-associated spectrum E, we have isomorphisms*

$$\pi_*(X \wedge E) \cong \pi_*(X \wedge BP) \otimes_{BP_*} E_*, \quad \text{and}$$
$$\pi_*(BP \wedge E \wedge X) \cong BP_*(BP) \otimes_{BP_*} E_* \otimes_{BP_*} BP_*(X).$$

PROOF. First we show that there exists a canonical isomorphism $\psi\colon \pi_*(X \wedge E) \cong \pi_*(X \wedge BP) \otimes_{BP_*} E_*$. Here the right action $\varphi\colon \pi_*(X \wedge BP) \otimes_{BP_*} BP_* \to \pi_*(X \wedge BP)$ is defined by $\varphi(x \otimes v) = (X \wedge \mu)(x \wedge v)$ for $x \in \pi_*(X \wedge BP)$ and $v \in BP_*$. Now consider a map $T'\colon E_* \otimes_{BP_*} BP_*(X) \to \pi_*(X \wedge BP) \otimes_{BP_*} E_*$ defined by $T'(e \otimes x) = c(x) \otimes e$ for $c = T_*\colon BP_*(X) = \pi_*(BP \wedge X) \to \pi_*(X \wedge BP)$, the induced map of the switching one. Then the definition of the right action shows that the map T' is an isomorphism, and define the desired isomorphism by the commutative diagram

$$\begin{array}{ccc} E_* \otimes_{BP_*} BP_*(X) & \xrightarrow{T'} & \pi_*(X \wedge BP) \otimes_{BP_*} E_* \\ \| & & \downarrow \psi \\ E_*(X) & \xrightarrow{c} & \pi_*(X \wedge E). \end{array}$$

We further consider the commutative diagram

$$\begin{array}{ccc} (BP \wedge E)_*(X) & \xrightarrow{T_*} & (E \wedge BP)_*(X) \\ \| & & \| \\ BP_*(E \wedge X) & \xrightarrow{T_*} & E_*(BP \wedge X) \\ \downarrow \Psi & & \| \\ BP_*(E) \otimes_{BP_*} BP_*(X) & \xrightarrow{T_*} & E_* \otimes_{BP_*} BP_*(BP \wedge X) \end{array}$$

to define a natural isomorphism Ψ. q.e.d.

COROLLARY 2.9. *For a BP-associated spectrum E, we have isomorphisms*

$$E_*(E) = E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_*, \quad \text{and}$$
$$E_*(E \wedge X) \cong E_*(E) \otimes_{E_*} E_*(X).$$

PROOF. The first equation is obtained as follows:

$$\begin{aligned} E_*(E) &= E_* \otimes_{BP_*} BP_*(E) = E_* \otimes_{BP_*} \pi_*(BP \wedge E) \\ &= E_* \otimes_{BP_*} \pi_*(BP \wedge BP) \otimes_{BP_*} E_* \\ &= E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_* \end{aligned}$$

by using the isomorphism of Lemma 2.8.

The second isomorphism of Lemma 2.8 gives us as follows:

$$\begin{aligned} E_*(E \wedge X) &= E_* \otimes_{BP_*} BP_*(E \wedge X) \\ &= E_* \otimes_{BP_*} \pi_*(BP \wedge E \wedge X) \\ &= E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_* \otimes_{BP_*} BP_*(X) \\ &= E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_* \otimes_{E_*} E_* \otimes_{BP_*} BP_*(X). \end{aligned}$$

q.e.d.

This corollary immediately implies the following

COROLLARY 2.10. *Let E be a BP-associated spectrum. Then we have*

$$E_*(E) = E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_*, \quad \textit{and}$$
$$E_*(E \wedge E) = E_*(E) \otimes_{E_*} E_*(E).$$

By this corollary, we obtain a Hopf algebroid $(E_*, E_*(E))$ whose structure is induced from that of $(BP_*, BP_*(BP))$. We have canonical homomorphism $i\colon (BP_*, BP_*(BP)) \to (E_*, E_*(E))$ defined by $i(v) = 1 \otimes v \in E_*$ and $i(x) = 1 \otimes x \otimes 1 \in E_*(E)$ for $v \in BP_*$ and $x \in BP_*(BP)$, respectively. Then this map i is a map of Hopf algebroids.

