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Note on the Bousfield Localization with Respect to $E(n)$

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§1. Introduction

For a generalized homology theory E_*, Bousfield [2] defines the localization $L_E X$ of a spectrum X with respect to E_*, and discuss about convergence of a generalized Adams spectral sequence based on E_*. Consider the Johnson-Wilson spectrum $E(n)$ ([3]) for each positive integer n and a prime p, whose coefficient ring is $E(n)_* = \boldsymbol{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$. This spectrum induces a generalized homology theory $E(n)_*$. Then there exists the generalized Adams spectral sequence converging to a homotopy group $\pi_*(L_{E(n)}X)$ of $E(n)$-localization of a spectrum X with E_2-term $E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(X))$ which we denote $E(n)_2^{s,t}(X)$. On the other hand, we also have the Adams-Novikov spectral sequence converging to a homotopy group $\pi_*(X)$ of a p-local connected spectrum X with E_2-term $E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(X))$ which we denote $BP_2^{s,t}(X)$.

Let X be a p-local connected spectrum and $\eta_X: X \to L_{E(n)}X$ be the localization map, that is, the homology theory $E(n)_*(-)$ induces an isomorphism $E(n)_*(\eta_X): E(n)_*(X) \to E(n)_*(L_{E(n)}X)$. Then this gives rise to a map $\eta_{X*}: BP_2^{s,t}(X) \to E(n)_2^{s,t}(X)$. We also have a map $\Phi_*: BP_2^{s,t}(X) \to E(n)_2^{s,t}(X)$ induced by the Thom map $\Phi: BP \to E(n)$. Here we have

THEOREM. *The Thom map $\Phi: BP \to E(n)$ induces the localization map $\eta_X: X \to L_{E(n)}X$ via the generalized Adams spectral sequences.*

This theorem means that the maps between the E_2-terms given above are the same. It seems that it is a forklore but there does not seem to appear anywhere.

§2. Bousfield localization

Throughout this paper we work on the homotopy category of spectra.

We begin with the definition of the Bousfield localization ([2], see also [4]). Let $E_*(-)$ denote a generalized homology theory. We call X *E_*-local* if $[C, X]_* = 0$ for any C with $E_*(C) = 0$. This definition implies immediately

COROLLARY 2.1. *Let L be E_*-local spectrum. Then each map $f: X \to Y$ with*

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$E_*(f)$ isomorphism induces an isomorphism

$$f^*: [Y, L]_* \cong [X, L]_*.$$

PROOF. Consider the cofibering $X \xrightarrow{f} Y \to C$. Then the assumption on f implies $E_*(C) = 0$, which shows $[C, L]_* = 0$ since L is E_*-local. Now the corollary follows from the exact sequence induced from the cofibering above. q.e.d.

COROLLARY 2.2. *Suppose that both of spectra X and Y are E_*-local. If a map $f: X \to Y$ induces an isomorphism $E_*(f)$, then f is a homotopy equivalence.*

PROOF. The corollary above shows an isomorphism $f^*: [Y, X]_* \cong [X, X]_*$ and define a map $g: Y \to X$ by $g = (f^*)^{-1}(1_X)$. Then $gf = f^*(g) = 1_X$.

Similarly the map g gives rise to a map f' defined by $f' = g^{-1}(1_Y)$ using the isomorphism $g^{-1}: [X, Y]_* \cong [Y, Y]_*$ shown by Corollary 2.1. Therefore we have $f'g = 1_Y$.

Note that $f = 1_Y f = f'gf = f' 1_X = f'$, and we see that f is a homotopy equivalence. q.e.d.

We call a E-local spectrum $L_E X$ the *localization* of a spectrum X with respect to $E_*(-)$ if there exists a map $\eta_X: X \to L_E X$, which is called the *localization map*, such that:

(i) the induced map $E_*(\eta_X)$ is an isomorphism, and

(ii) if there is a map $f: X \to Y$ with Y E-local such that $E_*(f)$ is an isomorphism, then there exists uniquely a map $g: Y \to L_E X$ such that $gf = \eta_X$.

