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On Sets of Hyperreal Numbers

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Abstract

We construct a field $*R$ of hyperreal numbers so that the field R of real numbers is embeded as a subfield of $*R$. The set $*R$ can be regarded as a metric space containing the set R as a discrete subset and we can generalize this property.

1. Introduction and preliminaries

In the 1960s, Abraham Robinson showed that the set R of real numbers can be regarded as a subset of a set $*R$ of hyperreal numbers which contains infinitely small and large numbers. We can contend the set $*R$ is a metric space containing the set R as a discrete subset. In the present paper we would like to generalize this property.

We shall first state some fundamental properties on filters. According to Comfort and Negrepontis [1], we shall give the following definition.

DEFINITION 1.1. Let I_i be an infinite set and $\mathscr{F}_i$ be a filter on I_i for each $i=1, 2, 3$. We define sets $\mathscr{F}_1\cdot\mathscr{F}_2$, $(\mathscr{F}_1\cdot\mathscr{F}_2)\cdot\mathscr{F}_3$, $\mathscr{F}_1\cdot\mathscr{F}_2\cdot\mathscr{F}_3$, and $\mathscr{F}_1\cdot(\mathscr{F}_2\cdot\mathscr{F}_3)$ as follows.

$$\mathscr{F}_1\cdot\mathscr{F}_2=\{A\in\mathscr{P}(I_1\times I_2)|\{y_1\in I_1|\{y_2\in I_2|(y_1, y_2)\in A\}\in\mathscr{F}_2\}\in\mathscr{F}_1\},$$

$$(\mathscr{F}_1\cdot\mathscr{F}_2)\cdot\mathscr{F}_3=\{A\in\mathscr{P}(I_1\times I_2\times I_3)|\{(y_1, y_2)\in I_1\times I_2|\{y_3\in I_3| (y_1, y_2, y_3)\in A\}\in\mathscr{F}_3\}\in\mathscr{F}_1\cdot\mathscr{F}_2\},$$

$$\mathscr{F}_1\cdot\mathscr{F}_2\cdot\mathscr{F}_3=\{A\in\mathscr{P}(I_1\times I_2\times I_3)|\{y_1\in I_1|\{y_2\in I_2|\{y_3\in I_3| (y_1, y_2, y_3)\in A\}\in\mathscr{F}_3\}\in\mathscr{F}_2\}\in\mathscr{F}_1\},$$

$$\mathscr{F}_1\cdot(\mathscr{F}_2\cdot\mathscr{F}_3)=\{A\in\mathscr{P}(I_1\times I_2\times I_3)|\{y_1\in I_1|\{(y_2, y_3)\in I_2\times I_3| (y_1, y_2, y_3)\in A\}\in\mathscr{F}_2\cdot\mathscr{F}_3\}\in\mathscr{F}_1\}.$$

We immediately have the following proposition.

PROPOSITION 1.2. (1.1) *$\mathscr{F}_1\cdot\mathscr{F}_2$ is a filter on $I_1\times I_2$.*
(1.2) *$(\mathscr{F}_1\cdot\mathscr{F}_2)\cdot\mathscr{F}_3$, $\mathscr{F}_1\cdot\mathscr{F}_2\cdot\mathscr{F}_3$ and $\mathscr{F}_1\cdot(\mathscr{F}_2\cdot\mathscr{F}_3)$ are filters on $I_1\times I_2\times I_3$ and we have*

$$(\mathscr{F}_1\cdot\mathscr{F}_2)\cdot\mathscr{F}_3=\mathscr{F}_1\cdot\mathscr{F}_2\cdot\mathscr{F}_3=\mathscr{F}_1\cdot(\mathscr{F}_2\cdot\mathscr{F}_3).$$

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The following proposition was given in [3].

PROPOSITION 1.3. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be ultrafilters on I_1 and I_2 respectively. Then $\mathscr{F}_1 \cdot \mathscr{F}_2$ is an ultrafilter on $I_1 \times I_2$.*

PROPOSITION 1.4. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be ultrafilters on I_1 and I_2 respectively, and let $\mathscr{F}_1$ or $\mathscr{F}_2$ be a free (ω-incomplete) ultrafilter, then $\mathscr{F}_1 \cdot \mathscr{F}_2$ is a free ultrafilter on $I_1 \times I_2$.*

In order to prove Proposition 1.4, we use a lemma.

