Computation of Obstructions for a Spectrum with BP_*-Homology $(BP_*/I_n)[t_1, t_2, \cdots, t_k]$

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§1. Introduction

The Toda-Smith spectrum $V(n)$ [5], [6] for $n = 0, 1, 2,$ and 3 at a prime p greater than $2n+1$ plays such an important role as giving periodic families in stable homotopy groups of spheres. The BP_*-homology of $V(n)$ is known as

$$BP_*/I_{n+1}$$

for the invariant ideal $I_{n+1} = (p, v_1, v_2, \cdots, v_n)$ for each n, where BP stands for the Brown-Peterson spectrum at a prime p whose coefficient ring is the polynomial ring $Z_{(p)}[v_1, v_2, \cdots]$ with Hazewinkel's generators v_i (*cf*, [3]). Ravenel's spectrum $T(k)$ [2] (*cf.* [3]) for each integer $k \geq 0$ satisfies:

$$BP_* T(k) = BP_*[t_1, \cdots, t_k] \subset BP_*BP$$

as a comodule algebra, where BP_*-homology of BP, BP_*BP, is the polynomial ring $BP_*[t_1, t_2, \cdots]$. Similarly to these spectra, we define a spectrum $W_k(n)$ to be

$$BP_* W_k(n) = (BP_*/I_{n+1})[t_1, \cdots, t_k]$$

as a subcomodule of BP_*BP/I_{n+1}. Note then that $W_0(n) = V(n)$ and $W_k(n) = T(k) \wedge V(n)$ if these spectra exist. We further have a ring spectrum $P(n)$ with BP_*-homology

$$BP_*P(n) = (BP_*/I_{n+1})[t_1, t_2, \cdots] = BP_*BP/I_{n+1}$$

(*cf.* [2, Cor. 2.14], whose $P(n)$ is our $P(n-1)$). We call $W_k(n)$ *good* if it is a ring spectrum and if there exists a map $i\colon W_k(n) \to P(n)$ of ring spectra which induces the canonical inclusion $i_*\colon (BP_*/I_{n+1})[t_1, \cdots, t_k] \subsetneq (BP_*/I_{n+1})[t_1, t_2, \cdots]$ on BP_*-homology.

If $n \leq 3$, then it is known [5], [6] that the spectrum $V(n)$ exists if and only if the prime $p \geq 2n+1$. Though we know nothing about the existence of $V(n)$ for $n \geq 4$, the first named author shows the existence of the spectrum $W_1(4)$ [4]. Here we investigate obstructions of the existence of $W_k(n)$ in the E_2-term of the Adams-Novikov spectral sequence and we have

THEOREM A. *Let k and n be non-negative integers with $k \geq 2$ and $n \leq k+3$. If a*

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good spectrum $W_k(n)$ *exists, then there exists a spectrum* $W_k(n+1)$.

In [6], Toda showed that $V(3)$ is a ring spectrum if $p \geq 11$. Therefore the spectrum $W_k(3) = T(k) \wedge V(3)$ get a ring structure with the canonical inclusion $i: W_k(3) \to BP \wedge V(3)$, which is a map of ring spectra, in other word, $W_k(3)$ is good. Thus we have

COROLLARY B. $W_k(4)$ *exists for* $k \geq 2$ *if* $p \geq 11$.

In §2 we prepare some Hopf algebroids which are used as computational tools, and we show that there is no obstruction in our case in §3. We then prove Theorem A in §4 by showing that the induced map satisfies the desired property.

