The Chromatic E_1-Term $H^1 M_2^1$ and its Application to the Homotopy of the Toda-Smith Spectrum $V(1)$

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Introduction

Let p denote a fixed prime number. The Brown-Peterson ring spectrum BP at the prime p gives rise to the Hopf algebroid

$$(A, \Gamma) = (BP_*, BP_*BP),$$

and the BP_*-homology theory with the coefficient ring $BP_* = \pi_* BP = Z_{(p)}[v_1, v_2, \cdots]$ (*cf.* [8], see also §2). The BP_*-homology BP_*X of a spectrum X turns into a Γ-comodule. We can consider homological algebra over the Hopf algebroid. We shall denote

$$H^{*,*}M = \mathrm{Ext}_\Gamma^{*,*}(A, M),$$

the derived functor of $\mathrm{Hom}_\Gamma(A,\ \ )$, for a Γ-comodule M (*cf.* §2). One of the typical examples of right Γ-comodules is A/I_n for each $n \geq 0$ whose structure map is the induced one from the right unit η_R of the Hopf algebroid. Here I_n denotes the invariant prime ideal $(p, v_1, \cdots, v_{n-1})$. The Toda-Smith spectrum $V(n)$ for $n \geq -1$ is defined to be the one satisfying

$$BP_* V(n) = A/I_{n+1},$$

which is known to exist if $n \leq 3$ and $p \geq 2n+1$ ($V(-1) = S$, the sphere spectrum) ([13], [14]). We have the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_* V(n)$ with

$$E_2^{*,*} = H^{*,*} BP_* V(n).$$

We get a family $\{N_n^i, M_n^i\}$ of comodules out of the comodule A/I_n, whose members are defined inductively by

$$N_n^0 = A/I_n,\ M_n^i = v_{n+i}^{-1} N_n^i \quad \text{and} \quad N_n^{i+1} = M_n^i / N_n^i.$$

In their paper [2], Miller, Ravenel, and Wilson constructed the chromatic spectral sequence from the family $\{N_n^i, M_n^i\}$, which converges to the E_2-term of the above

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spectral sequence and have the trigraded E_1-term

$$E_1^{s,t,*} = H^{t,*} M_{n+1}^s,$$

and they determined

$$H^t M_n^s$$

in the following cases:

a) $t \geq 0$, $s = 1$, and $n = 0$ at any prime p,

b) $t = 0$, $s = 1$, and $n \geq 0$ at any prime p, and

c) $t = 0$, $s = 2$, and $n = 0$ at an odd prime p,

by using the results in the cases:

d) $t \geq 0$, $s = 0$, and $n = 0$ or 1 at any prime p, and

e) $t = 0$ or 1, $s = 0$ and $n \geq 0$ at any prime p

determined by Ravenel [7] (*cf.* [8]). For other modules, the author determined the E_1-term in the case:

c)' $t = 0$, $s = 2$, and $n = 0$ at the prime 2

in [9], and in the case:

f) $t \geq 0$, $s = 1$, and $n = 1$ at a prime $p > 3$

in [10] by using the result

f)' $t = 1$, $s = 1$, and $n = 1$ at a prime $p > 3$

given in [12].

In this paper we determine the E_1-term $H^1 M_2^1$ at a prime > 3 by using the short exact sequence $0 \longrightarrow M_3^0 \longrightarrow M_2^1 \xrightarrow{v_2} M_2^1 \longrightarrow 0$ and the determined modules $H^i M_3^0$ for $i = 0$, 1 and 2. Since this is the E_1-term, we get some information about its E_∞-term, in other words, the E_2-term of the Adams-Novikov spectral sequence converging to $\pi_* V(1)$. As an application, we apply the first results on the chromatic E_1-term to give some families of non-trivial elements of the homotopy groups $\pi_* V(1)$ of the Toda-Smith spectrum $V(1)$ relating to the product of a β-element and a γ-element of $\pi_* S$.

§ 1. Statement of results

Before stating our result, we begin with preparing some notations. We denote the polynomial algebras

(1.1) $$k(n)_* = Z/p[v_n] \quad \text{and} \quad K(n)_* = Z/p[v_n, v_n^{-1}],$$

in which the generator v_n originates from those of the polynomial $A = BP_*$

$= Z_{(p)}[v_1, v_2, \cdots]$, and $k(2)_*$-modules by

$L_n(x)$ *the cyclic* $k(2)_*$*-module generated by the element* x *with* $v_2^n x = 0$.

(1.2) $L\{x_j\}$ *the* $k(2)_*$*-module isomorphic to*

$$K(2)_*/k(2)_*$$

with Z/p*-basis* $\{x_j\}$ *such that* $v_2 x_j = x_{j-1}$ *and* $x_0 = 0$.

We have integers

(1.3) $$\nu(k) = \max\{n : p^n | k\} \quad \text{and} \quad \varepsilon(k) = 2^{-1}(1 - (-1)^k)$$

Recall [11, (3.3.2)] the integers

$$\begin{aligned}
a(i) &= p^i + (p-1)(p^{i-1} - 1)/(p^2 - 1) && \text{for odd } i \geq 1, \text{ and} \\
&= pa(i-1) && \text{for even } i \geq 2; \\
b(i) &= p^{i-2}(p^2 + p + 1) && \text{for } i \geq 2; \\
c(i) &= p^i - 2(p^2 - p - 1); \text{ and} && \\
e(i) &= (p^i - 1)/(p - 1); &&
\end{aligned} \tag{1.4}$$

for each non-negative integer i. We further prepare integers:

$$\begin{aligned}
\lambda(k) &= 2 && \textit{for } k \textit{ with } p \nmid k(k-1), \\
&= 2p && \textit{for } k = up + 1 \textit{ with } p \nmid u(u-1), \\
&= a(l) + 1 && \textit{for } k = up^l + e(l) \textit{ with } l \textit{ even} \geq 2, \textit{ and } p \nmid u - 1, \textit{ or} \\
& && \textit{for } k = up^l + 1 \textit{ with } l \textit{ even} \geq 2 \textit{ and } p \nmid u, \\
&= a(l) + 2 && \textit{for } k = up^l + e(l) \textit{ with } l \textit{ odd} \geq 3 \textit{ and } p \nmid u - 1, \\
&= a(l) + p && \textit{for } k = up^l + 1 \text{ with } l \textit{ odd} \geq 3 \textit{ and } p \nmid u, \\
&= p + 1 && \textit{for } k = up \textit{ with } p \nmid u, \\
&= b(l) - 1 && \textit{for } k = up^l \textit{ with } l = 2, 4, \textit{ and } p \nmid u, \\
&= b(l) - p + 1 && \textit{for } k = up^l \textit{ with } l \textit{ odd} \geq 3 \textit{ and } p \nmid u, \\
&= b(l) - p && \textit{for } k = up^l \textit{ with } l \textit{ even} \geq 6 \textit{ and } p \nmid u; \\
\mu(k) &= 2 && \textit{for } k \textit{ with } p \nmid k(k+1), \\
&= 2p && \textit{for } k = up \textit{ with } p \nmid u(u+1), \\
&= 2a(l) - p + 1 && \textit{for } k = up^l \textit{ with } l \textit{ even} \geq 2, \textit{ and } p \nmid u(u+1),
\end{aligned} \tag{1.5}$$

$$= 2a(l) - p + 2 \qquad \textit{for } k = up^l \textit{ with } l \textit{ odd} \geq 3, \textit{ and } p \nmid u(u+1),$$

$$= (p-1)a(r+1) \qquad \textit{for } k = (up^l + p^2 - 1)p^r \textit{ with } r \geq 0 \textit{ and } l \geq 2; \text{ and}$$

$$\bar{a}(k) = a(l) \qquad \textit{for } k = up^l \textit{ with } p \nmid u.$$

We are now ready to state our main result of this paper.

THEOREM **A**. *Let p be a prime ≥ 5. Then the E_1-term $H^1M_2^1$ of the chromatic spectral sequence is the direct sum of $k(2)_*$-modules*:

(a) $L\{z_j\}$, $L\{x_{\varepsilon,j}\}$ $(\varepsilon = 0, 1)$, and $L\{\tilde{z}_j\}$;

(b) $L_{p-1}\langle\chi(k)\rangle$ *for* $k \in Z$;

(c) $L_{\lambda(k)}\langle\varphi(k)\rangle$ *for* $k \in Z$;

(d) $L_{\mu(k)}\langle\psi(k)\rangle$ *for* $k \in Z$; *and*

(e) $L_{\bar{a}(k)}\langle\zeta(k)\rangle$ *for* $k \in Z$.