§3. A generalized Adams SS based on a *BP*-associated spectrum

Let E be a BP-associated spectrum and $i\colon S^0 \to E$ denotes a map of spectra representing the unit element $1 \in E_*$, which is denoted by i_E in the previous section. In the same way as the case for a ring spectrum, we can construct a generalized Adams spectral sequence. Let $\bar{E}$ denotes a cofiber of i. Then we have the cofiber sequences

$$\bar{E}^{\wedge n} = S^0 \wedge \bar{E}^{\wedge n} \xrightarrow{i \wedge 1_n} E \wedge \bar{E}^{\wedge n} \xrightarrow{j_n} \bar{E}^{\wedge n+1} \xrightarrow{k_n} \Sigma \bar{E}^{\wedge n}$$

for integers $n > 0$. Put $D_1^{n,*} = \pi_*(\bar{E}^{\wedge n})$ and $E_1^{n,*} = E_*(\bar{E}^{\wedge n})$, and we have an exact couple $D_1 \xrightarrow{i} E_1 \xrightarrow{j} D_1 \xrightarrow{k} D_1$. This exact couple gives rise to the generalized Adams spectral sequence based on E. Now we observe the E_2-term.

LEMMA 3.1. *The cofiber sequence $S^0 \xrightarrow{i} E \xrightarrow{j} \bar{E}$ induces a short exact sequence*

$$0 \longrightarrow E_*(X) \xrightarrow{i_*} E_*(E \wedge X) \xrightarrow{j_*} E_*(\bar{E} \wedge X) \longrightarrow 0$$

for a spectrum X, which is split as E_*-modules.

PROOF. By Corollary 2.9, $E_*(E \wedge X) = E_*(E) \otimes_{E_*} E_*(X)$. Furthermore this is natural on X. For $X = S^0$, i_* is the map induced from $\eta_R\colon E_* \to E_*(E)$, and so we have $i_* = \eta_R \otimes 1$. Now we have a splitting $s\colon E_*(E) \otimes_{E_*} E_*(X) \to E_*(X)$ by setting $s = \varepsilon \otimes 1$ for the structure map $\varepsilon\colon E_*(E) \to E_*$.
q.e.d.

This lemma gives a resolution of the module $E_*(X)$:

$$0 \longrightarrow E_*(X) \longrightarrow E_*(E \wedge X) \xrightarrow{(ij)_*} E_*(E \wedge \bar{E} \wedge X)$$
$$\xrightarrow{(ij)_*} E_*(E \wedge \bar{E}^{\wedge 2} \wedge X) \xrightarrow{(ij)_*} E_*(E \wedge \bar{E}^{\wedge 3} \wedge X) \xrightarrow{(ij)_*} \cdots.$$

In fact, $E_*(E \wedge Y)$ is an extended comodule by Corollary 2.9 and the sequence is E_*-split exact by Lemma 3.1 and the definition of the map $(ij)_*$. Noticing that

$$E_1^{n,*}(X) = \mathrm{Hom}_{E_*}(E_*, E_*(\bar{E}^{\wedge n} \wedge X))$$
$$= \mathrm{Hom}_{E_*(E)}(E_*, E_*(E) \otimes_{E_*} E_*(\bar{E}^{\wedge n} \wedge X)).$$

Here the second isomorphism is defined by sending f to $(1 \otimes f)\Delta$, whose inverse map sends f to $(\varepsilon \otimes 1)f$, for the structure maps $\Delta\colon E_*(E) \to E_*(E) \otimes_{E_*} E_*(E)$ and $\varepsilon\colon E_*(E) \to E_*$ of the Hopf algebroid $E_*(E)$. Now apply the functor $\mathrm{Hom}_{E_*(E)}(E_*, -)$ to the resolution, and we get the E_2-term

$$E_2^{s,*}(X) = \mathrm{Ext}_{E_*(E)}^{s,*}(E_*, E_*(X)).$$

Summarizing these we have

PROPOSITION 3.2. *Let E be a BP-associated spectrum and X a spectrum. Then we have a spectral sequence $\{E_r^{s,*}(X)\}$ abutting to $\pi_*(X)$ with E_2-term*

$$E_2^{s,*}(X) = \mathrm{Ext}_{E_*(E)}^{s,*}(E_*, E_*(X)).$$

§4. E-nilpotent resolution

We begin with defining the E-nilpotent spectrum, which was first defined only for a ring spectrum. Let E be a spectrum. Then consider the smallest class $\mathscr{C}$ of spectra such that

