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On differentials of a generalized Adams spectral sequence

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§1. Introduction

A spectral sequence is a sequence of modules $\{E_r^*\}$ provided with a differential $d_r\colon E_r^s \to E_r^{s+r}$ for each $r > 0$, which satisfies

$$E_{r+1}^s = \mathrm{Ker}\{d_r\colon E_r^s \longrightarrow E_r^{s+r}\}/\mathrm{Im}\{d_r\colon E_r^{s-r} \longrightarrow E_r^s\}.$$

If there is an integer $r_s > 1$ for each s such that $E_r^s = E_{r+1}^s$ for any $r \geq r_s$, then we write $E_\infty^s = E_{r_s}^s$. Consider a filtration $0 \subset \cdots \subset F_{s+1} \subset F_s \subset \cdots \subset F_0 = M$ of a module M with $\bigcap_s F_s = 0$. We say that a spectral sequence $\{E_r, d_r\}$ *converges* to a module M, if there exists E_∞^s for each s and the filtration satisfies

$$F_s/F_{s+1} = E_\infty^s.$$

We call the module $\bigoplus_s F_s/F_{s+1} = \bigoplus_s E_\infty^s$ *associated graded module.* If all differentials d_r are null for $r > 1$, it is easy to study about the spectral sequence, since $E_r^s = E_{r+1}^s$ for $r > 1$. In this case tne spectral sequence is said to *collapse.* If a spectral sequence collapses and converges, we get almost all information on the target module M from its E_2-term. Generally E_2-terms E_2^* has a computable expression and so the target module M is computable up to extension problem if it collapses.

It is well known that an exact couple gives a spectral sequence. Let E be a ring spectrum. Then the cofiber sequence obtained from the unit map gives rise to the exact couple of homotopy groups, which defines the *E-Adams spectral sequence* converging to the homotopy $\pi_*(X)$ of a spectrum X under some conditions on E and X (*cf.* §3). The generalized Adams spectral sequence is a powerful tool to compute homotopy groups of a spectrum especially when it collapses at E_2-term. Contrary to the E_2-term, the differentials are not known how to compute algebraically. So it is hard to see how the differential behaves and we usually use some facts on homotopy to tell the behavior. When we have a cofiber sequence involving a spectrum whose homotopy groups are known, we can dig out more information on the homotopy groups from the spectral sequence for each spectrum in the cofiber sequence, that is, we can get formulae on the differential even if it is not collapse. The well known example of this kind is the Geometric Boundary Theorem ([4]) and generalized one (*cf.* [5]). Here we give some more formulae for the differentials out of information on the spectral sequence whose homotopy is

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known as well as the differentials of the spectral sequence.

A homotopy element ξ of $\pi_*(X)$ is said to be *detected* by x of the E_2-term $E_2^*(X)$ if x is a permanent cycle and the corresponding element to x in the E_∞-term is a corresponding one to ξ in the associated graded module $\mathscr{G}\pi_*(X)$.

In order to state our results, we give some notation and assumption. We consider a cofiber sequence

$$\longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \longrightarrow . \tag{1.1}$$

Suppose that

$$E_*(h) = 0: E_*(Z) \longrightarrow E_*(X) \tag{1.2}$$

induced from h in (1.1) and

$$E_*(E) \text{ is flat over } E_* \tag{1.3}$$

for the spectrum E. Then it induces the exact sequence

$$\longrightarrow E_2^*(X) \xrightarrow{f_*} E_2^*(Y) \xrightarrow{g_*} E_2^*(Z) \xrightarrow{h_*} E_2^{*+1}(X) \longrightarrow$$

of E_2-terms. In fact the E_2-term is given by the Ext-groups in this case. For example, it is satisfied for the case $E = BP$, the Brown-Peterson spectrum, and a cofiber sequence (1.1) such that $BP_{2t+1}(U) = 0$ for $t \in \boldsymbol{Z}$ and for $U = X$, Y and Z. We further suppose that

$$E_2^{s,2t-1}(W) = 0 \text{ for } W = X, Y, \text{ and } Z, \text{ and } E_4^*(Y) = E_\infty^*(Y). \tag{1.4}$$

Note that this means that the differential d_{2t} is null for $t > 0$, and in particular $E_2^* = E_3^*$. Our results are on the map $g_*: \pi_*(Y) \to \pi_*(Z)$. Consider an element $\eta \in \pi_*(Y)$ such that η is detected by $f_*(x)$ for some $x \in E_2^*(X)$ with

$$d_{2t+1}(x) = x_0 \neq 0 \in E_{2t+1}(X). \tag{1.5}$$

Moreover suppose that

(1.6) *E-Adams spectral sequences for these spectra converge to the homotopy groups of them.*

These assumptions imply that $f_*(x_0) = 0 \in E_r^*(Y)$ for some r, and so

$$f_*(x_0) = 0, \text{ or } f_*(x_0) = d_3(y_0) \text{ for some } y_0 \in E_2^*(Y). \tag{1.7}$$

Then under these circumstances, we have

THEOREM. (1) *If* $f_*(x_0) = d_3(y_0) \neq 0$ *in* $E_2^*(Y) = E_3^*(Y)$, *then* $g_*(\eta) \in \pi_*(Z)$ *is detected by* $g_*(y_0) \in E_3^*(Z)$.

(2) *Suppose that* $f_*(x_0) = 0$ *in* $E_2^*(Y) = E_3^*(Y)$. *Then we have an element* z *in* $E_2^*(Z)$ *with* $h_*(z) = x_0$ *and* $g_*(\eta) \in \pi_*(Z)$ *is detected by* z.