Here degrees of these generators are given by:

$$|z_j| = -2j(p^2 - 1),$$

$$|x_{\varepsilon,j}| = 2p^{\varepsilon}(p-1) - 2j(p^2-1) \qquad \textit{for} \quad \varepsilon = 0, 1,$$

$$|\tilde{z}_j| = 2(p+1-j)(p^2-1),$$

$$|\chi(k)| = 2(kp-1)(p^3-1) + 2p^2(p-1) - 2(p-1)(p^2-1),$$

$$|\varphi(k)| = 2k(p^3-1) + 2p^{\varepsilon(\nu(k)+1)}(p-1) - 2\lambda(k)(p^2-1),$$

$$|\psi(k)| = 2k(p^3-1) + 2p^{\varepsilon(\nu(k))}(p-1) - 2\mu(k)(p^2-1),$$

$$|\zeta(k)| = 2k(p^3-1) - 2\bar{a}(k)(p^2-1).$$

Let S denote the sphere spectrum localized at the prime p, $\alpha_1 \in \pi_q S$ $(q = 2p-2)$ be the Hopf invariant one element (*cf.* [8]), and consider the γ-families $\{\gamma'_{[s]}: s \geq 1\}$ and $\{\gamma'_{[sp/2]}: s \geq 1\}$ in $\pi_* V(1)$ for a prime ≥ 7 given by results of Toda [14] and Oka [4]. Then Theorem A and a standard argument of spectral sequences imply

PROPOSITION **B**. *Let p be a prime ≥ 5 and r and s be non-negative integers such that $p \nmid s > 0$. Then we have*

$$\gamma'_{[sp^r]}\alpha_1 \neq 0 \quad \textit{in} \quad \pi_* V(1)$$

if r is odd, or if r is even and either $p \nmid s+1$ or $p^2 | s+1$.

By a result of Smith [13], we also have the β-elements β_1, β_2 and β_3 of the stable homotopy $\pi_* S$ for a prime ≥ 5. Using a proposition which appears when we prove Theorem A brings another information to show the following

THEOREM C. *Let p be a prime ≥ 7, and r and s non-negative integers with $p \nmid s$. Then in $\pi_* V(1)$ we have*

$$\gamma'_{[sp^r]}\beta_1 \neq 0 \neq \gamma'_{[sp^r/2]}\beta_1$$

if r is even or $p \nmid s+1$,

$$\gamma'_{[sp^r]}\beta_2 \neq 0 \neq \gamma'_{[sp^r/2]}\beta_3$$

if $r \neq 0, 2$ or $p^2 \nmid s+p+1$, and

$$\gamma'_{[sp^r/2]}\beta_2 \neq 0$$

Here the integer r in $\gamma'_{[sp^r/2]}$ is positive.

§2. Some elements of $v_3^{-1}BP_*$

In this section and the next we set the prime p odd. Let BP denote the Brown-Peterson spectrum BP at a prime p. Then we have the Hopf algebroid

$$(A, \Gamma) = (BP_*, BP_*BP) = (Z_{(p)}[v_1, v_2, \cdots], BP_*[t_1, t_2, \cdots])$$

($|v_i| = |t_i| = 2p^i - 2$) with the right and the left units η_R, $\eta_L\colon A \to \Gamma$, the coproduct $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$, the counit $\varepsilon\colon \Gamma \to A$ and the conjugation $c\colon \Gamma \to \Gamma$ (*cf.* [8]). These structure maps are characterized by: $\varepsilon\eta_R = \varepsilon\eta_L = 1_A$, $(1_\Gamma \otimes \varepsilon)\Delta = (\varepsilon \otimes 1_\Gamma)\Delta = 1_\Gamma$, $(1_\Gamma \otimes \Delta)\Delta = (\Delta \otimes 1_\Gamma)\Delta$, $c\eta_R = \eta_L$, $c\eta_L = \eta_R$ and $cc = 1_\Gamma$. The explicit formulae for this Hopf algebroid will appear later where they are needed.

A (right) A-module M is said to be a (right) Γ-comodule M if there exists a counitary and coassociative (right) A-linear map $\psi\colon M \to M \otimes_A \Gamma$, *i.e.*, A-linear map ψ with $(1_M \otimes \varepsilon)\psi = 1_M$ (which denotes the identity of M) and $(1_M \otimes \Delta)\psi = (\psi \otimes 1_\Gamma)\psi$.

We note here that we have $\Delta\eta_R = (\eta_R \otimes 1_\Gamma)\eta_R$ for the Hopf algebroid $(A, \Gamma) = (BP_*, BP_*BP)$ and so A is a Γ-comodule with the structure map η_R (*cf.* [8]).

Let M be a Γ-comodule with the structure map ψ. Then the cobar complex (Ω^*M, d_*) is a pair of comodules given by

$$\Omega^tM = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \tag{2.1}$$

(t copies of Γ) and differentials $d_t\colon \Omega^tM \to \Omega^{t+1}M$ given by:

$$d_0m = \psi m - m \otimes 1,\ d_1 m \otimes x = d_0 m \otimes x + m \otimes d_1 x$$

$$d_1x = 1 \otimes x - \Delta x + x \otimes 1 \text{ and} \tag{2.2}$$

$$d_t m \otimes x \otimes y = (d_1 m \otimes x) \otimes y - m \otimes x \otimes d_{t-1}y$$

for $m \in M$, $x \in \Omega^1A$ and $y \in \Omega^{t-1}A$. Here we note that $\Omega^tM = M \otimes_A \Omega^tA$ as a comodule. The cohomology of the complex (Ω^*M, d_*) is denoted by

$$H^*M = \mathrm{Ext}^*_\Gamma(A, M). \tag{2.3}$$

From here on we use the following notation:

$$A_n = v_n^{-1}BP_*, \text{ the } \Gamma\text{-comodule with coaction } \eta_R; \text{ and}$$

(2.4) $$I_k = (p, v_1, \cdots, v_{k-1}) \text{ and } J(k) = (p, v_1, v_2^k),$$

the invariant ideals generated by the every entry.

We recall [11] the definition of the elements $u_{n,i}$ of A_n and $w_{n,i}$ of $\Omega^1 A_n$:

(2.5)[11, (2.8)] $$u_{n,0} = v_n^{-1} \text{ and } \textstyle\sum_{i+j=r} v_{n+i} u_{n,j}^{p^i} = 0 \text{ for } r \geq 1,$$

which inductively gives

(2.6)[11, (2.8)] $$\textstyle\sum_{i+j=r} u_{n,i} v_{n+j}^{p^i} \equiv 0 \mod (p).$$

(2.7)[11, (2.9)] *For an element $x \in A_n$, the element $e_n(x) \in \Omega^1 A_n/I_n$ is the one which satisfies*

$$\eta_R x \equiv e_n(x) \mod I_n.$$

(2.8)[11, (2.10)] $$w_{n,0} = 0, \text{ and } w_{n,r} = \textstyle\sum_{j=1}^r e_n(u_{n,r-j}^{p^{j-1}}) T_j^{p^{n-2}} (n \geq 2, r \geq 1),$$

where T_j is the element of Γ which satisfies ([11, p. 78])

$$\textstyle\sum_{i=0}^n t_i \eta_R v_{n-i}^{p^i} \equiv \sum_{i=0}^n v_i T_{n-i}^{p^{i-1}} \mod (p) \text{ and}$$

$$T_j \equiv t_j^p \mod I_j.$$

The relation between these elements is:

(2.9)[11, Prop. 2.2] *For non-negative integers n and r with $n \geq 2$,*

$$\eta_R u_{n,r} \equiv \textstyle\sum_{i+j=r} u_{n,i} t_j^{p^i} - w_{n,r}^p - v_{n-1} w_{n,r+1} \eta_R v_n^{-1} \mod I_{n-1} + (v_{n-1}^p).$$

For $n = 2$, we define

(2.10) $$w_i' = v_2^{e(i)} w_i = v_2^{e(i)} w_{2,i} \in \Omega^1 A_3$$

by using the notations given in (1.4), (2.4) and (2.8), and especially we have

(2.11) $$w_1' = t_1^p \text{ and } w_2' \equiv v_2^p t_1^{p+1} - v_2 c t_2^p - v_3 t_1^p \mod I_2$$

by (2.8) and (2.5). The d_1-image is:

(2.12)
$$d_1 w_r' \equiv -v_2^{p^{r-1}} w_{r-1}' \otimes t_1^{p^{r-2}} \mod J(p^{r-1} + p^{r-2}) \text{ and,}$$
$$d_1 w_3' \equiv -v_2^{p^2} w_2' \otimes t_1^p - v_2^{p^2+p} t_1^p \otimes t_2 + b_0 \mod I_2$$

which is obtained from

(2.13)[11, Prop. 2.3][1] *For positive integers n and r with $n \geq 2$,*

[1] In this proposition of [11] the definition of $C_{n,r}$ must be changed to $v_{n-1}C_{n,r} \equiv \Delta t_{r-1} - \Delta_n t_{r-1} \mod I_{n-1}$, and $C(n)$ in (2.12) of [11] must be replaced by $C(n)' = C(n) + Z/p\{v_{n-1}\}[t_1, t_2, \cdots]$.

$$d_1 w_{n,r} \equiv -\textstyle\sum_{0<i<r} w_{n,i} \otimes t_{r-i}^{p^{i-1}} + C_{n,r} \mod I_n,$$

where $C_{n,r}$ *is a certain element in* $\Omega^2 A_n$, *and especially*

$$C_{n,n+1} = b_{n-2} = -p^{-1}\textstyle\sum_{0<i<p^{n-1}} \binom{p^{n-1}}{i} t_1^i \otimes t_1^{p^{n-1}-i}.$$

LEMMA **2.14.** *Let* r *and* s *be integers with* $r \geq s \geq 0$. *Then*

$$u'_k = v_2^{e(k+1)} u_{2,k} \in A_3 \text{ and } u'_r \equiv (-1)^{r-s} v_3^{e(r-s)p^s} u'_s \mod J(p^s),$$

and hence for the element of (2.10),

$$w'_r \equiv (-1)^{r-s} v_3^{e(r-s)p^{s-1}} w'_s \mod J(p^{s-1}).$$

For $r = 3$, *we further have*

$$w'_3 \equiv -v_3^p w'_2 + v_2^p(v_3^p t_1^{p+1} - v_4 t_1^p) \mod J(p+1).$$

Here $J(k) = (p, v_1, v_2^k)$ *denotes the ideal of* $A_3 = v_3^{-1}BP_*$.