(i) $E \in \mathscr{C}$;
(ii) If N and X are spectra and $N \in \mathscr{C}$, then $N \wedge X \in \mathscr{C}$;
(iii) If $X \to Y \to Z$ is a cofiber sequence and two of the spectra X, Y and Z are in $\mathscr{C}$, then so is the third; and
(iv) If $N \in \mathscr{C}$ and M is a retract of N, then $M \in \mathscr{C}$.

Each element of $\mathscr{C}$ is called *E-nilpotent.*

We call a spectrum X *E-prenilpotent* if there is a map $f: X \to N$ with $N \in \mathscr{C}$ such that $f_*: E_*(X) \cong E_*(N)$ is an isomorphism. The following is given in [2, Prop. 3. 9] for a ring spectrum E.

PROPOSITION 4.1. *If the sphere spectrum S^0 is E-prenilpotent, then so is any spectrum.*

PROOF. Since S^0 is E-prenilpotent, we have a E-nilpotent spectrum N and a map $f: S^0 \to N$ such that $f_*: E_*(S^0) \cong E_*(N)$ is an isomorphism. Take any spectrum X, and we have a map $(f \wedge X)_*: E_*(X) \to E_*(N \wedge X)$. We prove that $(f \wedge X)_*$ is an isomorphism by a well known method. For a finite spectrum X, it can be proved by induction on the number of the cells of X by using the Five lemma. For the infinite spectrum X, we have a set $\{X_\alpha\}_{\alpha\in\Lambda}$ of finite spectra such that $X = \mathrm{colim}_{\alpha\in\Lambda} X_\alpha$. Then $E_*(X) = E_*(\mathrm{colim}_\alpha X_\alpha) = \mathrm{colim}_\alpha E_*(X_\alpha)$ is isomorphic to $\mathrm{colim}_\alpha E_*(N \wedge X_\alpha) = E_*(\mathrm{colim}_\alpha N \wedge X_\alpha) = E_*(N \wedge X)$ as desired. q.e.d.

We call a sequence $\{W_s\}$ of maps $k_s: W_s \to W_{s-1}$ *a tower under X* if there are maps $a_s: X \to W_s$ such that $k_s a_s = a_{s-1}$.

DEFINITION 4.2. Let E be a BP-associated spectrum. An *E-nilpotent resolution* of a spectrum X is a tower $\{W_s\}_{s\geq 0}$ of spectra under X such that

(i) W_s is E-nilpotent for each $s \geq 0$.
(ii) For each E-nilpotent spectrum N, the map $\mathrm{colim}_s [W_s, N]_* \to [X, N]_*$ is isomorphic.

Now consider again the cofiber sequences

$$\bar{E}^{\wedge n} \wedge X = S^0 \wedge \bar{E}^{\wedge n} \wedge X \xrightarrow{i \wedge id} E \wedge \bar{E}^{\wedge n} \wedge X$$

$$\xrightarrow{j_n} \bar{E}^{\wedge n+1} \wedge X \xrightarrow{k_n} \Sigma \bar{E}^{\wedge n} \wedge X$$

for integers $n > 0$, obtained from the above one by taking smash product with X. Using the map k_n, we define a map

$$k^n = k_0 k_1 \cdots k_{n-1} \colon \bar{E}^{\wedge n} \wedge X \longrightarrow X$$

and denote the cofiber of k_n by $\bar{E}_n \wedge X$. We then have a tower $\{\bar{E}_s \wedge X\}$ under X, whose maps are the induced ones v from the commutative diagram of cofiber sequences:

$$\begin{array}{ccccccc} \bar{E}^{\wedge n} \wedge X & \xrightarrow{k^n} & X & \xrightarrow{a_n} & \bar{E}_n \wedge X & \longrightarrow & \bar{E}^{\wedge n} \wedge X \\ \downarrow{\scriptstyle k_{n-1}} & & \| & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle k_{n-1}} \\ \bar{E}^{\wedge n-1} \wedge X & \xrightarrow{k^{n-1}} & X & \xrightarrow{a_{n-1}} & \bar{E}_{n-1} \wedge X & \longrightarrow & \bar{E}^{\wedge n-1} \wedge X. \end{array} \tag{4.3}$$