The second part of this holds under some weaker conditions and may be known to some experts. This theorem, especially the first part, will be applied to the case for $E = BP$, the Brown-Peterson spectrum at the prime 2, which will appear somewhere. Furthermore we note that this is still true for the case that $d_r = 0$ on Y in (1.1) other than $r = 2p - 1$ for an odd prime p, instead of $r = 3$ in our case. But I write down this only for the case $r = 3$ for simplicity.

In the next section we give a well known result, whose proof here is due to M. Hikida, which is applied to prove the theorem in the last section. In §3, we restate the construction of generalized Adams spectral sequences whose notation we use in the proof of Theorem.

§2. Key lemma

We first recall [3, Lemma 6.2] the well known fact on the cofiber sequences. Consider a commutative diagram

$$\begin{array}{ccccccccc}
 & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle y_3} & & \downarrow{\scriptstyle z_3} & & \downarrow{\scriptstyle x_3} & \\
\longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z_1 & \xrightarrow{h_1} & \Sigma X_1 & \longrightarrow \\
 & \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle y_1} & & \downarrow{\scriptstyle z_1} & & \downarrow{\scriptstyle x_1} & \\
\longrightarrow & X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{g_2} & Z_2 & \xrightarrow{h_2} & \Sigma X_2 & \longrightarrow \\
 & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle y_2} & & \downarrow{\scriptstyle z_2} & & \downarrow{\scriptstyle x_2} & \\
\longrightarrow & X_3 & \xrightarrow{f_3} & Y_3 & \xrightarrow{g_3} & Z_3 & \xrightarrow{h_3} & \Sigma X_3 & \longrightarrow , \\
 & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle y_3} & & \downarrow{\scriptstyle z_3} & & \downarrow{\scriptstyle x_3} &
\end{array} \tag{2.1}$$

whose rows and columns are all cofiber sequences of spectra. Applying the homotopy functor $\pi_*(-)$ to this diagram, we get the induced one

$$\begin{array}{ccccccccc}
 & \downarrow{\scriptstyle x_{3*}} & & \downarrow{\scriptstyle y_{3*}} & & \downarrow{\scriptstyle z_{3*}} & & \downarrow{\scriptstyle x_{3*}} & \\
\longrightarrow & \pi_*(X_1) & \xrightarrow{f_{1*}} & \pi_*(Y_1) & \xrightarrow{g_{1*}} & \pi_*(Z_1) & \xrightarrow{h_{1*}} & \pi_{*-1}(X_1) & \longrightarrow \\
 & \downarrow{\scriptstyle x_{1*}} & & \downarrow{\scriptstyle y_{1*}} & & \downarrow{\scriptstyle z_{1*}} & & \downarrow{\scriptstyle x_{1*}} & \\
\longrightarrow & \pi_*(X_2) & \xrightarrow{f_{2*}} & \pi_*(Y_2) & \xrightarrow{g_{2*}} & \pi_*(Z_2) & \xrightarrow{h_{2*}} & \pi_{*-1}(X_2) & \longrightarrow \\
 & \downarrow{\scriptstyle x_{2*}} & & \downarrow{\scriptstyle y_{2*}} & & \downarrow{\scriptstyle z_{2*}} & & \downarrow{\scriptstyle x_{2*}} & \\
\longrightarrow & \pi_*(X_3) & \xrightarrow{f_{3*}} & \pi_*(Y_3) & \xrightarrow{g_{3*}} & \pi_*(Z_3) & \xrightarrow{h_{3*}} & \pi_{*-1}(X_3) & \longrightarrow , \\
 & \downarrow{\scriptstyle x_{3*}} & & \downarrow{\scriptstyle y_{3*}} & & \downarrow{\scriptstyle z_{3*}} & & \downarrow{\scriptstyle x_{3*}} &
\end{array} \tag{2.2}$$

where the rows and the columns are all exact. We now rewrite [3, Lemma 6.2] to fit our purpose:

LEMMA 2.3. *Suppose that elements* $\zeta_1 \in \pi_*(Z_1)$ *and* $\eta_2 \in \pi_*(Y_2)$ *satisfy*

$$z_{1*}(\zeta_1) = g_{2*}(\eta_2)$$

in the above diagram (2.2). *Then there exists an element* $\xi_3 \in \pi_*(X_3)$ *satisfying*

$$f_{3*}(\xi_3) = y_{2*}(\eta_2) \quad \text{and} \quad h_{1*}(\zeta_1) = -x_3(\xi_3).$$

PROOF. First of all we will show that we can replace the maps f_1, f_2, x_1 and y_1 in the diagram (2.1) by inclusions. Consider $W = Y_1 \cup_{f_1} (X_1 \wedge I^+) \cup_{x_1} X_2$, in which I is the unit interval $[0, 1]$, I^+, the disjoint union of I and a point, and $f_1(x) \sim (x, 0)$ and $x_1(x) \sim (x, 1)$ for $x \in X_1$, and define a map $j\colon W \to Y_2$ by $j|_{Y_1} = y_1$, $j|_{X_2} = f_2$ and $j|_{X_1 \wedge I^+} =$ *a* homotopy between $y_1 f_1$ and $f_2 x_1$. Set

$$X_1' = X_1 \wedge \{1/2\}^+,\ X_2' = (X_1 \wedge I_1^+) \cup_{x_1} X_2,\ Y_1' = Y_1 \cup_{f_1} (X_1 \wedge I_0^+) \text{ and}$$
$$Y_2' = Y_2 \cup_j (W \wedge I^+),$$