PROOF. In this proof we use the abbreviation $u_r = u_{2,r}$. If $k = 0$, then $v_2^{e(1)}u_0 = 1 \in A_3$. By definition (2.5), we have

$$u'_k = v_2^{e(k+1)} u_k = -\textstyle\sum_{i=1}^{k} v_{2+i} v_2^{e(i-1)p} u'^{p^i}_{k-i},$$

which belongs to A_3 under the inductive hypothesis $u'_j \in A_3$ for $j < k$. This shows the first statement by induction. We also have a similar equality

$$u'_k = v_2^{e(k+1)} u_k \equiv -\textstyle\sum_{i=0}^{k-1} v_2^{e(k-i-1)p^{i+1}} u'_i v_{2+k-i}^{p^i}$$

by (2.6). This shows $u'_k \equiv -v_3^{p^{k-1}} u'_{k-1} \mod J(p^{k-1})$, and so inductively we obtain

(2.15) $$u'_r \equiv (-1)^{r-s} v_3^{e(r-s)p^s} u'_s \mod J(p^s).$$

Now apply this to the equality in the definition (2.8) of $w_{2,r}$, and we get

$$\begin{aligned} w'_r &\equiv \textstyle\sum_{j=1}^{r} v_2^{e(j-1)} \eta_R u'^{p^{j-1}}_{r-j} T_j \mod I_2 \\ &\equiv \textstyle\sum_{j=1}^{s} (-1)^{r-s} v_2^{e(j-1)} \eta_R v_3^{e(r-s)p^s} u'^{p^{j-1}}_{s-j} \mod J(p^{s-1}) \\ &\equiv w'_s \mod J(p^{s-1}) \end{aligned}$$

as desired. Use the congruence

$$u'_2 \equiv -v_3^p u_1 - v_2^p v_4 \mod J(p+1)$$

instead of (2.15) to get the case for $r = 3$. q.e.d.

We also consider the elements ζ_n of $\Omega^1 A_n/I_n$ relating to $u_{n,n}$ by

(2.16)[3, Th. 6.2.1.1] $$d_0 u_{n,n} \equiv \zeta_n - \zeta_n^p \mod I_n$$

for $n \geq 1$, which is one of generators of $H^1M_n^0$([7], *cf.* [3]) (see (4.1) for the definition of M_n^0). In particular we have

(2.17)[2, Prop. 3.18] $\zeta_2 = u_0t_2 - u_0^p ct_2^p + u_1 t_1^p = -u_0 ct_2 + w_2 \in \Omega^1 A_2/I_2$, and $d_0u_{2,2} \equiv \zeta_2 - \zeta_2^p$.

In [11], we farther define the elements X_i of A_3 to satisfy:

PROPOSITION **2.18** ([11, Prop. 3.1]).

1) $$X_i \equiv v_3^{p^i} \bmod J(p+1) = (p, v_1, v_2^{p+1}) \quad \textit{for } i \geq 2,$$

and

$$X_0 = v_3 \textit{ and } X_1 = v_3^p - v_2^p v_3^{-1} v_4$$

2) Mod $J(1+b(i))$,

$$\begin{aligned}
d_0 X_i &\equiv v_2 t_1^{p^i} \equiv v_2 v_3 W_1 && i = 0,\\
&\equiv v_2^p v_3^{p-1} t_1 - v_2^{p+1} v_3^{p-1} W_2 && i = 1,\\
&\equiv v_2^{a(2)} X_1^{p-1} t_1^p - v_2^{b(2)} v_3^{c(2)} (v_2^{-1} t_2 - W_3) && i = 2,\\
&\equiv v_2^{a(3)} X_2^{p-1} t_1 - v_2^{b(3)} X_1^{c(2)} (w_2 + v_2^{-1} t_2 - C - W_3) && i = 3,\\
&\equiv v_2^{a(i)} X_{i-1}^{p-1} t_1^p - v_2^{b(i)} v_3^{c(i)} (\zeta_2 - \zeta_3 + v_2^{-1} t_2) && \textit{even } i \geq 4,\\
&\equiv v_2^{a(i)} X_{i-1}^{p-1} t_1 - v_2^{b(i)} v_3^{c(i)} (\zeta_2 - \zeta_3 + w_2) && \textit{odd } i \geq 5,
\end{aligned}$$

where $b(0) = 1$ *and* $b(1) = p+1$, $w_2 = w_{2,2} = v_2^{-p} t_2^p - \eta_R(v_2^{-p-1} v_3) t_1^p$, $C = v_3^{-1}(t_2 c t_1^{p^2} + t_3)$ *and* $W_n = w_{3,n}$.

§3. Some elements of $v_3^{-1} BP_* BP$

In this section the prime p is also odd and we also use the notation given in (2.4). We shall give the elements which will generate our E_1-term.

Hereafter we use same notation for both a homology class and its representing cycle. Consider the elements

$$\begin{aligned}
&h_i \zeta_3 = t_1^{p^i} \otimes \zeta_3,\\
&g_i = 2 t_1^{p^i} \otimes t_2^{p^i} + t_1^{2p^i} \otimes t_1^{p^{i+1}},\\
(3.1)\qquad &k_i = 2 t_2^{p^i} \otimes t_1^{p^{i+1}} + t_1^{p^i} \otimes t_1^{2p^{i+1}}, \textit{ and}\\
&b_i = -\textstyle\sum_{0<k<p^{i+1}} \frac{1}{p} \binom{p^i}{k} t_1^k \otimes t_1^{p^{i+1}-k} = \frac{1}{p} d_1 t_1^{p^{i+1}}.
\end{aligned}$$

of $\Omega^2 A_3$ ($A_3 = v_3^{-1} BP_*$). We remark here that (k_i here) $= -2$(k_i in [7]). Let $K(3)_*$ denotes the field $Z/p[v_3, v_3^{-1}]$. Then,

(3.2)[7, Th. 2.4] $H^2 A_3/J(1)$ *is isomorphic to the* $K(3)_*$*-vector space spanned by the* *cycles* $h_i\zeta_3$, g_i, k_i *and* b_i *for* $i=0$, 1 *and* 2.

We also have the cycles

(3.3) $$g_i'=2ct_2^{p^i}\otimes t_1^{p^i}+t_1^{p^{i+1}}\otimes t_1^{2p^i} \text{ and } k_i'=2t_1^{p^{i+1}}\otimes ct_2^{p^i}+t_1^{2p^{i+1}}\otimes t_1^{p^i}$$

which are homologous to g_i and k_i in $\Omega^2A_3/J(1)$, respectively, where c denotes the conjugation and its explicit formulae are:

(3.4)[*cf.* [7]] $ct_1=-t_1$ *and* $ct_2=t_1^{p+1}-t_2$.

In fact, we have the elements

(3.5) $$G_i=2t_2^{p^i}t_1^{p^i}-t_1^{2p^i+p^{i+1}} \quad \text{and,}$$
$$K_i=2t_2^{p^i}t_1^{p^{i+1}}-t_1^{p^i+2p^{i+1}}$$

such that

(3.6) $$d_1G_i\equiv g_i'-g_i \mod I_2 \text{ and } d_1K_i\equiv k_i'-k_i \mod I_2.$$

This can be checked by the first two equalities of the following:

(3.7)(*cf.* [6, Th. 8]) $$d_1t_1=0,\ d_1t_2\equiv -t_1\otimes t_1^p \mod I_2, \text{ and}$$
$$d_1t_3\equiv -t_1\otimes t_2^p-t_2\otimes t_1^{p^2}-v_2b_1 \mod I_2.$$

This also gives

(3.8) $d_1ct_1=0$ *and* $d_1ct_2\equiv -t_1^p\otimes t_1 \mod I_2$.