LEMMA 4.4. *The spectrum $\bar{E}_s$ defined above is E-nilpotent.*

PROOF. Note that $\bar{E}_1 \wedge X = E \wedge X$, and so it is E-nilpotent. Suppose that $\bar{E}_s \wedge X$ is E-nilpotent. The above diagram gives rise to a cofiber sequence $\bar{E}_s \wedge X \to E \wedge \bar{E}^{\wedge s} \wedge X \to \bar{E}_{s+1} \wedge X$ by 3×3-Lemma. In the cofiber sequence, the first and the second terms are both E-nilpotent, and so is the other. Thus we have the lemma inductively. q.e.d.

LEMMA 4.5. *For a BP-associated spectrum E and a spectrum X, the tower $\{\bar{E}_s \wedge X\}$ is an E-nilpotent resolution of X.*

PROOF. The first condition to be the resolution is certified by Lemma 4.4. Now we consider the second one. Suppose that N be an E-nilpotent spectrum. By the definition of $\bar{E}_s$, we have an exact sequence

$$\cdots \longrightarrow [\bar{E}^{\wedge s} \wedge X, N]_{*+1} \longrightarrow [\bar{E}_s \wedge X, N]_* \xrightarrow{k^s} [X, N]_* \longrightarrow [\bar{E}^{\wedge s} \wedge X, N]_* \longrightarrow \cdots.$$

The diagram (4.3) enables us to take colimit of the above sequence, and we have another exact sequence

$$\cdots \longrightarrow \operatorname{colim} [\bar{E}^{\wedge s} \wedge X, N]_{*+1} \longrightarrow \operatorname{colim} [\bar{E}_s \wedge X, N]_* \xrightarrow{k^s} [X, N]_*$$

$$\longrightarrow \operatorname{colim} [\bar{E}^{\wedge s} \wedge X, N]_* \longrightarrow \cdots.$$

If $N = E \wedge Y$ for some Y, Lemma 2.7 shows that the map $[\bar{E}^{\wedge s} \wedge X, N] \to [\bar{E}^{\wedge s+1} \wedge X, N]$ is null. Therefore we have $\operatorname{colim}_s [\bar{E}^{\wedge s} \wedge X, N] = 0$ for the case $N = E \wedge Y$. Every E-nilpotent spectrum is obtained after finite steps of a construction using a cofibering and a retracting out of spectra of the form $N = E \wedge X$. Suppose that if N is obtained after k steps in such a way, then we have $\mathrm{colim}_s [\bar{E}^{\wedge s} \wedge X, N] = 0$. If N is a retract of E-nilpotent spectrum N' obtained after k steps, then we have an inclusion $\mathrm{colim}_s [\bar{E}^{\wedge s} \wedge X, N] \subset \mathrm{colim}_s [\bar{E}^{\wedge s} \wedge X, N'] = 0$. Thus it follows the desired equation. If N is a cofiber of a map $f: N_1 \to N_2$ of spectra obtained after k steps, then we have an exact sequence

$$\mathrm{colim}_s [\bar{E}^{\wedge s} \wedge X, \Sigma^{-1} N_2] \longrightarrow \mathrm{colim}_s [\bar{E}^{\wedge s} \wedge X, N] \longrightarrow \mathrm{colim}_s [\bar{E}^{\wedge s} \wedge X, N_1].$$

By the inductive hypothesis, the outer modules are null, and so is the center. This completes the proof for the spectrum of the $(k+1)$-st step. Therefore we have

$$\mathrm{colim}\, [\bar{E}^{\wedge s} \wedge X, N]_* = 0$$

for any E-nilpotent spectrum N, inductively, and we have the desired isomorphism.
q.e.d.