where I_0 and I_1 are the closed intervals $[0, 1/2]$ and $[1/2, 1]$, respectively. Then we have homotopy equivalences $X_1' \simeq X_1$, $X_2' \simeq X_2$, $Y_1' \simeq Y_1$ and $Y_2' \simeq Y_2$. Note that $X_1' = X_2' \cap Y_1'$. Therefore replacing spectra in (2.1) by the spectra with primes and maps by the canonical inclusions, we see that the homotopy commutative diagram (2.1) are homotopy equivalent to the strict commutative diagram

$$
(2.4)\qquad
\begin{array}{ccccccccccc}
 & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle y_3} & & \downarrow{\scriptstyle z_3} & & \downarrow{\scriptstyle x_3} & \\
\longrightarrow & X_1 & \overset{f_1}{\hookrightarrow} & Y_1 & \xrightarrow{g_1} & Z_1 & \xrightarrow{h_1} & \Sigma X_1 & \longrightarrow \\
 & \cap{\scriptstyle x_1} & & \cap{\scriptstyle y_1} & & \cap{\scriptstyle z_1} & & \cap{\scriptstyle x_1} & \\
\longrightarrow & X_2 & \overset{f_2}{\hookrightarrow} & Y_2 & \xrightarrow{g_2} & Z_2 & \xrightarrow{h_2} & \Sigma X_2 & \longrightarrow \\
 & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle y_2} & & \downarrow{\scriptstyle z_2} & & \downarrow{\scriptstyle x_2} & \\
\longrightarrow & X_3 & \overset{f_3}{\hookrightarrow} & Y_3 & \xrightarrow{g_3} & Z_3 & \xrightarrow{h_3} & \Sigma X_3 & \longrightarrow \\
 & \downarrow{\scriptstyle x_3} & & \downarrow{\scriptstyle y_3} & & \downarrow{\scriptstyle z_3} & & \downarrow{\scriptstyle x_3} &
\end{array}
$$

with $X_1 = X_2 \cap Y_1$, in which we omitted the primes on the spectra,

$$Z_i = Y_i/X_i \text{ for } i = 1, 2, 3, \text{ and } U_3 = U_2/U_1 \text{ for } U = X, Y, Z.$$

Furthermore consider diagrams of the cofiber sequences

$$
(2.5)\qquad
\begin{array}{ccccccccc}
\longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z_1 & \xrightarrow{h_1} & \Sigma X_1 & \longrightarrow \\
 & \downarrow\| & & \cap\, i_1 & & \cap\, i'_1 & & \downarrow\| & \\
\longrightarrow & X_1 & \overset{i}{\subset\!\!\longrightarrow} & Y_1 \cup X_2 & \xrightarrow{p} & Z_1 \vee X_3 & \xrightarrow{d} & \Sigma X_1 & \longrightarrow \\
 & \uparrow\| & & \cup\, i_2 & & \cup\, i'_2 & & \uparrow\| & \\
\longrightarrow & X_1 & \xrightarrow{x_1} & X_2 & \xrightarrow{x_2} & X_3 & \xrightarrow{x_3} & \Sigma X_1 & \longrightarrow,
\end{array}
\qquad \text{and}
$$

$$
(2.6)\qquad
\begin{array}{ccccccccc}
\longrightarrow & Z_1 & \overset{z_1}{\subset\!\!\longrightarrow} & Z_2 & \xrightarrow{z_2} & Z_3 & \xrightarrow{z_3} & \Sigma Z_1 & \longrightarrow \\
 & \uparrow g'_1 & & \uparrow g_2 & & \uparrow\| & & \uparrow & \\
\longrightarrow & Y_1 \cup X_2 & \overset{inc}{\subset\!\!\longrightarrow} & Y_2 & \xrightarrow{pr} & Z_3 & \xrightarrow{\partial} & \Sigma Y_1 \cup X_2 & \longrightarrow \\
 & \downarrow x'_2 & & \downarrow y_2 & & \downarrow\| & & \downarrow & \\
\longrightarrow & X_3 & \xrightarrow{f_3} & Y_3 & \xrightarrow{g_3} & Z_3 & \xrightarrow{h_3} & \Sigma X_3 & \longrightarrow.
\end{array}
$$

Then (2.6) gives the commutative diagram

$$
(2.7)\qquad
\begin{array}{ccccccccc}
\longrightarrow & \pi_*(Z_1) & \xrightarrow{z_{1*}} & \pi_*(Z_2) & \xrightarrow{z_{2*}} & \pi_*(Z_3) & \xrightarrow{z_{3*}} & \pi_{*-1}(Z_1) & \longrightarrow \\
 & \uparrow g'_{1*} & & \uparrow g_{2*} & & \uparrow\| & & \uparrow g'_{1*} & \\
\longrightarrow & \pi_*(Y_1 \cup X_2) & \xrightarrow{inc_*} & \pi_*(Y_2) & \xrightarrow{pr_*} & \pi_*(Z_3) & \xrightarrow{\partial} & \pi_{*-1}(Y_1 \cup X_2) & \longrightarrow
\end{array}
$$