§2. Camputatinal tools

Let p be a prime number and K denote the prime field of characteristic p. A pair (A, Γ) of commutative K-algebras is said to be a *Hopf algebroid* if it provides structure maps: a left and a right unit $\eta_L, \eta_R: A \to \Gamma$, a coproduct $\Delta: \Gamma \to \Gamma \otimes_A \Gamma$, a counit $\varepsilon: \Gamma \to A$, and a conjugation $c: \Gamma \to \Gamma$ satisfying

$$\varepsilon\eta_L = \varepsilon\eta_R = 1_A, \quad (\Gamma \otimes \varepsilon)\Delta = (\varepsilon \otimes \Gamma)\Delta = 1_\Gamma, \tag{2.1}$$
$$(\Gamma \otimes \Delta)\Delta = (\Delta \otimes \Gamma)\Delta, \quad c\eta_R = \eta_L, \quad c\eta_L = \eta_R \quad \text{and } cc = 1_\Gamma.$$

Here, $\Gamma \otimes_A \Gamma$ is the tensor product of A-bimodules given by η_R and η_L, and Δ and ε are A-bimodules maps. A right A-module M is said to be a *right Γ-comodule* if it provides a structure map $\psi_M: M \to M \otimes_A \Gamma$ which is a right A-linear map satisfying

$$(1_M \otimes \varepsilon)\psi_M = 1_M \quad \text{and} \quad (1_M \otimes \Delta)\psi_M = (\psi_M \otimes 1_\Gamma)\psi_M. \tag{2.2}$$

From here on, we assume that Γ is flat over A.

For a (right) Γ-comodule M, $\mathrm{Ext}^*_\Gamma(A, M)$ denotes the homology of the cobar complex $(\Omega^*_\Gamma M, d_*)$ with

$$\Omega^r_\Gamma M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma \quad (r \text{ copies of } \Gamma) \tag{2.3}$$

and differential $d_r: \Omega^r_\Gamma M \to \Omega^{r+1}_\Gamma M$ given by

$$\begin{aligned} d_r(m \otimes \gamma_1 \otimes \cdots \otimes \gamma_r) = {} & \psi_M(m) \otimes \gamma_1 \otimes \cdots \otimes \gamma_r \\ & + \textstyle\sum_{i=1}^r m \otimes \gamma_1 \otimes \cdots \otimes \Delta(\gamma_i) \otimes \cdots \otimes \gamma_r \\ & + (-1)^{r+1} m \otimes \gamma_1 \otimes \cdots \otimes \gamma_r \otimes 1. \end{aligned} \tag{2.4}$$

This Ext is computed by another complex.

(2.5)[1, Lemma 1.1] *Let*

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

be an exact sequence in which each entry is Γ-comodule and $\mathrm{Ext}^r_\Gamma(A, I^i) = 0$ *for all* r

> 0. *Then* $\mathrm{Ext}_{\Gamma}^{*}(A, M)$ *is the homology of the complex* $\mathrm{Hom}_{\Gamma}(A, I^{*})$.

Put

$$\Gamma = K[t_1, t_2, \cdots] \tag{2.6}$$

with $|t_i| = 2p^i - 2$, and define

$$\eta_R = \eta_L \colon K \hookrightarrow \Gamma \tag{2.7}$$

to be the canonical inclusion, and K-algebra maps given by

$$\Delta(t_n) = \textstyle\sum_{i=0}^{n} t_i \otimes t_{n-i}^{p^i}, \quad \varepsilon(t_n) = \delta_{n,0}, \quad \textit{and} \quad \textstyle\sum_{i=0}^{n} t_i c t_{n-i}^{p^i} = \delta_{n,0} \tag{2.8}$$

($t_0 = 1$) for $n \geq 0$, where $\delta_{n,j}$ is Kronecker's delta. Then the pair (K, Γ) is a Hopf algebroid. In a similar way, we see that

$$(K, \Sigma_i) = (K, K[t_i, t_{i+1}, \cdots]) \quad \textit{and} \quad (K, \Phi_{i,j}) = (K, K[t_i, \cdots, t_j]) \tag{2.9}$$

$(i \geq j)$ are Hopf algebroids. Furthermore, the cocentral extension

$$(K, \Phi_{i,j}) \xrightarrow{\iota_i} (K, \Sigma_i) \xrightarrow{f_i} (K, \Sigma_{j+1})$$

gives rise to the Cartan-Eilenberg spectral sequence

$$E_2 = H^{*,*}(\Phi_{i,j}) \otimes H^{*,*}(\Sigma_{j+1}) \Longrightarrow H^{*,*}(\Sigma_i). \tag{2.10}$$

Hereafter, we denote

$$H^{*,*}(L) = \mathrm{Ext}_L^{*,*}(K, K), \tag{2.11}$$

for a Hopf algebroid (K, L), in which the first affix denotes the homology degree and the second is the inner degree.