Define elements $P_{r,i}$ and $P'_{r,i}$ for $r\ge 2$ and $i=0$, 1 of Ω^1A_3 and Y_t for $t\in Z$ by

(3.9)
$$P_{2,0}=ct_2\eta_Rv_3-v_2t_1t_2^p+v_2t_3-v_2^pG_0,$$
$$2P_{2,1}=t_1^{2p}\eta_Rv_3-2v_2t_1^pt_2^p+v_2^pt_1^{2p+1},$$
$$P_{r,i}=v_3^{e(r-2)p}P_{2,i},$$
$$P'_{r,i}=P_{r,i}+(-1)^rw_r't_1^{p^{e(i)}}, \quad \text{and}$$
$$Y_t=X_1^tv_3t_1^p+v_2v_3^{tp}ct_2^p+(t-1)v_2^pX_1^tt_2-tv_2^{p+1}v_3^{tp-p}ct_3^p.$$

Here the elements w_i and w_i' are those given in (2.10). Then we have following lemmas.

Lemma 3.10. *Let i and r be non-negative integers with $i=0$, 1 and $r\ge 2$. Then we have*

$$d_1P_{2,0}\equiv -v_2^2b_1+v_2^pg_0+z\otimes t_1 \mod I_2,$$
$$d_1P_{2,1}\equiv 2^{-1}(v_2g_1-v_2^pk_0)+z\otimes t_1^p \mod I_2,$$
$$d_1P'_{2,0}\equiv -v_2^2b_1+v_2^pg_0-t_1\otimes w_2' \mod J(p+1),$$
$$d_1P_{r,0}\equiv -v_2^2v_3^{e(r-2)p}b_1+(-1)^rw_r'\otimes t_1 \mod J(3),$$

$$d_1P_{r,1} \equiv 2^{-1}v_2v_3^{e(r-2)p}g_1 + (-1)^r w'_r \otimes t_1^p \ \bmod\ J(2),$$
$$d_1P'_{r,0} \equiv -v_2^2v_3^{e(r-2)p}b_1 - (-1)^r t_1 \otimes w'_r \ \bmod\ J(3), \textit{ and}$$
$$d_1P'_{r,1} \equiv 2^{-1}v_2v_3^{e(r-2)p}g_1 - (-1)^r t_1^p \otimes w'_r \ \bmod\ J(2),$$

where

$$z = v_2^{p+1}\zeta_2 = v_2^p t_2 - v_2 c t_2^p - v_3 t_1^p$$
$$\equiv w'_2 \ \bmod\ J(p).$$

PROOF. By (2.2) we have

(3.11)(*cf.* [12, (2.3.2)]) $\quad d_1(x\eta_R v) = d_1x(1 \otimes \eta_R v) - x \otimes d_0 v$

for $x \in \Omega^1 A_3$ *and* $v \in A_3$, *and*

$$d_1(xy) = d_1x\Delta y + (x \otimes 1 + 1 \otimes x)d_1 y - x \otimes y - y \otimes x.$$

We also have $v_2t_2^p - t_1^p\eta_R v_3 \equiv z + v_2^p c t_2 \bmod I_2$ by (2.17). Here we have Landweber's formula:

(3.12) $$\eta_R v_n \equiv v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \ \bmod\ I_{n-1}.$$

Note that the structure map of the Γ-comodule A_3 is the one associated to the right unit $\eta_R\colon A \to \Gamma$ of the Hopf algebroid. Then a routine calculation with (3.12), (3.7) and (3.4) brings the case $r = 2$. For $r > 2$, use the congruence

$$w'_r \equiv (-1)^{r-s} v_3^{e(r-s)p^{s-1}} w'_s \ \bmod\ J(p^{s-1})$$

given in Lemma 2.14. The case with a prime follows from (2.12), (3.11) and the one with no prime. q.e.d.

LEMMA **3.13.** $d_1Y_t \equiv \binom{t}{2} v_2^{2p}v_3^{tp-1}g_0 \ \bmod\ J(2p+1)$.

PROOF. Noticing that $X_1^t v_3 = v_3^{tp+1} - tv_2^p v_3^{tp-p}v_4 + \binom{t}{2} v_2^{2p}v_3^{tp-2p-1}v_4^2$, we compute

$$d_1X_1^t v_3 t_1^p \equiv \xi_1 + \xi_2 + \xi_3 + \xi_4 - \xi_5 \ \bmod\ J(2p+1),$$

where $\xi_1 = v_2v_3^{tp}t_1^{p^2} \otimes t_1^p$, $\xi_2 = (t-1)v_2^p v_3^{tp} t_1 \otimes t_1^p$, $\xi_3 = tv_2^{p+1}v_3^{tp-p}ct_2^{p^2} \otimes t_1^p$, $\xi_4 = \binom{t}{2} v_2^{2p} v_3^{tp-1}t_1^2 \otimes t_1^p$ and $\xi_5 = t(t-1)v_2^{2p}v_3^{tp-p-1}v_4 t_1 \otimes t_1^p$ by the formulae (3.11), (3.12) and

$$\eta_R v_4 \equiv v_4 + v_2 t_1^{p^3} + v_2 t_2^{p^2} - t_1\eta_R v_3^p - t_2 v_2^{p^2} \quad \bmod\ (p, v_1).$$

If we put $\xi_6 = tv_2^{p+1}v_3^{tp-p}t_1^{p^3} \otimes ct_2^p$ and $\xi_7 = t(t-1)v_2^{2p}v_3^{tp-1}t_1 \otimes t_2$, then we get $d_1v_2v_3^{tp}ct_2^p \equiv \xi_6 - \xi_1$, $d_1(t-1)v_2^p X_1^t t_2 \equiv \xi_7 - \xi_2 + \xi_5$ and $d_1tv_2^{p+1}v_3^{tp-p}ct_3^p \equiv \xi_3 + \xi_6$ mod $J(2p+1)$. Collect terms and we obtain the lemma. q.e.d.

Let x be an element of $A_3 = v_3^{-1}BP_*$ such that

$$d_0x \equiv v_2^a v t_1^{p^i} \mod J(a+k) \tag{3.14}$$

for some $v \in A_3$ and non-negative integers i, a and k with $a \geq k$, and define elements ∂x^s, ρx, $\tilde{\rho}x$, βx and $\sigma_s x$ of $\Omega^1 A_3/I_2$ by

$$\begin{gathered}\partial x^s = d_0x^s + sx^{s-p+1}d_0x^{p-1},\ v_2^a\rho x = v_2^a v t_1^{p^i} - d_0x,\\ pv_2^a\tilde{\rho}x = pv_2^a x^{p-1}vt_1^{p^i} - d_0x^p + (d_0x)^p,\\ \beta x = x^{p-1}vt_1^{p^i} - \tilde{\rho}x,\ \textit{and}\ \sigma_s = x^s t_1^{p^i} + 2^{-1}sv_2^a x^{s-1}vt_1^{2p^i}.\end{gathered} \tag{3.15}$$

Here ρx and $\tilde{\rho}x$ are both well defined since the maps

$$v_2^a: \Omega^1 A_3/I_2 \longrightarrow \Omega^1 A_3/I_2 \text{ and } pv_1^{p-1}v_2^a: \Omega^1 A_3/I_2 \longrightarrow \Omega^1 A_3/(p^2, v_1^p)$$

are monomorphic. Furthermore, if $v = w^{p-1}$, then we define

$$\tau_s x = w^s t_1^{p^i} + sw^{s-p+1}\rho x. \tag{3.16}$$

Then,

LEMMA 3.17. *For the above elements, we have the following in $\Omega^* A_3$:*

(a) $\partial x^s \equiv \binom{s+1}{2} v_2^{2a} v^2 x^{s-2} t_1^{2p^i} \mod J(2a+k)$,

(b) $d_1\rho x \equiv d_0 v \otimes t_1^{p^i} \mod I_2$,

(c) $d_1\tau_s x \equiv \partial w^s \otimes t_1^{p^i} + sd_0w^{s-p+1} \otimes \rho x \mod I_2$, *and*

(d) $d_1\sigma_s x \equiv 2^{-1}sv_2^a x^{s-1}(-2\rho x \otimes t_1^{p^i} + d_0v \otimes t_1^{2p^i}) \mod J(2a)$.

If we assume that $d_0 v \equiv 0 \mod J(k)$, then

(e) $d_1\tilde{\rho}x \equiv d_0x^{p-1}v \otimes t_1^{p^i} + v_2^{ap-a}v^p b_i \mod J(ap-a+k)$, *and*

(f) $d_1\beta x \equiv -v_2^{ap-a}v^p b_i \mod J(ap-a+k)$.

