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Lifting of Functions and Hyperfunctions

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1. Introduction

It is very interesting to consider hyperfunctions using nonstandard analysis. In the present paper we would like to consider hyperfunctions using the notion of lifting of functions in the theory of nonstandard analysis.

The following definition was given in [3].

DEFINITION 1.1. Let $R^+ = \{y \in R \mid y > 0\}$, and $F = \{(0, y) \mid y \in R^+\}$. Then F has the finite intersection property. We denote by $\mathscr{F}$ one of the ultrafilters containing F.

Let K be a nonempty set and $x_1(y), x_2(y) \in \prod_{y \in R^+} K$. We define a relation $\sim$ as follows:
$x_1(y) \sim x_2(y)$, if and only if a condition $\{y \in R^+ \mid x_1(y) = x_2(y)\} \in \mathscr{F}$ is satisfied. The relation $\sim$ is an equivalence relation. We define $^*K$ to be the quotient set $\prod_{y \in R^+} K/\sim$.

An element of the set $^*R$ (resp. $^*C$) is called a hyperreal (resp. hypercomplex) number, and an element of the set $^*\mathrm{Map}(R, C)$ is called a generelized function. The equivalence class determined by a function $x(y) \in \prod_{y \in R^+} K$ will be denoted by $[x(y)]$.

We can consider the set $^*R$ is a subset of the set $^*C$. The sets $^*R$ and $^*C$ are made into commutative fields by defining the addition, the subtraction, the product, and the quotient in the usual way.

Similarly we define a set $^*\mathrm{Map}(K \times R^+)$. Let $\alpha = [x(y)]$ be a hyperreal (resp. hypercomplex) number. Then the element $[(x(y), y)]$ of the set $^*(R \times R^+)$ (resp. $^*(C \times R^+)$) is uniquely determined. The element $[(x(y), y)]$ is called a graph of $[x(y)]$.

According to M. Saito [4], we shall give the following definition.

DEFINITION 1.2. Let f be a complex valued function on an open interval I, and F be a complex valued function on the set $R \times R^+$. The function F is called a uniform lifting of the function f if the following condition is satisfied.

$$\mathrm{st}\, F(\alpha) = f(\mathrm{st}\, \alpha) \qquad \text{for every } \alpha \in {}^*I \tag{1.1}$$

If condition (1.1) is satisfied except a null set, a set which is Loeb measure zero, we say that F is a lifting of f.

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2. Lifting of functions and hyperfunctions

We consider a real variable and complex valued function f and a complex valued function F defined on the set $R \times R^{+}$. We would like to use a notation $(x, y) \to (x_0, +0)$, which means (x, y) tends to $(x_0, 0)$ satisfying $y > 0$.

THEOREM 2.1. *Suppose that f and F satisfy the following condition at $x_0 \in R$.*

$$\lim_{(x,y)\to(x_0,+0)} F(x, y) = f(x_0) \tag{2.1}$$

Then we have, if $\alpha = [x(y)] \in {}^*R$ and $\operatorname{st} \alpha = x_0$, then $\operatorname{st}[F(x(y), y)] = f(x_0)$.

PROOF. For a given positive real number ε, we can find a positive real number δ such that $|F(x, y) - f(x_0)| < \varepsilon$ for all x and y satisfying $|(x, y) - (x_0, 0)| < \delta$ and $y > 0$.

We put $A = \{y \in R^{+} \mid |x(y) - x_0| < \delta/2\}$ and $B = (0, \delta/2)$. Since $\operatorname{st} \alpha = x_0$ we have $A \in \mathscr{F}$, and clearly we have $B \in \mathscr{F}$. Hence we have $A \cap B \in \mathscr{F}$.

If $y \in A \cap B$, then $|(x(y), y) - (x_0, 0)| < \delta$, and it follows that

$$|F(x(y), y) - f(x_0)| < \varepsilon.$$

Finally, we have

$$\{y \in R^{+} \mid |F(x(y), y) - f(x_0)| < \varepsilon\} \supset A \cap B.$$

Since $A \cap B \in F$, we have $\operatorname{st}[F(x(y), y)] = f(x_0)$.