THEOREM 2.3 ([2]). *Every homology theory $E_*(-)$ has its localization $L_E X$ of a spectrum X. Furthermore L_E is functorial.*

By the definition of the localization map η_X, we give $E_*(L_E X)$ the same structure as $E_*(X)$ by η_X. Therefore we assume that $E_*(\eta_X)$ is the identity $1: E_*(X) = E_*(L_E X)$. The localization has the following elementary properties:

Proposition 2.4. *For a spactrum X, we have*

(i) *$L_E X$ is uniquely defined.*

(ii) *$L_E L_E = L_E$.*

(iii) *if there is a map $f: X \to Y$ with Y E-local such that $E_*(f)$ is an isomorphism, then there exists a map $h: L_E X \to Y$ such that $h\eta_X = f$.*

PROOF. (i) Suppose that there exists another E_*-localization $L'_E X$ of a spectrum X. Then we have an E_*-equivalence $\eta'_X: X \to L'_E X$. The second condition of the definition indicates the existence of the map $g: L'_E \to L_E$ such that $g\eta'_X = \eta_X$. Since both η'_X and η_X are E_*-equivalences, g is also an E_*-equivalence. Therefore the map g turns out to be a homotopy equivalent by Corollary 2.2.

(ii) Both $L_E X$ and $L_E L_E X$ are E_*-local and the map $\eta_{L_E X}: L_E X \to L_E L_E X$ is an

E_*-equivalence. This case again follows from Corollary 2.2.

(iii) Consider the fiber $F_E X$ of the map $\eta_X: X \to L_E X$, and we have the long exact sequence $\cdots \to [L_E X, Y]_* \xrightarrow{\eta_X^*} [X, Y]_* \to [F_E X, Y]_* \to \cdots$. Note that $E_*(F_E X) = 0$ since η_X^* is E_*-equivalent. Thus $[F_E X, Y]_* = 0$ by the definition of the E_*-local spectrum, which shows that η_X^* is an epimorphism. Hence we obtain the desired map h. q.e.d.

PROPOSITION 2.5. *Suppose that $X \to Y \to Z$ is a cofiber sequence. If any two of X, Y and Z are E_*-local, so is the other.*

PROOF. Let W be any E_*-acyclic spectrum and suppose that X and Y are E_*-local. Then $[W, X]_* = 0 = [W, Y]_*$. Furthermore the cofibering induces the exact sequence $[W, Y]_* \to [W, Z]_* \to [W, X]_*$. Therefore we see that $[W, Z]_* = 0$, which shows that Z is E_*-local. q.e.d.

PROPOSITION 2.6. *If $W \to X \to Y$ is a cofiber sequence, then so is $L_E W \to L_E X \to L_E Y$.*

PROOF. Let f denote the map $W \to X$ and C denote the cofiber of $L_E f: L_E W \to L_E X$. Then the proposition 2.5 shows that C is E_*-local. Consider the diagram

$$\begin{array}{ccccc} L_E W & \xrightarrow{L_E f} & L_E X & \longrightarrow & C \\ \| & & \| & & \downarrow h \\ L_E W & \xrightarrow{L_E f} & L_E X & \xrightarrow{L_E g} & L_E Y, \end{array}$$

in which g stands for the map $X \to Y$. We also see that $L_E g L_E f = 0$ since $gf = 0$ and L_E is functorial. Hence we have a map h in the diagram above. Applying $E_*(-)$ to the diagram, we obtain a commutative diagram with two exact rows, which gives us an isomorphism $E_*(h)$ by the five lemma, since $E_*(L_E ?) = E_*(?)$, where "?" may be substituted by a spectrum or a map. Therefore Corollary 2.2 implies that h is a homotopy equivalence. q.e.d.

§3. Generalized Adams spectral sequence

Next recall [1] the generalized Adams spectral sequence. For a ring spectrum E, we have an E-Adams resolution $X \leftarrow X_{E,1} \leftarrow X_{E,2} \leftarrow \cdots$ of a spectrum X, in which $X_{E,n+1}$ is a fiber of the induced map $X_{E,n} \to E \wedge X_{E,n}$ from the unit map $i: S \to E$ of E. Then the E-Adams spectral sequence $\{E_r^{s,t}(X)\}$ for a spectrum X is the one associated with the exact couple $\pi_*(X_{E,n+1}) \to \pi_*(X_{E,n}) \to E_*(X_{E,n})$ induced from the E-Adams resolution. Bousfield [2] gives a different resolution from this E-Adams one. Let X_E^n be the cofiber of the map $X_{E,n} \to X$ and consider the associated resolution $pt \leftarrow X_E^1 \leftarrow X_E^2 \leftarrow \cdots$. Then we construct the homotopy inverse limit

$\lim_{\leftarrow} X_E^n$ of this resolution. We call this *E-nilpotent completion* of X and denote it by $E^\wedge X$. Then we have

THEOREM 3.1 ([2]). *Let E and X be connective spectra. Suppose that for each s and t, there exists a finite r such that* $E_r^{s,t}(X)=0$. *Then the E-Adams spectral sequence* $\{E_r^{s,t}(X)\}$ *converges to* $\pi_{t-s}(E^\wedge X)$.