PROOF. (a) follows immediately from the equality $d_0X^t = (X + d_0X)^t - X^t$ for any integer t and element $X \in A_3$, given by the definition (2.2) of d_0. Since $(1_M \otimes \Delta)\psi = (\psi \otimes 1_\Gamma)\psi$, the definition (2.2) also implies that $d_1d_0Y = 0$ for $Y \in A_3$, and that $d_1Vt_1^{p^i} \equiv d_0V \otimes t_1^{p^i} \mod (p)$ in $\Omega^2 A_3$ for $V \in A_3$ since $d_1t_1^{p^i} \equiv 0 \mod (p)$. Furthermore, we see that

$$d_1v_2x \equiv v_2d_1x \mod I_2$$

by (3.12) and (2.4). Use these equalities to verify (b) and (c). Notice that $d_1x^s \equiv sv_2^a x^{s-1}(vt_1^{p^i} - \rho x) \mod J(2a)$ and $d_1t_1^{2p^i} \equiv -2t_1^{p^i} \otimes t_1^{p^i} \mod (p)$, and we have (d). We further have $d_1(d_0V)^p = (d_0V)^p \otimes 1 - (d_0V \otimes 1 + 1 \otimes d_0V)^p + 1 \otimes (d_0V)^p$ since $\Delta d_0V = d_0V \otimes 1 + 1 \otimes d_0V$ followed from $d_1d_0Y = 0$. Noticing the equality $1 \otimes v$

$= v\otimes 1$ given by the hypothesis $d_0 v = 0$, we have $p^{-1}d_1(d_0x)^p \equiv v_2^{ap}v^p b_i \mod J(ap+k)$. Combining these gives (e) and (f). For (b) and (e), we also consider the monomorphisms v_2^a and $pv_1^{p-1}v_2^a$ as we define those elements. q.e.d.

We next consider the following elements:

$$(3.18)\qquad \begin{aligned} 2\omega_3 &= 2w'_3 + v_2^{p^2}v_3t_1^{2p} - v_2^{e(3)-1}t_1^{2p+1} - v_2^{p^2+1}G_1 - v_2^{p^2+p}K_0,\\ Z_s &= 2(\omega_3 - v_2^{b(2)-1}t_1^pt_2)\eta_R X_2^{c(s,2)}, \textit{ and}\\ Y_{4,s} &= \sigma_s X_4 - 4^{-1}sv_2^{b(4)-b(2)}(Z_s - 2v_2^{b(2)-p-1}v_3^{c(s,4)}P_{2,1}) \end{aligned}$$

for an integer s and the integer:

$$(3.19)\qquad c(s, i) = sp^i - p^{i-1} - p^{i-2}.$$

In a same way as the above lemmas, we easily obtain the following lemma from (3.6), (2.11) and (2.12):

LEMMA **3.20.** $2d_1\omega_3 \equiv v_2^{p^2+1}g_1 + v_2^{p^2+p}k_0 + 2v_2^{p^2+p+1}b_0 \mod I_2$.

LEMMA **3.21.** $d_1Z_s \equiv v_2^{b(2)}v_3^{c(s,4)}(v_2^{-p}g_1 + 3v_2^{-1}k_0 - 2v_2^{-p-1}z\otimes t_1^p) \mod J(b(2))$ *and* $d_1Y_{4,s} \equiv -\frac{3}{2}sv_2^{b(4)-1}v_3^{c(s,4)}k_0 \mod J(b(4))$.

PROOF. Notice that

$$\omega_3 \equiv -v_3^p w'_2 + v_2^p(v_3^p t_1^{p+1} - v_4t_1^p) \quad \mod J(p+1)$$

by Lemma 2.14 and

$$X_1^{c(s,3)-1} \equiv v_3^{c(s,4)-p} + v_2^p v_3^{c(s,4)-2p-1}v_4 \quad \mod J(p+1)$$

by definition in Proposition 2.18, and we have

$$\begin{aligned} X_1^{c(s,3)-1}\omega_3 &\equiv -v_3^{c(s,4)}w'_2 + v_2^pv_3^{c(s,4)}t_1^{p+1} \quad \mod J(p+1) \text{ (by (2.11))},\\ &\equiv -v_3^{c(s,4)}z + v_2^pv_3^{c(s,4)}t_2 \quad \mod J(p+1) \text{ (by (2.17))}, \end{aligned}$$

and then we compute

$$\begin{aligned} d_1 2\omega_3\eta_R X_2^{c(s,2)} \equiv v_2^{b(2)}v_3^{c(s,4)}(&v_2^{-p}g_1 + v_2^{-1}k_0 - 2v_2^{-p-1}z\otimes t_1^p\\ &+ 2v_2^{-1}(t_2\otimes t_1^p - t_1^{p+1}\otimes t_1^p - t_1^p\otimes t_2)) \end{aligned}$$

mod $J(b(2))$ by (3.11), Proposition 2.18 and Lemma 3.20. We get

$$-2d_1v_2^{b(2)-1}t_1^pt_2\eta_R X_2^{c(s,2)} \equiv -2v_2^{b(2)-1}v_3^{c(s,2)}d_1t_1^pt_2 \mod J(b(2))$$

from Proposition 2.18, which equals

$$2v_2^{b(2)-1}v_3^{c(s,2)}(t_1^{p+1}\otimes t_1^p + t_1\otimes t_1^{2p} + t_1^p\otimes t_2 + t_2\otimes t_1^p) \mod J(b(2))$$

by (3.11). Adding these shows the first congruence of the lemma. In Lemma 3.17 (d), take $x = X_4$, and we see $a = a(4)$, $k = p^3 - 1$, $v = X_3^{p-1}$ and $i = 1$ by Proposition

2.18. Therefore we obtain

$$d_1\sigma_s X_4 \equiv -sv_2^{b(4)-p-1}v_3^{c(s,4)}z\otimes t_1^p - 2^{-1}sv_2^{b(4)-1}v_3^{c(s,4)}k_0 \mod J(b(4)),$$

and so we have the last congruence by Lemma 3.10. q.e.d.

We further consider the elements y_i of $A = BP_*$ defined by

$$y_2 = v_3,\ y_i = y_{i-2}^{p^2} + v_2^{a(i-3)p^2-1}v_3 \quad \text{for even } i \geq 4, \text{ and}$$

$$y_i = y_{i-1}^p \quad \text{for odd } i \geq 3.$$

Notice that $a(r-1) - a(r-3) = p^{r-1} - p^{r-4}$ seen by (1.4). Then we see the following inductively by (3.12):

LEMMA **3.22.** $d_0 y_i \equiv v_2^{p^{i-2}}t_1^{p^i} - v_2^{a(i-1)}t_1^{p^{\varepsilon(i)}} \mod I_2$, *for* $i \geq 2$.

We lastly define elements ω_r for $r \geq 4$ of $\Omega^1 A_3$ by

(3.23) $$\omega_r = (-1)^{r+1}(w_r' - v_2^{p^{r-1}-p^{r-4}}w_{r-1}'\eta_R y_{r-2}) + v_2^{a(r-1)}P_{r-1,\varepsilon(r)}.$$

Then we have the following by (2.12), Lemmas 3.10, 3.22 and (3.11).

PROPOSITION **3.24.** *If an integer r is odd* ≥ 3, *then*

$$d_1\omega_r \equiv 2^{-1}v_2^{a(r-1)+1}v_3^{e(r-3)p}g_1 \mod J(a(r-1)+2).$$

If r is even ≥ 4, *then*

$$d_1\omega_r \equiv -v_2^{a(r-1)+2}v_3^{e(r-3)p}b_1 \mod J(a(r-1)+3).$$

§4. $H^1M_2^1$

Here the prime p is greater than 3. We also use the notation $(A, \Gamma) = (BP_*, BP_*BP)$.

A is a Γ-comodule with coaction η_R, and we obtain the Γ-comodules given by

(4.1) $$N_n^0 = A/I_n,\ M_n^i = v_{n+i}^{-1}N_n^i \text{ and the exact sequence:}$$

$$0 \longrightarrow N_n^i \xrightarrow{\subset} M_n^i \longrightarrow N_n^{i+1} \longrightarrow 0.$$

By definition every element of M_n^i is a linear combination of following elements:

(4.2) x/v *for* $x \in v_{n+i}^{-1}A/I_n$ *and* $v = v_n^{e_1}v_{n+1}^{e_2}\cdots v_{n+i-1}^{e_i} \in A$ $(e_j > 0)$, *such that*

$$x/v = 0 \text{ if and only if } v_{n+j-1}^{e_j} \text{ divides } x \text{ for some } j\ (0 < j \leq i).$$

The definition of the comodule M_n^i give rise to the short exact sequence

$$0 \longrightarrow M_{n+1}^{i-1} \xrightarrow{\iota_n} M_n^i \xrightarrow{v_n} M_n^i \longrightarrow 0 \quad (\iota_n x = x/v_n).$$

This sequence induces the long exact one:

$$
(4.3)\qquad
\begin{aligned}
0 \longrightarrow H^0 M^{i-1}_{n+1} \xrightarrow{\iota_n} \cdots \xrightarrow{v_n} H^{k-1} M^i_n \\
\xrightarrow{\delta} H^k M^{i-1}_{n+1} \xrightarrow{\iota_n} H^k M^i_n \xrightarrow{v_n} H^k M^i_n \xrightarrow{\delta} \cdots,
\end{aligned}
$$

where δ is the boundary homomorphism.

In this section we determine the structure of $H^1 M^1_2$ by virtue of the following

LEMMA **4.4.** *Let ι_n: $H^k M^{i-1}_{n+1} \to H^k M^i_n$ be the map in* (4.3) *and B a direct sum of the $k(n)_*$-submodules $L\{g_{\lambda,j}\}$ ($\lambda \in \Lambda$) and $L\langle k_\mu \rangle$ ($\mu \in M$) of $H^k M^i_n$. If B satisfies*

(a) $\operatorname{Im} \iota_n \subset B$, *and*

(b) $\{\delta k_\mu : \mu \in M\}$ *is linearly independent,*

then

$$B = H^k M^i_n.$$

This is proved in the same manner as [12, Lemma 3.9] using [2, (3.12)].