Consider the homotopy limit $E^\wedge X$ of the tower $\{\bar{E}_s \wedge X\}$. We call the limit $E^\wedge X$ *E-nilpotent completion*, which provides a map $\alpha: X \to E^\wedge X$ such that each map $a_s: X \to \bar{E}_s$ is obtained from α.

Here we recall [3, Ch. 8] the category $\mathcal{T}ow$-$\mathcal{A}$ associated to a category $\mathcal{A}$. By a tower $\{V_s\}$ in a category $\mathcal{A}$, we mean a sequence of objects $V_s \in \mathcal{A}$ for $s \geq 0$ together with maps $V_{s+1} \to V_s$ for $s \geq 0$. An object of the category $\mathcal{T}ow$-$\mathcal{A}$ associated to $\mathcal{A}$ is a tower in the category $\mathcal{A}$ and the set of morphism is defined by

$$\mathrm{Hom}\,(\{V_s\}, \{W_s\}) = \lim_t \mathrm{colim}_s \mathrm{Hom}\,(V_s, W_t).$$

Note that if $\{V_{i(s)}\}$ is a cofinal subtower of $\{V_s\}$, then $\{V_{i(s)}\}$ is isomorphic to $\{V_s\}$ in $\mathcal{T}ow$-$\mathcal{A}$. Furthermore a tower $\{W_s\}$ under Y in $\mathcal{A}$ corresponds to a map $v: \{Y\} \to \{W_s\}$ in $\mathcal{T}ow$-$\mathcal{A}$ for a constant tower $\{Y\}$.

LEMMA 4.6. *Assume that E is a BP-associated spectrum. If $\{W_s\}$ is an E-nilpotent resolution of Y in the stable homotopy category $\mathcal{S}$ of spectra, then there exists a unique isomorphism $e: \{\bar{E}_s \wedge Y\} \to \{W_s\}$ in $\mathcal{T}ow$-$\mathcal{S}$ such that*

$$\begin{array}{ccc} \{Y\} & \xrightarrow{1} & \{Y\} \\ \downarrow & & \downarrow \\ \{\bar{E}_s \wedge Y\} & \xrightarrow{e} & \{W_s\} \end{array}$$

PROOF. Since $\{W_s\}$ is a E-nilpotent resolution of Y, we have an isomorphism $\mathrm{colim}_s [\bar{E}_s \wedge Y, N] \cong [Y, N]$ for any E-nilpotent spectrum N. Take $N = W_s$. Then the isomorphism pulls back each map $a_s: Y \to W_s$ to a map $e_s: \bar{E}_{i(s)} \wedge Y \to W_s$ for an integer $i(s)$. These maps give rise to the morphism e that makes the diagram commutative. In a similar fashion we get the inverse of e and then e is an isomorphism.
q.e.d.

We call a morphism $f\colon \{V_s\} \to \{W_s\}$ in $\mathcal{T}ow\text{-}\mathcal{S}$ *a weak equivalence* if it induces an isomorphism $f_*\colon \{\pi_i(V_s)\} \to \{\pi_i(W_s)\}$ in $\mathcal{T}ow\text{-}\mathcal{A}b$ for each i, where $\mathcal{A}b$ is the category of abelian groups. Then we have

LEMMA 4.7. *Let $f\colon \{V_s\} \to \{W_s\}$ be a weak equivalence in $\mathcal{T}ow\text{-}\mathcal{S}$. If V_∞ and W_∞ are homotopy limit of towers $\{V_s\}$ and $\{W_s\}$, respectively, then there exists an equivalence $u\colon V_\infty \simeq W_\infty$ such that*

$$\begin{array}{ccc} \{V_\infty\} & \xrightarrow{\{u\}} & \{W_\infty\} \\ \downarrow & & \downarrow \\ \{V_s\} & \xrightarrow{f} & \{W_s\} \end{array}$$

commutes in $\mathcal{T}ow\text{-}\mathcal{S}$.