by applying the functor $\pi_*(-)$. In this diagram, since

$$z_{1*}(\zeta_1) = g_{2*}(\eta_2)$$

by the hypothesis, we see that

$$pr_*(\eta_2) = z_{2*}g_{2*}(\eta_2) = z_{2*}z_{1*}(\zeta_1) = 0.$$

The exactness gives an element $\nu \in \pi_*(Y_1 \cup X_2)$ such that

(2.8) $$inc_*(\nu) = \eta_2,$$

and we compute

$$z_{1*}g'_{1*}(\nu) = g_{2*}\, inc_*(\nu) = g_{2*}(\eta_2) = z_{1*}(\zeta_1).$$

This also gives an element $\zeta_3 \in \pi_*(Z_3)$ such that

$$z_{3*}(\zeta_3) = \zeta_1 - g'_{1*}(\nu).$$

Put now

(2.9) $$v' = v + \partial_*(\zeta_3) \in \pi_*(Y_1 \cup X_2),$$

and we get

(2.10) $$g'_{1*}(v') = \zeta_1.$$

We note that this is also proved using a Mayer-Vietoris type exact sequence induced from (2.7). We also have an element $\xi_3 \in \pi_*(X_3)$ defined by

(2.11) $$\xi_3 = x'_{2*}(v').$$

Then we calculate to show

$$\begin{aligned} y_{2*}(\eta_2) &= y_{2*}\, inc_*(v) && \text{(by (2.8))} \\ &= y_{2*}\, inc_*(v') && \text{(by (2.9))} \\ &= f_{3*}x'_{2*}(v') && \text{(by (2.6))} \\ &= f_{3*}(\zeta_3) && \text{(by (2.11))}. \end{aligned}$$

Thus we get the first equation in the lemma.

Consider next the commutative diagram

(2.12) $$\begin{array}{ccccccccc} \longrightarrow & \pi_*(X_1) & \xrightarrow{f_{1*}} & \pi_*(Y_1) & \xrightarrow{g_{1*}} & \pi_*(Z_1) & \xrightarrow{h_{1*}} & \pi_{*-1}(X_1) & \longrightarrow \\ & \downarrow \| & & \downarrow i_{1*} & & \downarrow i'_{1*} & & \downarrow \| & \\ \longrightarrow & \pi_*(X_1) & \xrightarrow{i_*} & \pi_*(Y_1 \cup X_2) & \xrightarrow{p_*} & \pi_*(Z_1 \vee X_3) & \xrightarrow{d_*} & \pi_{*-1}(X_1) & \longrightarrow \\ & \uparrow \| & & \uparrow i_{2*} & & \uparrow i'_{2*} & & \uparrow \| & \\ \longrightarrow & \pi_*(X_1) & \xrightarrow{x_{1*}} & \pi_*(X_2) & \xrightarrow{x_{2*}} & \pi_*(X_3) & \xrightarrow{x_{3*}} & \pi_{*-1}(X_1) & \longrightarrow, \end{array}$$

induced from (2.5). Then we see that

$$p_*(v) = (g'_{1*}(v'),\ x'_{2*}(v')) = (\zeta_1,\ \xi_3)$$

in $\pi_*(Y_1 \cup X_2/X_1) = \pi_*(Z_1 \vee X_3) = \pi_*(Z_1) \oplus \pi_*(X_3)$, by (2.10) and (2.11) and so

$$0 = d_*(p_*(v)) = d_*((\zeta_1,\ \xi_3)) = h_{1*}(\zeta_1) + x_{3*}(\xi_3).$$

Thus we have the other equation. q.e.d.

§3. Generalized Adams spectral sequence

Here we recall [1] the construction of a generalized Adams spectral sequence to argue about the differential in a closer look.

Let E be a ring spectrum such that $E_*(E) = \pi_*(E \wedge E)$ is flat over $E_* = \pi_*(E)$ as a right module whose structure is induced from the multiplication $\mu: E \to E \wedge E$.

Let $i: S^0 \to E$ denote the unit of the ring spectrum E, where S^0 means the sphere spectrum, and the unit i induces the cofiber sequence $\to S^0 \xrightarrow{i} E \xrightarrow{j} \bar{E} \xrightarrow{k} S^1 \to$. For a spectrum X, define

$$X_n = \bar{E}^{\wedge n} \wedge X \quad \text{and} \quad EX_n = E \wedge X_n,$$

and

$$i_n = i \wedge id: X_n = S^0 \wedge X_n \longrightarrow E \wedge X_n = EX_n,$$
$$j_n = j \wedge id: EX_n = E \wedge X_n \longrightarrow \bar{E} \wedge X_n = X_{n+1}, \quad \text{and}$$
$$k_n = k \wedge id: X_{n+1} = \bar{E} \wedge X_n \longrightarrow S^1 \wedge X_n = \Sigma X_n,$$

where $Z^{\wedge n} = Z \wedge \cdots \wedge Z$ (n copies of Z), and id denotes the identity maps. Then we have the *E-Adams tower*

(3.1) $$X = X_0 \xleftarrow{k_0} \Sigma^{-1} X_1 \xleftarrow{k_1} \cdots \xleftarrow{k_{n-1}} \Sigma^{-n} X_n \xleftarrow{k_n} \cdots$$

and a cofiber sequence

(3.2) $$X_n \xrightarrow{i_n} EX_n \xrightarrow{j_n} X_{n+1} \xrightarrow{k_n} \Sigma X_n,$$

on which applying the homotopy functor $\pi_*(-)$, we obtain the exact couple

(3.3) $$\begin{array}{ccc} \pi_*(X_{n+1}) & \xrightarrow{k_{n*}} & \pi_*(X_n) \\ & {}_{j_{n*}}\nwarrow \quad \swarrow_{i_{n*}} & \\ & \pi_*(EX_n). & \end{array}$$

The exact couple gives the spectral sequence by setting $E_1^{n,*}(X) = \pi_*(EX_n)$ in a usual fashion. That is to say, take