LEMMA 1.5. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be filters on I_1 and I_2 respectively. Then we have the following.*

(1.3) *Let $X_1 \in \mathscr{F}_1$ and $X_2 \in \mathscr{F}_2$, then $X_1 \times X_2 \in \mathscr{F}_1 \cdot \mathscr{F}_2$.*

(1.4) *Let $X_1 \subset I_1$ and $X_2 \subset I_2$, and let $X_1 \notin \mathscr{F}_1$ or $X_2 \notin \mathscr{F}_2$, then $X_1 \times X_2 \notin \mathscr{F}_1 \cdot \mathscr{F}_2$.*

PROOF OF PROPOSITION 1.4. Let $\mathscr{F}_1$ be a free ultrafilter, then there exists A_n such that $A_n \in \mathscr{F}_1$ for each $n \in N$ and $\bigcap_{n=1}^{\infty} A_n \notin \mathscr{F}_1$, where N is the set of all positive integers.

Using Lemma 1.5, we have $A_n \times I_2 \in \mathscr{F}_1 \cdot \mathscr{F}_2$ for each $n \in N$. Since

$$\bigcap_{n=1}^{\infty} (A_n \times I_2) = (\bigcap_{n=1}^{\infty} A_n) \times I_2 \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n \notin \mathscr{F}_1,$$

we have

$$(\bigcap_{n=1}^{\infty} A_n) \times I_n \notin \mathscr{F}_1 \cdot \mathscr{F}_2.$$

Therefore $\mathscr{F}_1 \cdot \mathscr{F}_2$ is a free ultrafilter. Q.E.D.

Let K be a nonempty set and

$$a(y_1, \cdots, y_n),\ b(y_1, \cdots, y_n) \in \prod_{(y_1, \cdots, y_n) \in I_1 \times \cdots \times I_n} K.$$

We define a relation $R(n)$, by

$a(y_1, \cdots, y_n) R(n) b(y_1, \cdots, y_n)$, if and only if

$\{(y_1, \cdots, y_n) \in I_1 \times \cdots \times I_n | a(y_1, \cdots, y_n) = b(y_1, \cdots, y_n)\} \in \mathscr{F}_1 \cdot \mathscr{F}_2 \cdots \mathscr{F}_n$.

The relation $R(n)$ is an equivalence relation.

We would like to use a notation

$$\prod_{(y_1, \cdots, y_n) \in I_1 \times \cdots \times I_n} K / \mathscr{F}_1 \cdot \mathscr{F}_2 \cdots \mathscr{F}_n$$

in order to express the quotient space

$$\prod_{(y_1,\cdots,y_n)\in I_1\times\cdots\times I_n} K/R(n).$$

The quotient class determined by a function $a(y_1,\cdots,y_n)$ will be denoted by $[a(y_1,\cdots,y_n)]$.

THEOREM 1.6. *The following formula is valid.*

$$\prod_{y_1\in I_1}(\prod_{y_2\in I_2} K/\mathscr{F}_2)/\mathscr{F}_1 = \prod_{(y_1,y_2)\in I_1\times I_2} K/\mathscr{F}_1\cdot\mathscr{F}_2.$$

PROOF. If

$$(\tilde{a}(y_2))(y_1)\in\prod_{y_1\in I_1}(\prod_{y_2\in I_2} K),$$

then we can contend

$$(\tilde{a}(y_2))(y_1)\in\prod_{(y_1,y_2)\in I_1\times I_2} K,$$

and we write

$(\tilde{a}(y_2))(y_1)=a(y_1,y_2)$.

We immediately have this result by the following fact.