PROOF. This proof is the same as that of [2, Lemma 5.11]. First suppose that f is represented by a strict tower map $\{f_s\}$, where $f_s\colon V_s \to W_s \in \mathcal{S}$ for $s \geq 0$. Then choose $u\colon V_\infty \to W_\infty$ so that

$$\begin{array}{ccccccc} V_\infty & \longrightarrow & \Pi V_s & \longrightarrow & \Pi V_s & \longrightarrow & \Sigma V_\infty \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \Pi f_s} & & \downarrow{\scriptstyle \Pi f_s} & & \downarrow{\scriptstyle \Sigma u} \\ W_\infty & \longrightarrow & \Pi W_s & \longrightarrow & \Pi W_s & \longrightarrow & \Sigma W_\infty \end{array}$$

commutes. In the induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \lim^1 \pi_{*+1}(V_s) & \longrightarrow & \pi_*(V_\infty) & \longrightarrow & \lim \pi_*(V_s) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle u_*} & & \downarrow & & \\ 0 & \longrightarrow & \lim^1 \pi_{*+1}(W_s) & \longrightarrow & \pi_*(W_\infty) & \longrightarrow & \lim \pi_*(W_s) & \longrightarrow & 0, \end{array}$$

the outer vertical maps are isomorphisms by the hypothesis. Thus $u_*\colon \pi_*(V_\infty) \cong \pi_*(W_\infty)$ and $u\colon V_\infty \simeq W_\infty$ as desired. In the general case, f can be factored as $\{V_s\} \xrightarrow{\{t_s\}^{-1}} \{V_{i(s)}\} \xrightarrow{\{\varphi_s\}} \{W_s\}$, where $\{t_s\}\colon \{V_{i(s)}\} \longrightarrow \{V_s\}$ is the canonical map from some cofinal subtower to $\{V_s\}$, and $\{\varphi_s\}$ is a strict tower map. Since $\{t_s\}$ and $\{\varphi_s\}$ are both strict weak equivalences, we can construct equivalences $v\colon V_{i(\infty)} \simeq V_\infty$ and $w\colon V_{i(\infty)} \simeq W_\infty$ as above. Here $V_{i(\infty)}$ is a homotopy inverse limit of $\{V_{i(s)}\}$. We have the lemma by letting $u = wv^{-1}$. q.e.d.

Now combining above two lemmas show the following

PROPOSITION 4.8. *For a BP-associated spectrum E and a spectrum Y, let $\{W_s\}$ be an E-nilpotent resolution of Y with homotopy inverse limit W_∞. Then $W_\infty \simeq E^\wedge Y$.*

PROOF. Lemma 4.6 guarantees the existence of an isomorphism $e\colon \{W_s\} \to \{\bar{E}_s \wedge Y\}$, which induces a weak equivalence. Therefore we obtain the proposition by Lemma 4.7. q.e.d.

§5. E-localization and E-nilpotent completion

In this section we rewrite Bousfield's theory for a BP-associated spectrum instead of a ring spectrum.

First we define the localization. Let E be a spectrum. A spectrum X is said to be *E-local* if the homotopy group $[C, X]_* = 0$ for any C with $\pi_*(E \wedge C) = 0$. The *E-localization functor* $L_E\colon \mathscr{S} \to \mathscr{S}$ is equipped with a natural transformation $\eta\colon 1 \to L_E$ providing that

(i) $\eta_X\colon X \to L_E X$ induces an isomorphism $\eta_{X*}\colon E_*(X) \cong E_*(L_E X)$, and

(ii) for any $f\colon X \to Y$ with an isomorphism $f_*\colon E_*(X) \cong E_*(Y)$, there is a unique $r\colon Y \to L_E X$ such that $rf = \eta_X$.