Let BP be the Brown-Peterson spectrum at a prime p. Then it is known [2] not only that BP satisfies the hypothesis of Theorem 3.1 but also that $BP^\wedge X = L_{BP}X = X_{(p)}$ for a connective spectrum X, and so we have

THEOREM 3.2 ([1] and [2]). *For each prime p we have the Adams-Novikov spectral sequence converging to a homotopy group* $\pi_*(X)$ *of a p-local connective spectrum X with* E_2*-term* $\mathrm{Ext}^{*,*}_{BP_*(BP)}(BP_*, BP_*(X))$.

Let $E(n)$ for $n>0$ and a prime p denote the ring spectrum introduced by Johnson and Wilson whose coefficient ring is $\boldsymbol{Z}_{(p)}[v_1,\cdots,v_n,v_n^{-1}]$(*cf.* [4, Cor. 2.16]). We also use the notation L_n for the Bousfield localization $L_{E(n)}$ following Ravenel. For small n, $E(n)$ satisfies the condition of Bousfield's Convergence Theorem and we have

THEOREM 3.3 ([1], [2], and [4]). *Let p be a prime number and n a positive integer with* $n<p-1$. *Then we have* $E(n)^\wedge X = L_n X$ *and the E(n)-Adams spectral sequence converging to* $\pi_*(L_n X)$ *with* E_2*-term* $\mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(X))$.

§4. Relation between the maps

Let E and F be ring spectra and $f\colon E\to F$ a map of ring spectra. Then this map f induces the map of resolutions $f^*\colon E^*X\to F^*X$ and so we have an induced map $f^\wedge\colon E^\wedge X\to F^\wedge X$ for a spectrum X. Here E^*X and F^*X denote the resolutions given in the previous section. That is, E^nX denotes X_E^n.

Let $\eta_X\colon X\to L_nX$ be the $E(n)$-localization map of a spectrum X and $\Phi\colon BP\to E(n)$ the Thom map. Suppose that X is p-local and connective. Then as noted in the previous section, we have

$$BP^\wedge X = X \quad \text{and} \quad E(n)^\wedge X = L_nX.$$

Therefore we see that the induced map $\Phi^\wedge\colon BP^\wedge X\to E(n)^\wedge X$ is $\Phi^\wedge\colon X\to L_nX$. Now the theorem in the introduction means the following

THEOREM 4.1. *Let n be positive integer and X* a *p-local connective spectrum. The E(n)-localization map* $\eta_X\colon X\to L_nX$ *is the same as the induced map* $\Phi^\wedge\colon X\to L_nX$.

PROOF. Suppose that the map $\Phi^\wedge\colon X\to L_kX$ induces an isomorphism $E(n)_*(\Phi^\wedge)\colon E(n)_*(X)\to E(n)_*(L_nX)$. Then by definition, we have a map $g\colon L_nX\to L_nX$ such that $g\Phi^\wedge=\eta_X$. Since $E(n)_*(\eta_X)$ and $E(n)_*(\Phi^\wedge)$ are both isomorphism, we have an isomorphism $E(n)_*(g)$. Now apply Corollary 2.2 to obtain the the homotopy equivalence g.

Therefore it is sufficient to show that the map $\Phi^\wedge$ induces an isomorphism $E(n)_*(\Phi^\wedge)$. The map Φ induces the map of E_2-terms

$$\Phi_*: \mathrm{Ext}^{*,*}_{BP_*(BP)}(BP_*, BP_*(E(n) \wedge X)) \longrightarrow \mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(E(n) \wedge X)).$$

By the change of rings theorem, we have isomorphisms

$$\mathrm{Ext}^{*,*}_{BP_*(BP)}(BP_*, BP_*(E(n) \wedge X)) = E(n)_*(X), \quad \text{and}$$

$$\mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(E(n) \wedge X)) = E(n)_*(X),$$

since we have isomorphisms $BP_*(E(n) \wedge X) = BP_*(BP) \otimes_{BP_*} E(n)_*(X)$ and $E(n)_*(E(n) \wedge X) = E(n)_*(E(n)) \otimes_{E(n)_*} E(n)_*(X)$. The Thom map Φ induces the canonical ring map:

$$BP_*(BP) \longrightarrow E(n)_*(E(n)) = E(n)_* \otimes_{BP_*} BP_*(BP_*) \otimes_{BP_*} E(n)_*,$$

where BP_* acts on $E(n)_*$ by sending v_i to v_i for $i \leq n$ and 0 for the other i. Therefore observing the change of rings theorem shows that the map Φ_* is an isomorphism. Since these spectral sequence for $E(n) \wedge X$ collapse, this isomorphism induces the desired one. q.e.d

References

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