We turn now to $H^1 M^1_2$. First recall the following statement:

(4.5)[7, Th. 2.2](*cf.* [2, Prop. 3.18]) $H^1 M^0_3$ *is the $K(3)_*$-vector space generated by elements*

$$\zeta_3 \text{ and } t_1^{p^i} \text{ for } i = 0, 1 \text{ and } 2.$$

Proposition 2.18 shows that

$$G_{s/1} = v_3^s / v_2 \text{ and } G(sp^i) = X_i^s / v_2^{a(i)} \text{ for } p \nmid s \text{ and } i \geq 1$$

belong to $H^0 M^1_2$, and implies

(4.6)[2, (5.18)] *Let δ: $H^0 M^1_2 \to H^1 M^0_3$ be the boundary homomorphism of* (4.3). *Then*

$$\delta(G_{s/1}) = s v_3^{s-1} t_1^{p^2} \text{ and } \delta(X_i^s / v_2^{a(i)}) = s v_3^{(sp-1)p^{i-1}} t_1^{p^{\varepsilon(i+1)}}$$

for $i \geq 1$, and ε of (1.3)

Observing the exact sequence (4.3) with the above results (4.5) and (4.6) brings the following

LEMMA **4.7.** $\operatorname{Im} \iota_2 \subset H^1 M^1_2$ *is the Z/p-vector space spanned by the bases represented by the cycles $\chi(up - 1/1)$, $\varphi(tp^i/1)$, $\psi(sp^i/1)$ and $\zeta(t/1)$ for $i \geq 0$, $u \in Z$, $t \in (Z - pZ) \cup \{0\}$ and $s \in S = \{s \in Z : p \nmid s(s+1) \text{ or } p^2 | s+1\}$, where the cycles are represented by*

$$
(4.8)\qquad
\begin{aligned}
\chi(up - 1/1) = v_3^{up-1} t_1^{p^2} / v_2, \quad \varphi(tp^i/1) = v_3^{tp^i} t_1^{p^{\varepsilon(i+1)}} / v_2, \\
\psi(sp^i/1) = v_3^{sp^i} t_1^{p^{\varepsilon(i)}} / v_2 \quad \text{and} \quad \zeta(u/1) = v_3^u \zeta_3 / v_2
\end{aligned}
$$

for ε of (1.3).

Now we define the elements $\chi(m/j)$, $\varphi(m/j)$, $\psi(m/j)$ and $\zeta(m/j)$ of $\Omega^1 M^1_2$ for $m \in Z$ and $j \geq 1$ which equal to those of (4.8) if $j = 1$, and some of which will give the generators of $H^1 M^1_2$.

(4.9) $$\chi(up-1/j) = v_3^{up-p}\beta v_3/v_2^j;$$

(4.10) $$\varphi(t/j) = v_3^{t-1}(v_3t_1^p + tv_2ct_2^p)/v_2^j \quad \text{if } p \nmid t(t-1),$$

$$\varphi(up+1/j) = Y_u/v_2^j \quad \text{if } p \nmid n(n-1),$$

$$\varphi(up^l + e(l)/j) = X_l^u\omega_{l+1}/v_2^j - uX_{l-1}^{up-1}P'_{l+1,\varepsilon(l+1)}/v_2^{j-a(l)} \quad \text{for } l \geq 2,$$

$$\varphi(up^l + 1/j) = X_l^u z/v_2^j + uX_{l-1}^{up-1}P'_{2,1}/v_2^{j-a(l)} \quad \text{for even } l \geq 2,$$

$$\varphi(up^l + 1/j) = X_l^u z/v_2^j + uX_{l-1}^{up-1}P'_{2,0}/v_2^{j-a(l)}$$
$$- uX_{l-1}^{up-p}\beta X_{l-1}/v_2^{j-a(l-1)-p-1} \quad \text{for odd } l \geq 3,$$

$$\varphi(tp/j) = \sigma_t X_1/v_2^j - 2^{-1}tv_3^{tp-2p-2}(t_1^{p^2}\eta_R v_4^2 - 2v_3 t_2^{p^2}\eta_R v_4)/v_2^{j-p-1},$$

$$\varphi(tp^2/j) = \sigma_t X_2/v_2^j,$$

$$\varphi(tp^4/j) = Y_{4,s}/v_2^j, \quad \textit{and}$$

$$\varphi(tp^r/j) = \sigma_t X_r/v_2^j + m_r t v_3^{c(t,r)} P_{2,\varepsilon(r+1)}/v_2^{j-b(r)+p+1} \quad \text{for } r = 3 \text{ or } \geq 5;$$

(4.11) $$\psi(s/j) = \tau_s X_1/v_2^j - \binom{s+1}{2} v_3^{s-p-2} t_1^{2p^2}\eta_R v_4/v_2^{j-2},$$

$$\psi(sp/j) = \tau_s X_2/v_2^j$$

$$\psi(sp^r/j) = \tau_s X_{r+1}/v_2^j + m_{r+1}s(s+1)v_3^{sp^r-2p^{r-1}}P'_{2,\varepsilon(r+1)}/v_2^{j-2a(r)+p}$$
$$\text{for } r \geq 2, \text{ and}$$

$$\psi((up^l + p^2 + 1)p^r/j) = X_{r+l}^u \beta X_{r+1}/v_2^j \quad \text{for } l \geq 2; \text{ and}$$

(4.12) $$\zeta(sp^r/j) = X_r^s \zeta_3^{p^{r+1}}/v_2^j.$$

Here

$$m_r = 1 \text{ if } r = 3 \text{ or even} \geq 4, \text{ and } = 2 \text{ if } r \text{ is odd} \geq 5.$$

By definition we can easily verify the following

LEMMA **4.13.** *For every above element $\xi(m/j)$ for $\xi = \chi, \varphi, \psi$ and ζ,*

$$v_2^{j-1}\xi(m/j) = \xi(m/1),$$

which is the element of (4.8).

Then the results of §§2–3 imply

PROPOSITION **4.14.** *Let $\delta\colon H^1M_2^1 \to H^2M_3^0 = H^2A_3/J(1)$ be the boundary homomorphism in* (4.3)(see (3.2) *for its range*). *For the elements in* (4.9), (4.10), (4.11) *and* (4.12), *we have the following*:

1. $\delta\chi(up-1/p-1) = -v_3^{up-p}b_2;$

2. $\delta\varphi(t/2) = \binom{t}{2} v_3^{t-2} k_1'$,

$\delta\varphi(up + 1/2p) = \binom{u}{2} v_3^{up-1} g_0$,

$\delta\varphi(up^l + e(l)/a(l) + 1) = 2^{-1}(1-u) v_3^{up^l + e(l-2)p} g_1$ *for even* $l \geq 2$,

$\delta\varphi(up^l + e(l)/a(l) + 2) = (u-1) v_3^{up^l + e(l-2)p} b_1$ *for odd* $l \geq 3$,

$\delta\varphi(up^l + 1/a(l) + 1) = 2^{-1} u v_3^{up^l - p^{l-1}} g_1$ *for even* $l \geq 2$,

$\delta\varphi(up^l + 1/a(l) + p) = u v_3^{up^l - p^{l-1}} g_0$ *for odd* $l \geq 3$,

$\delta\varphi(tp/p + 1) = -2^{-1} t v_3^{tp-2p} k_2$,

$\delta\varphi(tp^2/b(2) - 1) = -2^{-1} t v_3^{c(t,2)} k_0$,

$\delta\varphi(tp^4/b(4) - 1) = -\frac{3}{2} t v_3^{c(t,4)} k_0$,

$\delta\varphi(tp^r/b(r) - p + 1) = -m_r t v_3^{c(t,r)} b_1$ *for odd* $r \geq 3$, *and*

$\delta\varphi(tp^r/b(r) - p) = 2^{-1} t v_3^{c(t,r)} g_1$ *for even* $r \geq 6$;

3. $\delta\psi(s/2) = \binom{s+1}{2} v_3^{s-p-1} g_2$,

$\delta\psi(sp/2p) = \binom{s+1}{2} v_3^{sp-2} g_0$,

$\delta\psi(sp^r/2a(r) - p + 1) = m_{r+1} \binom{s+1}{2} v_3^{sp^r - 2p^{r-1}} g_1$ *for even* $r \geq 2$,

$\delta\psi(sp^r/2a(r) - p + 2) = -s(s+1) v_3^{sp^r - 2p^{r-1}} b_1$ *for* $r \geq 3$, *and*

$\delta\psi((up^l + p^2 - 1)p^r/(p-1)a(r+1)) = -v_3^{up^{r+l} + p^{r+2} - p^{r+1}} b_{\varepsilon(r)}$

for $r \geq 0$ *and* $l \geq 2$; *and*

4. $\delta\zeta(sp^r/a(r)) = s X_{r-1}^{sp-1} t_1^{p^{\varepsilon(r+1)}} \otimes \zeta_3^{p^r}$.