Then the following is well known (*cf.* [6]; see [9] for the proof):

PROPOSITION 5.1. *For a spectrum X, we have*

(i) *$L_E X$ is uniquely defined.*

(ii) *$L_E(L_E X) = L_E X$.*

(iii) *If there is a map $f\colon X \to Y$ with Y E-local such that $E_*(f)$ is an isomorphism, then there exists a map $h\colon L_E X \to Y$ such that $h\eta_X = f$.*

LEMMA 5.2. *For a BP-associated spectrum E and a spectrum X, $E \wedge X$ is E-local.*

PROOF. Let C be a spectrum such that $\pi_*(E \wedge C) = 0$. Take an element x in $[C, E \wedge X]$. Then we compute

$$\begin{aligned}
&(\nu \wedge X)(E \wedge \kappa \wedge X)(E \wedge x)(i_E \wedge C)\\
&\qquad = (\nu \wedge X)(E \wedge \kappa \wedge X)(i_E \wedge E \wedge X)x\\
&\qquad = (\nu \wedge X)(i_E \wedge BP \wedge X)(\kappa \wedge X)x\\
&\qquad = (\iota \wedge X)(\kappa \wedge X)x && \text{(by Lemma 2.6)}\\
&\qquad = x && \text{(by Lemma 2.4)},
\end{aligned}$$

in the commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{x} & E \wedge X & & \xrightarrow{\kappa \wedge 1} & & BP \wedge X\\
\downarrow{\scriptstyle i_E \wedge 1} & & \downarrow{\scriptstyle i_E \wedge 1} & & \nwarrow{\scriptstyle \nu \wedge 1} & & \downarrow{\scriptstyle i_E \wedge 1}\\
E \wedge C & \xrightarrow{1 \wedge x} & E \wedge E \wedge X & & \xrightarrow{1 \wedge \kappa \wedge 1} & & E \wedge BP \wedge X.
\end{array}$$

Since $E \wedge C$ is contractible, we get $x = 0$ in $[C, E \wedge X]_*$, and so $[C, E \wedge X]_* = 0$ as desired. q.e.d.

LEMMA 5.3. *Let E be a BP-associated spectrum. Then E-nilpotent spectrum is a E-local.*

PROOF. We here again follow Bousfield's proof. First filter the E-nilpotent class $\mathscr{C}$ as folows. Let $\mathscr{C}_0$ consist of all spectra equivalent to $E \wedge X$ for some spectrum X. Inductively suppose that $\mathscr{C}_m$ with $m \geq 0$ is given. Then $\mathscr{C}_{m+1}$ consists of all spectra N such that either N is a retract of a member of $\mathscr{C}_m$ or there is a cofiber sequence $X \to N \to Z$ with $X, Z \in \mathscr{C}_m$.

We now prove the lemma. For a spectrum of $\mathscr{C}_0$, we have the lemma by Lemma 5.2. If a spectrum X of $\mathscr{C}_{m+1}$ is a retract of a spectrum Y of $\mathscr{C}_m$, then $[C, X]_* \subset [C, Y]_*$ and so we have $[C, X]_* = 0$ for any spectrum C with $\pi_*(E \wedge C) = 0$ by the inductive hypothesis that Y is E-local. Consider now the case that X fits into the cofiber sequence $Y \to X \to Z$ with Y and Z in $\mathscr{C}_m$. Now applying the homotopy functor $[C, -]_*$ to it, we see that the hypothesis $[C, W]_* = 0$ for $W = Y, Z$ shows $[C, X]_* = 0$ as we desired. Now we complete the induction.

q.e.d.

We also see that a homòtopy limit of E-local spectra is also E-local, since we have the Milnor sequence $0 \to \lim^1_{\alpha *} [C, X_\alpha]_{*+1} \to [C, \lim_\alpha X_\alpha]_* \to \lim_{\alpha *} [C, X_\alpha]_* \to 0$. Therefore, we obtain

COROLLARY 5.4. *Let E be a BP-associated spectrum, and W_∞ denote a homotopy limit of an E-nilpotent resolution $\{W_s\}$. Then W_∞ is E-local.*

This together with Lemma 4.5 shows that the E-nilpotent resolution $E^\wedge X$ is E-local. Then by Proposition 5.1, we have a map $\beta: L_E X \to E^\wedge X$ such that $\beta \eta_X = \alpha: X \to E^\wedge X$.