(3.4) $$E_1^{s,*}(X) = \pi_*(EX_s) = E_*(X_s), \quad \text{and}$$
$$d_1 = i_{s+1*} j_{s*}: E_1^{s,*}(X) \longrightarrow E_1^{s+1,*}(X),$$

and inductively define

(3.5) $$E_{r+1}^{s,*} = \mathrm{Ker}\,\{d_r: E_r^{s,*} \longrightarrow E_r^{s+r,*}\}/\mathrm{Im}\,\{d_r: E_r^{s-r,*} \longrightarrow E_r^{s,*}\} \qquad \text{for } r \geq 1, \text{ and}$$
$$d_r = i_{s+r*} k_{s+r-1*}^{-1} \cdots k_{s+1}^{-1} j_{s*} \qquad \text{for } r > 1.$$

We call this spectral sequence *E-Adams spectral sequence*. This spectral sequence is natural with respect to a map $f: X \to Y$ of spectra, since it induces a map of exact couples obtained from X and Y. The induced maps $f_n: X_n \to Y_n$ and $Ef_n: EX_n \to EY_n$ are defined as follows:

$$f_n = id \wedge f: X_n = \bar{E}^{\wedge n} \wedge X \longrightarrow Y_n = \bar{E}^{\wedge n} \wedge Y \quad \text{and}$$
$$Ef_n = id \wedge f_n: EX_n = E \wedge X_n \longrightarrow E \wedge Y_n = EY_n.$$

Since $Ef_{n*}\colon \pi_*(EX_n) \to \pi_*(EY_n)$ is the same map as $f_{n*}\colon E_*(X_n) \to E_*(Y_n)$, we hereafter will use the notation f_{n*} instead of Ef_{n*}. Consider the filtration $\cdots \subset F_n \subset F_{n-1} \subset \cdots \subset F_0 = \pi_*(X)$ defined by

$$F_n = \operatorname{Im}\{k(n)_* = k_{0*}k_{1*}\cdots k_{n-1*}\colon \pi_*(X_n) \longrightarrow \pi_*(X_0) = \pi_*(X)\}.$$

Then the spectral sequence is said to *converge* to $\pi_*(X)$ if

$$\bigcap_n F_n = 0 \quad \text{and} \quad E^s_\infty(X) = F_s/F_{s+1},$$

for $E^s_\infty(X) = \operatorname{dirlim}_r E^s_r(X)$. Note that there is a canonical map $c\colon \pi_*(X) \to \pi_*(E^\wedge X)$. An element ξ of $\pi_*(X)$ is said to be *detected* by an element x of E_2-term if $c(\xi)(\neq 0)$ represents $\bar{x} \in \mathscr{G}\pi_*(E^\wedge X) = \bigoplus_n F_n/F_{n+1} = E_\infty(X)$ corresponding to x in the E_2-term $E_2(X)$.

By the assumption on E, which says that $E_*(E)$ is E_*-flat, the E_2-term is shown (*cf.* [1]) to be the Ext-group:

$$E_2^{s,t}(X) = \operatorname{Ext}^{s,t}_{E_*(E)}(E_*, E_*(X)).$$

Furthermore Bousfield showed [2] that it converges to $\pi_*(E^\wedge X)$. Here $E^\wedge X$ is the E-nilpotent completion of X defined as follows:

Consider the composition $k(n) = k_0k_1\cdots k_{n-1}\colon X_n \to X_0 = X$ and write its cofiber by X^n. Then 3×3 lemma gives the cofiber sequence

$$X^{n+1} \longrightarrow X^n \longrightarrow \Sigma^{-n+1} EX_n \longrightarrow \Sigma X^{n+1},$$

which gives another spectral sequence in the same way as stated above. Here by 3×3 lemma, we mean that in the commutative diagram (2.1), if 5 rows and columns out of 6 are cofiber sequences, then so is the other. It turns out that this spectral sequence is the same as the one given above. Now define the *E-nilpotent completion* by

$$E^\wedge X = \operatorname{holim}_{\leftarrow} X^n,$$

and we see that the spectral sequence converges to the homotopy groups $\pi_*(E^\wedge X)$ [2]. Furthermore assume that the core of $E_* = \pi_*(E)$ is the ring $\boldsymbol{Z}[J^{-1}]$ for a set of prime numbers J, or the cyclic ring $\boldsymbol{Z}/n$, and denote L_EX the Bousfield E-localization, where the core of a ring is the subring $\{r \in R \mid r \otimes 1 = 1 \otimes r \in R \otimes_{\boldsymbol{Z}} R\}$. Then if both of E and X are connective spectra, then the E-Adams spectral sequence converges to the homotopy $\pi_*(L_EX)$ of the localization L_EX. Besides, L_EX is also connective if so is X, by [2, Th.s 6.5, 6.6 and Prop. 2.4]. Since $L_EL_EX = L_EX$, the spectral sequence for $\pi_*(L_EX)$ converges to $\pi_*(L_EX)$. So the hypothesis (1.6) is not void.