LEMMA **2.12.** *Let s and t be non-negative integers. Then*

$$H^{s,t}(\Gamma) = 0 \quad \textit{if} \quad t < 2s(p-1), \textit{ and}$$

$$H^{s,t}(\Sigma_i) = 0 = H^{s,t}(\Phi_{i,j}) \quad \textit{if} \quad t < 2s(p^i - 1).$$

PROOF. We have the normalized cobar complex $(\tilde{\Omega}_L^{*} M, d_{*}) \subset (\Omega_L^{*} M, d_{*})$ for a Hopf algebroid (K, L) with $\tilde{\Omega}_L^{s} M = M \otimes_A \mathrm{Ker}\,\varepsilon \otimes_A \cdots \otimes_A \mathrm{Ker}\,\varepsilon$ (s copies of $\mathrm{Ker}\,\varepsilon$), whose homology is known to be same as that of the cobar complex. Since $(\mathrm{Ker}\,\varepsilon)^t = 0$ if $t < 2(p-1)$ for $L = \Gamma$, and if $t < 2(p^i - 1)$ for $L = \Sigma_i$ or $\Phi_{i,j}$, we see that $\tilde{\Omega}_L^{s,t} K = 0$ if $t < 2s(p-1)$ for $L = \Gamma$, and if $t < 2s(p^i - 1)$ for $L = \Sigma_i$ or $\Phi_{i,j}$. This implies the lemma. q.e.d.

COROLLARY **2.13**. *Let m be a positive integer and s and t non-negative integers with $t - s = 2p^m - 3$. Then*

$$H^{s,t}(\Sigma_i) = H^{s,t}(\Phi_{i,m})$$

for $i \leq m$.

PROOF. (2.10) says

$$H^{a,b}(\Phi_{i,m}) \otimes H^{c,d}(\Sigma_{m+1}) \Longrightarrow H^{s,t}(\Sigma_i) \quad (s = a + c,\ t = b + d).$$

By definition, the innerdegree is greater than the homological degree and so $b - a > 0$, which implies

$$d - c < t - s = 2p^m - 3.$$

Then $H^{c,d}(\Sigma_{m+1}) = 0$ for $c \geq 1$ since $d < 2p^m - 3 + c < 2c(p^{m+1} - 1)$. Thus the spectral sequence collapses and we have the corollary. q.e.d.

We here introduce another Hopf algebroid

(2.14) $$(K, \Phi_i) = (K, K[t_i])$$

with t_i primitive. Then we have the cocentral extension

$$\Phi_{i,m-1} \longrightarrow \Phi_{i,m} \longrightarrow \Phi_m$$

which give the Cartan-Eilenberg spectral sequence

(2.15) $$E_2 = H^{*,*}(\Phi_{i,m-1}) \otimes H^{*,*}(\Phi_m) \Longrightarrow H^{*,*}(\Phi_{i,m}).$$

This implies

$$\text{rank } (H^{*,*}(\Phi_{i,m-1}) \otimes H^{*,*}(\Phi_m))^{s,t} \geq \text{rank } H^{s,t}(\Phi_{i,m}).$$

Now Corollary 2.13 together with this reduces to

COROLLARY **2.16**. *Let m, s and t be such integers as those of Corollary* 2.13. *Then*

$$\text{rank } (H^{*,*}(\Phi_i) \otimes \cdots \otimes H^{*,*}(\Phi_m))^{s,t} \geq \text{rank } H^{s,t}(\Sigma_i).$$