§6. Convergence of GASS

In order to investigate the map $\beta: L_E X \to E^\wedge X$, we prepare some more notation on a generalized spectral sequence. Recall the convergence of a spectral sequence. First filter the homotopy groups $\pi_*(E^\wedge X)$ by letting

$$F^s \pi_*(E^\wedge X) = \mathrm{Ker}\,(\pi_*(E^\wedge X) \longrightarrow \pi_*(\bar{E}_{s-1} \wedge X)).$$

Then the map

$$\pi_*(E^\wedge X) \longrightarrow \lim_s \,(\pi_*(E^\wedge X)/F^s \pi_*(E^\wedge X)) \tag{6.1}$$

is always surjective. Note that $\bar{E}_s$ is defined only for nonnegative integers s. Thus the E_r-terms has the property that

$$E^{s,t}_{r+1}(X) \subset E^{s,t}_r(X) \qquad \text{for } r > s,$$

which enables us to define

$$E^{s,t}_\infty(X) = \bigcap_{r>s} E^{s,t}_r(X)$$

and gives a monomorphism

$$F^s\pi_*(E^\wedge X)/F^{s+1}\pi_*(E^\wedge X) \longrightarrow E_\infty^{s,s+*}(X). \tag{6.2}$$

We call the spectral sequence $\{E_r^{s,t}\}$ *converging completely* to $\pi_*(E^\wedge X)$ if both of (6.1) and (6.2) are isomorphisms. Then the following is the general result:

PROPOSITION 6.3. *The E-Adams spectral sequence* $\{E_r^{s,t}(X)\}$ *converges completely to* $\pi_*(E^\wedge X)$ *if for each s, t there is a finite r such that* $E_r^{s,t}(X) = E_\infty^{s,t}(X)$.

In fact, if the hypothesis $E_r^{s,t}(X) = E_\infty^{s,t}(X)$ for some finite r is valid, then the inverse system $\{\pi_*(E^\wedge X)/F^s\pi_*(E^\wedge X)\}_s$ satisfies the Mittag-Leffler condition and we see that (6.1) is an isomorphism. The monomorphism (6.2) is seen to be an isomorphism by a standard argument of a spectral sequences.

LEMMA 6.4. *If X is an E-prenilpotent spectrum, then* L_EX *is E-nilpotent.*

PROOF. Since X is E-prenilpotent, we have an E-nilpotent spectrum N and a map $f: X \to N$ such that $f_*: E_*(X) \to E_*(N)$ is an isomorphism. Noticing that $L_EN = N$ by Lemma 5.3, we have an induced map $L_Ef: L_EX \to N$ and an isomorphism $(L_Ef)_*: E_*(L_EX) \to E_*(X)$. Therefore L_EX is homotopic to N, since both of them are E-local. q.e.d.

PROPOSITION 6.5. *For an E-prenilpotent spectrum X, E-Adams spectral sequence converges completely to* $\pi_*(L_EX)$.

PROOF. The tower $\{L_EX\}$ is an E-nilpotent resolution of X by Lemma 6.4. Then by Lemma 4.6 we have an equivalence $\{L_EX\} \cong \{\bar{E}_s \wedge X\}$ and $L_EX \simeq E^\wedge X$ by Proposition 4.8. This equivalence gives the convergence, since $\{L_EX\}$ is a constant tower. In fact, a constant tower gives rise to trivial E_1-terms, other than $E_1^{0,*}$, of the spectral sequence associated to the tower. Therefore, it satisfies the condition of Proposition 6.3. q.e.d.

M. Hopkins and D. Ravenel show (*cf.* [8]) the following

THEOREM 6.6. *The sphere spectrum* S^0 *is* $v_n^{-1}BP$*-prenilpotent.*

By Lemma 2.4, $E(n)$ is a retract of $v_n^{-1}BP$ and so

COROLLARY 6.7. *The sphere spectrum* S^0 *is* $E(n)$*-prenilpotent.*

As we have noticed in the introduction, the Landweber exact functor theorem ([4]) shows that $E(n)$ is a BP-associated spectrum. Therefore we apply Propositions 3.2 and 6.5 and Corollary 6.7, and obtain our main result:

COROLLARY 6.8. $E(n)$*-Adams spectral sequence* $\{E_r^{s,t}(X)\}$ *converges to* $\pi_*(L_{E(n)}X)$ *for any spectrum X.*

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