We now consider a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Then it induces the maps of E-Adams towers (see (3.1))

$$X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \xrightarrow{h_n} \Sigma X_n$$

for each $n \geq 0$, which induces the natural maps of spectral sequences. In fact, we have exact sequences

(3.6) $$\pi_*(X_n) \xrightarrow{f_{n*}} \pi_*(Y_n) \xrightarrow{g_{n*}} \pi_*(Z_n) \xrightarrow{h_{n*}} \pi_{*-1}(X_n),$$

and

(3.7) $$E_*(X_n) \xrightarrow{f_{n*}} E_*(Y_n) \xrightarrow{g_{n*}} E_*(Z_n) \xrightarrow{h_{n*}} E_*(X_n)$$

of E_1-terms of the spectral sequence. The sequence (3.6) gives rise to the natural map between the filtrations of the spectral sequence. Furthermore if we assume that

$$E_*(E) \text{ is flat over } E_*$$

and

$$E_*(h) = 0,$$

then we have induced long exact sequence

(3.8) $$E_2^n(X) \xrightarrow{f_{n*}} E_2^n(Y) \xrightarrow{g_{n*}} E_2^n(Z) \xrightarrow{\delta} E_2^{n+1}(X)$$

of E_2-terms, since the exact sequence (3.7) is split and the resulting short exact sequence induces the long exact sequence of Ext groups which is the E_2-terms.

§4. Proof of theorem

In this section we fix a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

and a ring spectrum E such that

$$E_*(E) \text{ is flat over } E_*,$$

(4.1) $$E_{2t-1}(W) = 0 \text{ for } W = S^0, E, X, Y, \text{ and } Z, \text{ and}$$

$$E \wedge W = W \text{ for } W = X, Y, \text{ and } Z.$$

Here $E_t(W)$ stands for t dimensional E_*-homology group of W, not for E_t-term of the spectral sequence computing $\pi_*(W)$. The E_t-terms would involve superscripts and we can tell the difference. For the ring spectrum E, we have the E-Adams spectral sequence

$$E_2^{s,t}(W) = \mathrm{Ext}_{E_*(E)}^{s,t}(E_*, E_*(W)) \Longrightarrow \pi_*(E \wedge W),$$

for a spectrum W. Note that the second condition on E in (4.1) gives not only

$$E_*(h) = 0,$$

but also on the spectral sequence,

$$E_2^{s,2t-1}(W) = 0 \quad \text{for} \quad W = X,\ Y, \text{ and } Z,$$

which further gives a relation on the differentials of the spectral sequence:

$$d_{2t} = 0 \in E_{2t}^*(W) \quad \text{for} \quad W = X,\ Y, \text{ and } Z.$$

Furthermore we assume that the spectrum Y satisfies the following conditions on the spectral sequence:

$$d_r = 0 \qquad \text{for } r > 3.$$

Let η be an element of $\pi_*(Y)$ detected by an element $f_*(x) \in E_2^s(Y)$ for $x \in E_2^s(X)$ with

$$d_{2t+1}(x) = x_0 \neq 0.$$

Then there exists an element $\tilde{\eta} \in \pi_*(Y_s)$ such that

$$(4.2) \qquad i_{s*}(\tilde{\eta}) = f_*(x).$$

Hereafter we abuse names for elements of E_2-term and their representatives in the E_1-term. The equation $d_{2t+1}(x) = x_0$ is interpreted to give elements x_i's of $\pi_*(X_{s+i})$ for $1 \leq i \leq 2t+1$ such that

$$(4.3) \qquad j_{s*}(x) = x_1,\ k_{s+l-1*}(x_l) = x_{l-1} \quad \text{and} \quad i_{s+2t+1*}(x_{2t+1}) = x_0.$$

By the naturality of the differential of the spectral sequence, we see that $f_*(x_0) = f_*(d_{2t+1}(x)) = d_{2t+1}(f_*(x)) = 0$ since $f_*(x)$ is a permanent cycle. Therefore $f_*(x_0)$ is zero in the E_2-term or is hit by the differential d_3 by the hypothesis on Y that $d_r = 0$ for $r = 2$ and $r > 3$.

First we study the case that $f_*(x_0) = d_3(y_0) \neq 0$ in $E_2^{s+2t+1}(Y) = E_3^{s+2t+1}(Y)$. In this case, we see that

$$f_{s+2t+1*}(x_{2t+1}) \neq 0 \quad \text{in} \quad \pi_*(Y_{s+2t+1}),$$

since $i_{s+2t+1*}f_{s+2t+1*}(x_{2t+1}) = f_{s+2t+1*}i_{s+2t+1*}(x_{2t+1}) = f_*(x_0) \neq 0$. We also see that $f_{s+1*}(x_1) = f_{s+1*}j_{s*}(x) = j_{s*}f_*(x) = j_{s*}i_{s*}(\tilde{\eta}) = 0$, and so the hypothesis on Y that $E_4^*(Y) = E_\infty^*(Y)$ indicates that

$$(4.4) \qquad j_{s+2t-2*}(y_0) = f_{s+2t-1*}(x_{2t-1}).$$

This is shown as follows: if $f_{s+i*}(x_i) = 0$ and $f_{s+i+1*}(x_{i+1}) \neq 0$ for an integer $i < 2t-2$, then there exists a non-zero element $u \in E_3^{s+i}(Y)$ such that $j_{s+i*}(u) = f_{s+i+1*}(x_{i+1})$ by the exactness since $k_{s+i*}f_{s+i+1*}(x_{i+1}) = f_{s+i*}k_{s+i*}(x_{i+1}) = f_{s+i*}(x_i) = 0$. Besides, $d_3(u)$ is represented by $i_{s+i+3*}(v_3)$ for $v_3 \in \pi_*(Y_{s+i+3})$ such that $k_{s+i+1*}k_{s+i+2*}(v_3) = j_{s+i*}(u)$, and so we can take $v_3 = f_{s+i+3*}(x_{i+3})$. Then,

$$
\begin{aligned}
d_3(u) &= i_{s+i+3*}(v_3) \\
&= i_{s+i+3*}(f_{s+i+3*}(x_{i+3})) \\
&= i_{s+i+3*}(k_{s+i+3*}f_{s+i+4*}(x_{i+4})) \\
&= 0
\end{aligned}
$$

since $i_{s+i+3*}k_{s+i+3*} = 0$, where the existence of x_{i+4} follows from the hypothesis $i < 2t-2$. The hypothesis that $E_4^*(Y) = E_\infty^*(Y)$ then implies that $d_r(u) = 0$, which means that u is a permanent cycle and so we have $j_{s+i*}(u) = f_{s+i+1*}(x_{i+1}) = 0$. This contradicts to $f_{s+i+1*}(x_{i+1}) \neq 0$.