The structure of $H^{*,*}(\Phi_i)$ is well known to be:

(2.17) $$H^{*,*}(\Phi_i) = E(h_{i,j}) \otimes P(b_{i,j}).$$

Here E denotes exterior and P polynomial alegebra and bidegrees of $h_{i,j}$ and $b_{i,j}$ are $(1, 2p^j(p^i - 1))$ and $(2, 2p^{j+1}(p^i - 1))$, respectively. Consider the commutative graded free algebra F_n generated by $h_{i,j}$ and $b_{i,j}$ with $n \leq i \leq m$. Then,

COROLLARY **2.18.** *For integers m, s, t in Corollary* 2.13,

$$\text{rank } F_n^{s,t} \geq \text{rank } H^{s,t}(\Sigma_n).$$

Let (K, L) be a Hopf algebroid and M and N a right and a left L-comodule, respectivery. Then the cotensor $M \square_L N$ is defined to be the Kernel of the map

$$\psi_M \otimes 1_N - 1_M \otimes \psi_N \colon M \otimes N \longrightarrow M \otimes L \otimes N.$$

The map $\psi_\Gamma = (1_\Gamma \otimes f_i)\Delta: \Gamma \to \Gamma \otimes \Sigma_{i+1}$ makes Γ a right Σ_{i+1}-comodule and the inclusion $K \subsetneq \Sigma_{i+1} = \Sigma_{i+1} \otimes K$ makes K a left Σ_{i+1}-comodule, where f_i denotes the canonical projection. Then we see the following

LEMMA **2.19.** $\Gamma \square_{\Sigma_{i+1}} K = \Phi_{1,i}$.

PROOF. Consider the composition

$$\psi = \psi_\Gamma \circ i_i: \Phi_{1,i} \longrightarrow \Gamma \longrightarrow \Gamma \otimes \Sigma_{i+1}.$$

Then $\psi(x) = x \otimes 1$ for $x \in \Phi_i$ by definition of coproduct, which implies

$$\Phi_{1,i} \subset \Gamma \square_{\Sigma_{i+1}} K.$$

Take any element

$$x = \sum_F \lambda_F t^F \in \Gamma \quad (\lambda_F \in K),$$

where $F = (f_1, f_2, \cdots)$ is sequence of non-negative integers which are all zero except finite numbers, and

$$t^F = t_1^{f_1} t_2^{f_2} \cdots.$$

Since $\Delta(t_n) = \sum_{i=0}^n t_i \otimes t_{n-i}^{p^i}$, $\psi_\Gamma(t_j) = t_j \otimes 1$ if $j \leq i$ and $\psi_\Gamma(t_j) = \tilde{t}_j + 1 \otimes t_j$ for some element $\tilde{t}_j \in \Gamma \otimes \Sigma_{i+1}$ if $j \geq i+1$. Therefore

$$\psi_\Gamma(x) - x \otimes 1 = \sum_{E,F} \lambda_E (1 \otimes t^F) + \cdots,$$

which is zero if λ_F is zero or F is of the form $(f_1, f_2, \cdots, f_i, 0, \cdots)$. Hence $x \in \Gamma \square_{\Sigma_{i+1}} K$ implies

$$x \in \Phi_{1,i},$$

and we have the lemma. q.e.d.

THEOREM **2.20.** $\mathrm{Ext}_\Gamma^{*,*}(K, \Phi_{1,i}) = H^{*,*}(\Sigma_{i+1}) (= \mathrm{Ext}_{\Sigma_{i+1}}^{*,*}(K, K))$.