Let

$$[[\tilde{a}(y_2)](y_1)],\ [[\tilde{b}(y_2)](y_1)]\in\prod_{y_1\in I_1}(\prod_{y_2\in I_2} K/\mathscr{F}_2)/\mathscr{F}_1,$$

then we have

$$\begin{aligned}
&[[\tilde{a}(y_2)](y_1)]=[[\tilde{b}(y_2)](y_1)]\\
\Leftrightarrow&\{y_1\in I_1|[\tilde{a}(y_2)](y_1)=[\tilde{b}(y_2)](y_1)\}\in\mathscr{F}_1\\
\Leftrightarrow&\{y_1\in I_1|\{y_2\in I_2|(\tilde{a}(y_2))(y_1)=(\tilde{b}(y_2))(y_1)\}\in\mathscr{F}_2\}\in\mathscr{F}_1\\
\Leftrightarrow&\{(y_1,y_2)\in I_1\times I_2|a(y_1,y_2)=b(y_1,y_2)\}\in\mathscr{F}_1\cdot\mathscr{F}_2.
\end{aligned}$$

Q.E.D.

COROLLARY 1.7. *The fillowing formula is valid.*

$$\prod_{y_1\in I_1}(\prod_{y_2\in I_2}(\cdots(\prod_{y_n\in I_n} K/\mathscr{F}_n)\cdots)/\mathscr{F}_2)/\mathscr{F}_1=\prod_{(y_1,\cdots,y_n)\in I_1\times\cdots\times I_n} K/\mathscr{F}_1\cdots\mathscr{F}_n.$$

2. On sets of hyperreal numbers

Now we shall give the following definition.

DEFINITION 2.1. Let $R^+=\{y\in R|y>0\}$ and let $F=\{(0,y)|y\in R^+\}$. Then F has the finite intersection property. We shall denote by $\mathscr{F}$ one of the ultrafilters containing F. The filter $\mathscr{F}$ is a free ultrafilter.

We Shall use the following notations:

$$\mathscr{F}^1 = \mathscr{F} \text{ and } \mathscr{F} \cdot \mathscr{F}^n = \mathscr{F}^{n+1} \quad \text{for } n \in N,$$

$$(R^+)^1 = R^+ \text{ and } R^+ \times (R^+)^n = (R^+)^{n+1} \quad \text{for } n \in N,$$

$$^{0*}R = R,\ ^{1*}R = {}^*R = \prod_{y_1 \in R^+} R/\mathscr{F}, \text{ and}$$

$$^{n*}R = \prod_{(y_1, \cdots, y_n) \in (R^+)^n} R/\mathscr{F}_1 \cdots \mathscr{F}_n \quad \text{for } n \in N.$$

We immediately have the following theorem.

THEOREM 2.2. $^{n*}R = {}^*({}^{(n-1)*}R)$ *for* $n \in N$.

An element of the set $^{n*}R$ *is called a hyperreal number. The set* $^{n*}R$ *is made into a commutative ordered field by defining the addition, the subtraction, the product, the quotient and the order in the usual way.*

We define absolute value in $^{n*}R$ as follows.

DEFINITION 2.3. If $x = [x(y_1, \cdots, y_n)] \in {}^{n*}R$, then

$$|x| = [|x(y_1, \cdots, y_n)|].$$

We have the following theorem immediately.

THEOREM 2.4. $^{n*}R$ *is a metric space with a usual metric*

$$d(x, y) = |y - x| \qquad \textit{for } x, y \in {}^{n*}R.$$

DEFINITION 2.5 (Infinitesimal). We shall say that $\varepsilon = [\varepsilon(y_1, \cdots, y_n)] \in {}^{n*}R$ is infinitesimal or infinitesimal small if for every $\delta \in R^+$ we have

$$\{(y_1, \cdots, y_n) \in (R^+)^n \,|\, \varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F}^n. \tag{2.1}$$

When ε and δ satisfy condition (2.1), we write $|\varepsilon| < \delta$.

PROPOSITION 2.6. *Let* $\delta \in R^+$ *and* $\varepsilon = [\varepsilon(y_1, \cdots, y_n)] \in {}^{n*}R$, *then we have*

$$\{(y_1, \cdots, y_n) \in (R^+)^n \,|\, |\varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F}^n$$

$$\Leftrightarrow \{y_1 \in R^+ \,|\, \{(y_2, \cdots, y_n) \in (R^+)^{n-1} \,|\, |\varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F}^{n-1}\} \in \mathscr{F}$$

$$\Leftrightarrow \{y_1 \in R^+ \,|\, \{y_2 \in R^+ \,|\, \{(y_3, \cdots, y_n) \in (R^+)^{n-2};\ |\varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F}^{n-2}\} \in \mathscr{F}\} \in \mathscr{F}\}$$