Here we notice that the simbol '/' does not means the fraction and the right hand side of '/' in each element denotes the integer of (1.5).

PROOF. Since δ is defined by $\delta x = \iota_3^{-1}\{d_1 \tilde{x}\}$ for $\tilde{x}$ with $v_3 \tilde{x} = x$,

$$\delta\xi(m/j) = \iota_3^{-1}\{d_1 \xi(m/j+1)\} \text{ for } \xi = \chi,\ \varphi,\ \psi \text{ and } \zeta.$$

Noticing that $(d_1 x/v_2^j) = (d_1 x)/v_2^j$ and $(d_1 x)/v_2^j = y/v_2^j$ if $d_1 x \equiv y \mod J(j)$. Since $d_0 v_3 \equiv v_2 t_1^{p^2} - v_2^p t_1 \mod I_2$ by (3.12), we have $d_0 v_3^{up-p} \equiv 0 \mod J(p)$ by the binomial theorem and $d_1 \beta v_3 \equiv -v_2^{p-1} b_2 \mod J(p)$ by Lemma 3.17, which shows the equality 1.

Turn to the equalities of 2. We compute $d_1 v_3^t t_1^p / v_2^3 = t v_3^{t-1} t_1^{p^2} \otimes t_1^p / v_2^2$

$+\binom{t}{2}v_3^{t-2}t_1^{2p^2}\otimes t_1^p/v_2$ and $d_1tv_3^{t-1}ct_2^p/v_2^2=t(t-1)v_3^{t-1}t_1^{p^2}\otimes ct_2^p/v_2-tv_3^{t-1}t_1^{p^2}\otimes t_1^p/v_2^2$ by (2.2) and (3.12). Thus we obtain the first equality of 2 by refering the definition (3.3) of k'. The second one follows immediately from Lemma 3.13. We further see that

$$d_1X_l^u\omega_{l+1}/v_2^{a(l)+2}=uv_3^{up^l-p^{l-1}}t_1^p\otimes\omega_{l+1}/v_2^2+2^{-1}v_3^{up^l+e(l-2)p}g_1/v_2$$

for even $l\geq 2$ and

$$d_1X_l^u\omega_{l+1}/v_2^{a(l)+3}=uv_3^{up^l-p^{l-1}}t_1\otimes\omega_{l+1}/v_2^3-v_3^{up^l+e(l-2)p}b_1/v_2$$

for odd $l\geq 3$ by (2.2), Propositions 2.18 and 3.24, and

$$-ud_1X_{l-1}^{up-1}P'_{l+1,1}/v_2^2=-2^{-1}uv_3^{up^l-p^{l-1}+e(l-1)p}g_1/v_2-(-1)^luv_3^{up^l-p^{l-1}}t_1^p\otimes\omega'_{l+1}/v_2^2$$

for even $l\geq 2$ and

$$-ud_1X_{l-1}^{up-1}P'_{l+1,0}/v_2^3=uv_3^{up^l-p^{l-1}+e(l-1)p}b_1/v_2-(-1)^luv_3^{up^l-p^{l-1}}t_1\otimes\omega'_{l+1}/v_2^3$$

for odd $l\geq 3$ by Lemma 3.10. The definition (3.23) shows $\omega_{l+1}\equiv(-1)^lw'_{l+1}$ mod $J(3)$, so these equalities give rise to the third and fourth equalities of 2. We obtain the equalities

$$d_1X_l^uz/v_2^{a(l)+2}=uv_3^{up^l-p^{l-1}}t_1^p\otimes z/v_2^2$$

for even $l\geq 2$, and

$$d_1X_l^uz/v_2^{a(l)+p+1}=uv_3^{up^l-p^{l-1}}t_1\otimes z/v_2^{p+1}$$

for odd $l\geq 3$ from Proposition 2.18 and the fact that $d_1z\equiv 0$ mod I_2 given by (2.16), and the equalties

$$ud_1X_{l-1}^{up-1}P'_{2,1}/v_2^2=2^{-1}uv_3^{up^l-p^{l-1}}g_1/v_2-uv_3^{up^l-p^{l-1}}t_1^p\otimes w'_2/v_2^2,\ \text{and}$$
$$ud_1X_{l-1}^{up-1}P'_{2,0}/v_2^{p+1}=-uv_3^{up^l-p^{l-1}}b_1/v_2+uv_3^{up^l-p^{l-1}}g_0/v_2-uv_3^{up^l-p^{l-1}}t_1\otimes w'_2/v_2^{p+1}$$

from Lemma 3.10, and obtain

$$-ud_1X_{l-1}^{up-p}\beta X_{l-1}/v_2^{a(l)-a(l-1)}=uv_3^{up^l-p^{l-1}}b_1/v_2^{p-1}$$

from lemma 3.17 and the fact that $d_0X_{l-1}\equiv v_2^{a(l-1)}v_3^{p^{l-1}-p^{l-2}}t_1^p$ mod $J(a(l-1)+p)$ shown by Proposition 2.18 with $a(l)-a(l-1)=pa(l-1)-a(l-1)+p-1$. These equalities shows the fifth and sixth ones. Notice that we have $W_2\equiv -v_3^{-p-1}t_1^{p^2}\eta_Rv_4+v_3^{-p}t_2^{p^2}$ mod I_3 by (2.8), and $\rho X_1\equiv v_2v_3^{p-1}W_2$ mod $J(2)$ by Proposition 2.18 and (3.15). Then it follows from Lemma 3.17 that

$$d_1\sigma_tX_1/v_2^{p+2}=-tv_3^{tp-1}(v_3^{-p}t_2^{p^2}-v_3^{-p-1}t_1^{p^2}\eta_Rv_4)\otimes t_1/v_2+2^{-1}tv_3^{tp-2}t_1^{p^2}\otimes t_1^2/v_2.$$

The direct calculation with (3.7), (3.11) and (3.12) shows

$$\begin{aligned}
&- 2^{-1}td_1v_3^{tp-2p-2}(t_1^{p^2}\eta_R v_4^2 - 2v_3t_2^{p^2}\eta_R v_4)/v_2\\
&\quad= -2^{-1}v_3^{tp-2p-2}(-t_1^{p^2}\otimes(-2v_3^pv_4t_1 + v_3^2t_1^{2p^3} + v_3^{2p}t_1^2)\\
&\qquad + 2v_3^2t_1^{p^2}\otimes t_1^{2p^3} + 2v_3t_2^{p^2}\otimes(v_3t_1^{p^3} - v_3^pt_1))/v_2.
\end{aligned}$$

Therefore we obtain the seventh. The eighth follows from Lemma 3.17 since we have $\varphi X_2 \equiv v_2^p v_3^{c(2)} t_2$ and $d_0X_1^{p-1} \equiv v_2^p v_3^p t_1$ mod $J(p+1)$ by Proposition 2.18, and the ninth immediately from Lemma 3.21.

We see

$$d_1\sigma_tX_r/v_2^{b(r)-p+2} = -m_rtX_r^{t-1}w_2'\otimes t_1/v_2^3$$

for odd $r \geq 3$, and

$$d_1\sigma_tX_r/v_2^{b(r)-p+1} = -tX_r^{t-1}w_2'\otimes t_1^p/v_2^2$$

for even $r \geq 6$, since $\varphi X_r \equiv m_rv_2^{b(r)-a(r)-p-1}w_2'$ and $d_0X_{r-1}^{p-1} \equiv 0$ mod $J(b(r)-a(r)-1)$, where we note that $a(r-1) = b(r)-a(r)-1$. Now the last two equalities in 2 follow from the above equalities and Lemma 3.10.