We also see that $f_{s+2t-1*}(x_{2t-1}) \neq 0$ since if it does not hold, then $f_{s+2t+1*}(x_0)$ should be hit by d_1, which is a contradiction to $f_*(x_0) \neq 0$ in the E_2-term. Thus we have

$$f_{s+2t-2*}(x_{2t-2}) = 0 \quad \text{and} \quad f_{s+2t-1*}(x_{2t-1}) \neq 0.$$

Put $\xi = j_{s+2t-2*}(y_0) - f_{s+2t-1*}(x_{2t-1})$. Then we see that $k_{s+2t-2*}(\xi) = 0$, and so there exists an element $y_1 \in E_2^{s+2t-2}(Y)$ such that $j_{s+2t-2*}(y_1) = \xi$. Since $d_3(y_0) = f_*(x_0)$, we have an element ζ such that $k_{s+2t-1*}k_{s+2t*}(\zeta) = j_{s+2t-2*}(y_0)$ and $i_{s+2t+1*}(\zeta) = f_*(x_0)$, and put an element $\xi_3 = f_{s+2t+1*}(x_{2t+1}) - \zeta$. Then $i_{s+2t+1*}(\xi_3) = f_*(x_0) - f_*(x_0) = 0$, which shows $d_3(y_1) = 0$ and so y_1 is a permanent cycle by the hypothesis $E_4^*(Y) = E_\infty^*(Y)$ in the same way as that shown for u above. Hence $j_{s+2t-2*}(y_1) = \xi = 0$, and we have (4.4).

Consider now the commutative diagram

$$
(4.5) \qquad
\begin{array}{ccccccccc}
 & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} & \\
\longrightarrow & \Sigma^{-1}EZ_m & \xrightarrow{h_m} & EX_m & \xrightarrow{f_m} & EY_m & \xrightarrow{g_m} & EZ_m & \longrightarrow \\
 & \downarrow{\scriptstyle j_m} & & \downarrow{\scriptstyle j_m} & & \downarrow{\scriptstyle j_m} & & \downarrow{\scriptstyle j_m} & \\
\longrightarrow & \Sigma^{-1}Z_{m+1} & \xrightarrow{h_{m+1}} & X_{m+1} & \xrightarrow{f_{m+1}} & Y_{m+1} & \xrightarrow{g_{m+1}} & Z_{m+1} & \longrightarrow \\
 & \downarrow{\scriptstyle k_m} & & \downarrow{\scriptstyle k_m} & & \downarrow{\scriptstyle k_m} & & \downarrow{\scriptstyle k_m} & \\
\longrightarrow & Z_m & \xrightarrow{h_m} & \Sigma X_m & \xrightarrow{f_m} & \Sigma Y_m & \xrightarrow{g_m} & \Sigma Z_m & \longrightarrow \\
 & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} &
\end{array}
$$

of Adams towers, in which

$$m = s + 2t - 2.$$

We then apply Lemma 2.3 to this diagram and elements in (4.4), and obtain an element $\zeta \in \pi_*(Z_{s+2t-2})$ such that

(4.6) $\qquad h_{s+2t-2*}(\zeta)=k_{s+2t-2*}(x_{2t-1})$ and $g_{s+2t-2*}(y_0)=-i_{s+2t-2*}(\zeta)$.

Let $k_*=k_{s+1*}\cdots k_{s+2t-3*}: \pi_*(Z_{s+2t-2})\to\pi_*(Z_{s+1})$ and $\zeta_1=k_*(\zeta)$. Then $h_{s+1*}(\zeta_1)=h_{s+1*}k_*(\zeta)=k_*h_{s+2t-2*}(\zeta)=k_*k_{s+2t-2*}(x_{2t-1})=x_1$. We again apply Lemma 2.3 to the diagram

(4.7)
$$\begin{array}{ccccccccccc}
 & & \downarrow k_s & & \downarrow k_s & & \downarrow k_s & & \downarrow k_s & & \\
\longrightarrow & & \Sigma^{-1}Z_s & \xrightarrow{h_s} & X_s & \xrightarrow{f_s} & Y_s & \xrightarrow{g_s} & Z_s & \longrightarrow & \\
 & & \downarrow i_s & & \downarrow i_s & & \downarrow i_s & & \downarrow i_s & & \\
\longrightarrow & & \Sigma^{-1}EZ_s & \xrightarrow{h_s} & EX_s & \xrightarrow{f_s} & EY_s & \xrightarrow{g_s} & EZ_s & \longrightarrow & \\
 & & \downarrow j_s & & \downarrow j_s & & \downarrow j_s & & \downarrow j_s & & \\
\longrightarrow & & \Sigma^{-1}Z_{s+1} & \xrightarrow{h_{s+1}} & X_{s+1} & \xrightarrow{f_{s+1}} & Y_{s+1} & \xrightarrow{g_{s+1}} & Z_{s+1} & \longrightarrow & \\
 & & \downarrow k_s & & \downarrow k_s & & \downarrow k_s & & \downarrow k_s & & \\
\longrightarrow & & Z_s & \xrightarrow{h_s} & \Sigma X_s & \xrightarrow{f_s} & \Sigma Y_s & \xrightarrow{g_s} & \Sigma Z_s & \longrightarrow, &
\end{array}$$