$$\Leftrightarrow \cdots$$

$$\Leftrightarrow \{y_1 \in R^+ \,|\, \{y_2 \in R^+ \,|\, \cdots \{y_n \in R^+ \,|\, |\varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F} \cdots\} \in \mathscr{F}\} \in \mathscr{F}$$

$$\Leftrightarrow \{(y_1, y_2) \in (R^+)^2 \,|\, \{(y_3, \cdots, y_n) \in (R^+)^{n-2} \,|\, |\varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F}^{n-2}\} \in \mathscr{F}^2$$

$$\Leftrightarrow \cdots$$

$$\Leftrightarrow \{(y_1, \cdots, y_{n-1}) \in (R^+)^{n-1} \,|\, \{y_n \in R^+ \,|\, |\varepsilon(y_1, \cdots, y_n)| < \delta\} \in \mathscr{F}\} \in \mathscr{F}^{n-1}.$$

The proof is easy, and we omit it.

PROPOSITION 2.7. *Let* $\varepsilon = [\varepsilon(y_n)] \in {}^*R$ *and* $\delta \in R^+$ *which satisfy a condition* $\{y_n \in R^+ \mid |\varepsilon(y_n)| < \delta\} \in \mathscr{F}$. *Then we have*

$$\{(y_1, \cdots, y_n) \in (R^+)^n \mid |\varepsilon(y_n)| < \delta\} \in \mathscr{F}^n.$$

PROOF. Let $A = \{y_n \in R^+ \mid |\varepsilon(y_n)| < \delta\}$, then $A \in \mathscr{F}$. Since

$$\{(y_1, \cdots, y_n) \in (R^+)^n \mid |\varepsilon(y_n)| < \delta\} = (R^+)^{n-1} \times A,$$
$$(R^+)^{n-1} \in \mathscr{F}^{n-1} \quad \text{and} \quad A \in \mathscr{F},$$

we have

$$\{(y_1, \cdots, y_n) \in (R^+)^n \mid |\varepsilon(y_n)| < \delta\} \in \mathscr{F}^{n-1} \cdot \mathscr{F} = \mathscr{F}^n. \qquad \text{Q.E.D.}$$

Let $a, x, \varepsilon \in {}^{n*}R$ and $\varepsilon > 0$. We define a set $U(a, \varepsilon)$ by

$$U(a, \varepsilon) = \{x \in {}^{n*}R \mid |x - a| < \varepsilon\}.$$

PROPOSITION 2.8. *Let* $a = [a(y_1, \cdots, y_{n-1})]$, $x = [x(y_1, \cdots, y_{n-1})] \in {}^{(n-1)*}R$ *and let* $\varepsilon = [\varepsilon(y_n)]$ *be a positive infinitesimal, then*

$$U(a, \varepsilon) = \{a\}.$$

PROOF. Let $a \neq x$, and let

$$B = \{(y_1, \cdots, y_{n-1}) \in (R^+)^{n-1} \mid |x(y_1, \cdots, y_{n-1}) - a(y_1, \cdots, y_{n-1})| > 0\}$$

then we have $B \in \mathscr{F}^{n-1}$.
Since

$$\{(y_1, \cdots, y_{n-1}) \in (R^+)^{n-1} \mid \{y_n \in R^+ \mid |x(y_1, \cdots, y_{n-1}) - a(y_1, \cdots, y_{n-1})| > \varepsilon(y_n)\} \in \mathscr{F}\} \supset B,$$

we have

$$\{(y_1, \cdots, y_n) \in (R^+)^n \mid |x(y_1, \cdots, y_{n-1}) - a(y_1, \cdots, y_{n-1})| > \varepsilon(y_n)\} \in \mathscr{F}^n,$$

which shows $|x - a| > \varepsilon$.

Clearly we have $U(a, \varepsilon) \ni a$. Hence we have

$$U(a, \varepsilon) = \{a\}. \qquad \text{Q.E.D.}$$

Using Proposition 2.8 we have the following theorem.

THEOREM 2.9. *A metric space* ${}^{(n-1)*}R$ *is a discrete subspace of a metric space* ${}^{n*}R$, *for every* $n \in N$.

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