For the equalities in 3, we compute

$$\begin{aligned}
d_1\tau_sX_1/v_2^3 &= \binom{s+1}{2}v_3^{s-2}t_1^{2p^2}\otimes t_1/v_2\\
&\quad + s(s+1)v_3^{s-1}t_1^{p^2}\otimes(v_3^{-p}t_2^{p^2} - v_3^{-p-1}t_1^{p^2}\eta_Rv_4)/v_2,\\
d_1\tau_sX_2/v_2^{2p+1} &= \binom{s+1}{2}v_3^{sp-2}t_1\otimes t_1^p/v_2\\
&\quad + s(s+1)v_3^{sp-p^2+p-1}t_1\otimes v_3^{p^2-p-1}t_2/v_2, \text{ and}\\
d_1\tau_sX_{r+1}/v_2^{2a(r)-p+2} &= s(s+1)v_3^{sp-p^2+p-1}t_1^p\otimes(m_{r+1}v_3^{c(r+1)}w_2')/v_2^2
\end{aligned}$$

for even $r \geq 2$ and

$$d_1\tau_sX_{r+1}/v_2^{2a(r)-p+3} = s(s+1)v_3^{sp-p^2+p-1}t_1\otimes(m_{r+1}v_3^{c(r+1)}w_2')/v_2^3$$

for odd $r \geq 3$ from Proposition 2.18, Lemma 3.17 and (2.11). We further compute

$$\begin{aligned}
-\binom{s+1}{2}d_1v_3^{s-p-2}t_1^{2p^2}\eta_Rv_4/v_2 &= s(s+1)v_3^{s-p-2}t_1^{p^2}\otimes t_1^{p^2}\eta_Rv_4/v_2\\
&\quad + \binom{s+1}{2}v_3^{s-p-2}t_1^{2p^2}\otimes(v_3t_1^{p^3} - v_3^pt_1)/v_2
\end{aligned}$$

by (3.11), (3.12) and (3.7), and

$$m_{r+1}s(s+1)d_1v_3^{sp^r-2p^{r-1}}P'_{2,1}/v_2^2 = m_{r+1}s(s+1)v_3^{sp^r-2p^{r-1}}(2^{-1}v_2g_1 - t_1^p\otimes w_2')/v_2^2$$

and

$$m_{r+1}s(s+1)d_1v_3^{sp^r-2p^{r-1}}P'_{2,0}/v_2^3 = m_{r+1}s(s+1)v_3^{sp^r-2p^{r-1}}(-v_2^2b_1 - t_1\otimes w'_2)/v_2^3.$$

Combining these equalities leads us to the first four of 3. The other one follows immediately from Lemma 3.17 and Proposition 2.18.

The equality in 4 follows also from Proposition 2.18, since we get $d_1\zeta_3^{p^{r+1}} \equiv 0$ mod $J(p^{r+1})$ from the binomial theorem. q.e.d.

PROOF OF THEOREM A. All the elements in Proposition 4.14 are linearly independent by (3.2) and hence the theorem follows from lemmas 4.4 and 4.7 by setting

$$\chi(k) = \chi(kp-1/p-1),\ \varphi(k) = \varphi(k/\lambda(k)),$$

$$\psi(k) = \psi(k/\mu(k)) \quad \textit{and} \quad \zeta(k) = \zeta(k/\tilde{a}(k)).$$ q.e.d.

§5. Application to the stable homotopy

In this section we assume the prime $p \geq 7$. Let s denote the sphere spectrum, and $V(0)$, $V(1)$ and $V(2)$, the cofibers of the maps $p\in\pi_0S$, $\alpha\in[V(0), V(0)]_q$ and $\beta\in[V(1), V(1)]_{(p+1)q}$, respectively. Here $q = 2p-2$, α denotes the Adams map and β the map given in [13]. Then the map $\gamma\in[V(2), V(2)]_{e(3)q}$ with $BP_*\gamma = v_3$ exists by [14], and gives the γ-family $\{\gamma_{[s]} = j_2\gamma^s i_2\}$ of $[V(1), V(1)]_*$ where $j_2\in[V(2), V(1)]_{-(p+1)q-1}$ and $i_2\in[V(1), V(2)]_0$ are the canonical maps. Let $i\in\pi_0V(1)$ denotes the canonical map, and we have the γ-family $\{\gamma'_{[s]} = \gamma_{[s]}i\}$ of $\pi_*V(1)$. Consider the Adams-Novikov spectral sequence converging to $\pi_*V(1)$, whose E_2-term is $H^*N_2^0$. In the E_2-term we define the γ-element according to [2] by

$$\gamma'_{[s]} = \delta' v_3^s/v_2 \quad \text{for} \quad v_3^s/v_2\in H^0N_2^1$$

for the boundary homomorphism δ' associated to the short exact sequence $0 \longrightarrow N_2^0 \xrightarrow{\subset} M_2^0 \longrightarrow N_2^1 \longrightarrow 0$, which survives to the same named element of $\pi_*V(1)$. Similarly we have the β-family $\{\beta_s = j\beta^s i\}$ of π_*S for the canonical map $j\in[V(1), S]_{-q-2}$, and of the E_2-term surviving to the same named map. For the β-family of E_2-term we have

(5.1)[5, Lemma 4.4] $\quad \beta_s \equiv sv_2^{s-1}b_0 + \binom{s}{2}v_2^{s-2}k_0$ mod I_2 for $s > 0$.

THEOREM **5.2.** *Let r and s be non-negative integers with $p \nmid s > 0$. Then in the homotopy group $\pi_*V(1)$,*

$$\gamma'_{[sp^r]}\beta_1 \neq 0 \quad \textit{if } r \textit{ is even or } p\nmid s+1, \textit{ and}$$

$$\gamma'_{[sp^r]}\beta_2 \neq 0 \quad \textit{if } r \neq 0,\ 2 \textit{ or } p^2\nmid s+p+1.$$

PROOF. Since the filtration of these elements is three, nothing kills them in the Adams-Novikov spectral sequence converging to $\pi_*V(1)$. Therefore we prove the non-triviality in the E_2-term $H^3N_2^0$. Consider the diagram

$$\begin{array}{ccccc} & & H^2M_2^0 & \xrightarrow{f} & H^2N_2^1 & \xrightarrow{\delta'} & H^3N_2^0 \\ & & & & \downarrow{\scriptstyle\lambda} & & \\ H^1M_2^1 & \xrightarrow{\delta} & H^2M_3^0 & \xrightarrow{\iota_2} & H^2M_2^1 & & \end{array}$$

which has two exact rows associated to the short ones in the above and §4 respectively. λ is the localization map in (4.1). Put $GB_1 = v_3^{sp^r}b_0/v_2$ and $GB_2 = v_3^{sp^r}k_0/v_2$. Then we see that

$$\lambda GB_1 \neq 0 \quad \text{if } r \text{ is even or } p \nmid s+1, \text{ and}$$

$$\lambda GB_2 \neq 0 \quad \text{if } r \neq 0, 2 \text{ or } p^2 \nmid s+p+1,$$

since these do not belong to $\mathrm{Im}\,\delta$ by Proposition 4.14 and Theorem A. Furthermore we see that $\mathrm{Im}\,\lambda f$ is generated by b_k/v_2^j ($k = 1, 2$) and $t_1^{p^i} \otimes z/v_2^j$ ($i = 0, 1$) for $j \geq 1$ by [7, Th. 2.4] and (3.7). Therefore $GB_k \notin \mathrm{Im}\,f$ by Lemma 3.10, and so $\gamma'_{[sp^r]}\beta_k = \delta' GB_k \neq 0$ if r and s satisfy the above condition. q.e.d.

We define the γ-element $\gamma'_{[up/2]}$ ($u > 0$) of $\pi_* V(1)$ by the composition $S \longrightarrow L_2 \xrightarrow{f^u} L_2 \longrightarrow V(1)$ for the spectrum L_2 such that

$$BP_*L_2 = BP_*/J(2),$$

and the map f given in [4, Th. 4.2] which induces

$$f_* = v_3^p \colon BP_*/J(2) \longrightarrow BP_*/J(2),$$

and for the canonical maps $S \to L_2$ and $L_2 \longrightarrow V(1)$. We also have the γ-element in the E_2-term:

$$\gamma'_{[m/2]} = \delta' v_3^m/v_2^2 \quad (p|m),$$

and hence

(5.3) $$v_2\gamma'_{[m/2]} = \gamma'_{[m]} \quad \textit{and} \quad \gamma'_{[m/2]}\beta_3 = 3\gamma'_{[m]}\beta_2$$

in the E_2-term by (5.1). It is known from the Geometric Boundary Theorem [1] that these elements survive to the elements in $\pi_* V(1)$ of same name. Therefore we have

COROLLARY **5.4.** *Let r and s be positive integers with $p \nmid s$. Then in $\pi_* V(1)$,*

$$\gamma'_{[sp^r/2]}\beta_1 \neq 0 \textit{ if } r \textit{ is even or } p \nmid s+1, \textit{ and}$$

$$\gamma'_{[sp^r/2]}\beta_2 \neq 0 \neq \gamma'_{[sp^r/2]}\beta_3 \quad \textit{if } r \neq 2 \textit{ or } p^2 \nmid s+p+1.$$

Furthermore, $v_3^m\beta_2/v_2^2 = 2v_3^m b_0/v_2 + v_3^m k_0/v_2^2$ by (5.1), which is non-trivial by Proposition 4.14 and Theorem A. Thus in a same way as Theorem 5.2, we obtain

THEOREM **5.5.** *Let m be a positive integer with $p|m$. Then,*

$$\gamma'_{[m/2]}\beta_2 \neq 0 \quad \text{in } \pi_* V(1).$$

The localization map λ also induces the map $\lambda\colon H^1N_2^1 \to H^1M_2^1$ and we have the monomorphism $\delta\colon H^1N_2^1 \to H^2N_2^0$ at positive degree. Furthermore $\alpha_1 = t_1$ in H^1BP_* (*cf.* [2]). Then Lemma 4.7 similarly implies the following

PROPOSITION **5.6.** *For non-negative integers r and s with $p \nmid s > 0$, we have*

$$\gamma'_{[sp^r]}\alpha_1 \neq 0 \quad \textit{in } \pi_* V(1)$$

if r is odd, or if r is even and either $p \nmid s+1$ or $p^2 | s+1$.

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