and the relation

$$i_{s*}(\tilde{\eta})=f_*(x)$$

of (4.2). Then we have an element $\zeta'\in\pi_*(Z_s)$ such that

(4.8) $\qquad j_{s*}(x)=h_{s+1*}(\zeta')$ and $g_{s*}(\tilde{\eta})=k_{s*}(\zeta')$.

Here note that $j_{s*}(x)=x_1$ by (4.3). We can take ζ' to be ζ_1 in (4.6). In fact, if we put $v=\zeta'-\zeta_1$, then we compute $h_{s+1*}(v)=x_1-x_1=0$ by (4.8) and (4.6). We then have an element $\varphi\in\pi_*(Y_{s+1})$ such that $g_{s+1*}(\varphi)=v$. Put $\hat{\eta}=\tilde{\eta}-k_{s*}(\varphi)$, and we see that

$$i_{s*}(\hat{\eta})=f_*(x),$$

which enables us to use $\hat{\eta}$ instead of $\tilde{\eta}$. The element $\hat{\eta}$ also satisfies

$$\begin{aligned}
g_{s*}(\hat{\eta}) &= k_{s*}(\zeta')-k_{s*}g_{s+1*}(\varphi)\\
&= k_{s*}(\zeta')-k_{s*}(v)\\
&= k_{s*}(\zeta')-k_{s*}(\zeta'-\zeta_1)\\
&= k_{s*}(\zeta_1).
\end{aligned}$$

Thus we have shown the first statement in Theorem.

Next suppose that $f_*(x_0)$ is zero in the E_2-term. In this case the argument used in the proof is almost the same as that of the previous case. Then there exists an element $w\in E_1^{s+2t}(Y)=\pi_*(EY_{s+2t})$ such that $d_1(w)=i_{s+2t+1*}j_{s+2t*}(w)=f_{s+2t+1*}(x_0)$

which is not null since $h_{s+2t+1*}=0$ and $x_0 \neq 0$. If $f_{s+l*}(x_l)=0$ and $f_{s+l+1}(x_{l+1}) \neq 0$ for $l<2t-1$, then there is an element u such that $j_{s+l*}(u)=f_{s+l+1}(x_{l+1})$. Furthermore, $k_{s+l+1*}\cdots k_{s+2t*}(f_{s+2t+1*}(x_{2t+1})-j_{s+2t*}(w))=f_{s+l+1}(x_{l+1})$ implies $d_3(u)=0$ and hence u is a permanent cycle and $f_{s+l+1}(x_{l+1})=0$, which is a contradiction. Thus $f_{s+2t*}(x_{2t})=0$ and $f_{s+2t+1}(x_{2t+1}) \neq 0$, which shows

$$j_{s+2t*}(w)=f_{s+2t+1*}(x_{2t+1}).$$

Now apply Lemma 2.3 to this equation and we have an element ζ such that

$$g_{s+2t*}(w)=i_{s+2t*}(\zeta) \quad \text{and} \quad h_{s+2t*}(\zeta)=k_{s+2t*}(x_{2t+1}).$$

Moreover in the E_2-term, the connected homomorphism $\delta\colon E_2^{2t}(Z) \to E_2^{2t+1}(X)$ is shown to send $g_{s+2t*}(w)$ to x_0 by the definition of δ.

Denote the composition $k_*=k_{s+1*}\cdots k_{s+2t-1*}\colon \pi_*(Z_{s+2t}) \to \pi_*(Z_{s+1})$ and put

$$\zeta_1=k_*(\zeta). \tag{4.9}$$

Then $h_{s+1*}(\zeta_1)=h_{s+1*}k_*(\zeta)=k_*h_{s+2t*}(\zeta)=k_*k_{s+2t*}(x_{2t+1})=x_1$. We again apply Lemma 2.3 to the relation $i_{s*}(\tilde{\eta})=f_*(x)$ of (4.2). Then similarly to (4.8) we have an element $\zeta' \in \pi_*(Z_s)$ such that $x_1=j_{s*}(x)=h_{s+1*}(\zeta')$ and $g_{s*}(\tilde{\eta})=k_{s*}(\zeta')$. We can take ζ' to be ζ_1 in (4.9) also in this case. Hence $g_*(\eta)$ is detected by the element $z \in E_2^{2t}(Z)$ represented by $g_{s+2t*}(w)$ which satisfies $\delta(z)=x_0$.

References

[1] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, 1974.

[2] A. K. Bousfield, The localization of spectra with respect to homology, *Topology*, **18** (1979), 257–281.

[3] M. Hikida, Relations between several Adams spectral sequences, *Hiroshima Math. J.* **19** (1989), 37–76.

[4] D. C. Johnson, H. R. Miller, W. S. Wilson, and R. S. Zahler, Boundary homomorphisms in the generalized Adams spectral sequence and the nontriviality of infinitely many γ_t in the stable homotopy, *Reunion sobre teorie de homotopia*, Northwestern Univ., 1974, Soc. Mat. Mexicana (1975), 47–59.

[5] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, 1986.