THEOREM 2.2. *Let $x_0 \in R$. Suppose that there exists a positive real number δ such that*

$$\lim_{y \to +0} F(x, y) = f(x) \quad \textit{for} \quad x \in (x_0 - \delta, x_0 + \delta).$$

a) *Assume that condition* (2.1) *is satisfied. Then the function f is continuous at x_0.*
b) *We define F at $(x, 0)$ for each $x \in (x_0 - \delta, x_0 + \delta)$ by letting $F(x, 0) = f(x)$ and a set U_d^{+} for a positive real number d to be a set $U(x_0, d) \cap R \times R^{+}$.*

Assume that

(2.2) *there exists a real number d less than δ, and the function F is continuous on the set $\bar{U}_d^{+}$.*

Then the functions f and F satisfy condition (2.1).

PROOF. a) For a given positive real number ε, there exists a positive real number δ_1, which is smaller than δ, such that if $|(x, y) - (x_0, 0)| < \delta_1$ and $y > 0$, then

$$|F(x, y) - f(x_0)| < \varepsilon.$$

Let $|x - x_0| < \delta_1/2$, then there exists a positive real number δ_2 such that if $0 < y < \delta_2$, then

$$|F(x, y) - f(x)| < \varepsilon.$$

Hence, for every y with $0 < y < \min\{\delta_1/2, \delta_2\}$, we have

$$|f(x) - f(x_0)| \leqq |f(x) - F(x, y)| + |F(x, y) - f(x_0)|$$
$$< \varepsilon + \varepsilon = 2\varepsilon.$$

This completes the proof of a).

b) Since F is continuous on a compact set $\bar{U}_\alpha^+$, f and F satisfy condition (2.1). Let $\alpha = [x(y)]$ be a hyperreal number, then $[F(x(y), y)]$ is a hypercomplex number defined on the graph of α, so we shall write $[F(x(y), y)] = F(\alpha)$.

Assume that f and F satisfy condition (2.1), and $\mathrm{st}\,\alpha = x_0$ for a hyperreal number α. Then we have

$$\mathrm{st}\, F(\alpha) = f(x_0) = f(\mathrm{st}\,\alpha).$$

Hence we have the following theorem.

THEOREM 2.3. a) *Assume that f and F satisfy condition* (2.1) *at every point of an open interval I. Then F is a uniform lifting of f.*

b) *If condition* (2.1) *is satisfied, except a finite subset of I, then F is a lifting of f.*

Let Ω be a complex neighbourhood (see A. Kaneko [2]) of an open interval I having a property,

let $\Omega_+ = \Omega \cap R^+$ and $\Omega_- = \Omega \cap R^-$ where $R^- = \{y \in R \mid y < 0\}$, then

$$\Omega \setminus I = \Omega_+ \cap \Omega_-.$$

Let $U(z)$ be a horomorphic function defined on $\Omega \setminus I$, and let $U_\pm(z) = U(z)|_{\Omega_\pm}$.

Suppose that

$$f(x) = \lim_{y \to +0} \{U_+(x + iy) - U_-(x - iy)\} \qquad \text{for every } x \in I.$$

Then the function F defined by

$$\begin{aligned} F(x, y) &= U_+(x + iy) - U_-(x - iy) \quad \text{if} \quad x + iy \in \Omega_+ \text{ and } x - iy \in \Omega_-, \text{ and} \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{aligned}$$

has the following properties:

c) If F satisfies condition (2.2) at every point of I, then F is a uniform lifting of f.

d) If F satisfies condition (2.2) except a finite subset of I, then F is a lifting of f.

EXAMPLE 2.4 (Heaviside function). Let F be a function defined by

$$F(x, y) = \frac{1}{2\pi}\{\mathrm{Arg}(-x + iy) - \mathrm{Arg}(-x - iy)\},$$

and let f be a function defined by

$$f(x)=0 \quad \text{for} \quad x<0,\ f(0)=1/2,\ f(x)=1 \quad \text{for} \quad x>0,$$

then F is a lifting of f.

EXAMPLE 2.5 (Dirac's delta function). Let F be a function defined by

$$F(x, y) = -\frac{1}{2\pi i}\left(\frac{1}{x+iy} - \frac{1}{x-iy}\right) = \frac{y}{\pi(x^2+y^2)},$$

and let f be a function defined by

$$f(x)=0 \quad \text{for} \quad x \neq 0,\ f(0)=+\infty,$$

then F is a lifting of f.

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