PROOF. For the cobar complex $C^* = \Omega^*_{\Sigma_{i+1}} \Sigma_{i+1}$, we see that the complex $0 \to K \to C^*$ satisfies the condition of (2.5) and

$$H^{*,*}(\Sigma_{i+1}) = H(\mathrm{Hom}_{\Sigma_{i+1}}(K, C^*)).$$

On the other hand, apply $\Gamma \square_{\Sigma_{i+1}}$ on that complex, and we have

$$0 \longrightarrow \Phi_{1,i} \longrightarrow \Gamma \square_{\Sigma_{i+1}} C^*$$

by Lemma 2.19, and furthermore it is exact, since C^* is split exact. Therefore this also satisfies the condition of (2.5), and

$$\mathrm{Ext}_\Gamma^{*,*}(K, \Phi_{1,i}) = H(\mathrm{Hom}_\Gamma(K, \Gamma \square_{\Sigma_{i+1}} C^*)).$$

Here we get easily $\mathrm{Hom}_\Gamma(K, \Gamma \square_{\Sigma_{i+1}} C^*) = \mathrm{Hom}_{\Sigma_{i+1}}(K, C^*)$ by definition and we have the desired equality. q.e.d.

§ 3. Calculation of the E_2-term

From this section on we assume the prime p is greater than 7. The Brown Peterson spectrum BP at the prime p has the coefficient ring $BP_* = Z_{(p)}[v_1, v_2, \cdots]$ with $|v_i| = 2p^i - 2$. We define the spectrum $W_k(n)$ to be the one which satisfies

$$(3.1) \qquad BP_* W_k(n) = (BP_*/I_{n+1})[t_1, t_2, \cdots, t_k] \subset BP_*BP/I_{n+1}$$

as comodule algebras, where $BP_*BP = BP_*[t_1, t_2, \cdots]$ and I_n denotes the ideal $(p, v_1, \cdots, v_{n-1})$ of BP_* or BP_*BP. Consider the Adams-Novikov spectral sequence converging to the homotopy group $\pi_*(W_k(n-1))$ with the E_2-term

$$(3.2) \qquad E_2^{s,t}(k, n) = \mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_* W_k(n-1)).$$

In this section we shall compute the E_2-term $E_2^{s,t}(k, n)$ with $t - s = 2p^n - 3$ for integers k and n with $n \leq k + 4$.

Considering the cobar complexes, we have an isomorphism

$$(3.3) \qquad E_2^{s,t}(k, n) = \mathrm{Ext}_{BP_*BP/I_n}^{s,t}(BP_*/I_n, BP_* W_k(n-1))$$

and the vanishing line:

$$(3.4) \qquad E_2^{s,t}(k, n) = 0 \quad \text{if} \quad t < 2s(p-1).$$

The condition $t - s = 2p^n - 3$ together with (3.4) implies that $E_2^{s,t}(k, n)$ does not intersect $v_n(E_2^{a,b}(k, n))$ with $b - a = -1$. Therefore

$$(3.5) \qquad E_2^{s,t}(k, n) = \mathrm{Ext}_\Gamma^{s,t}(K, \Phi_{1,k})$$

at $t - s = 2p^n - 3$, and the right hand side is the one of Theorem 2.20.

LEMMA **3.6.** *Let* $t - s = 2p^n - 3$. *If* $E_2^{s,t}(k, n) \neq 0$, *then*

$$s \leq \frac{2p^n - 3}{2p^{k+1} - 3}.$$

PROOF. Lemma 2.12 and Theorem 2.20 induces that $E_2^{s,t}(k, n) \neq 0$ implies $t \geq s(2p^{k+1} - 2)$. Apply now $t = 2p^n - 3 + s$ to this inequality, and we get the desired one. q.e.d.

LEMMA **3.7.** $E_2^{s,t}(k, n) = 0$ *for* $t - s = 2p^n - 3$ *and* $k, s \geq 2$ *with* $n \leq k + 4$.

PROOF. We have

$$(3.8) \qquad \mathrm{rank}(H^{*,*}(\Phi_{k+1}) \otimes \cdots \otimes H^{*,*}(\Phi_n))^{s,t} \geq \mathrm{rank}\ E_2^{s,t}(k, n)$$

by Theorem 2.20, and Corollary 2.16 and (3.5). If $n \leq k$, then the left hand side is 0 in the above inequality. Therefore $E_2^{s,t}(k, n) = 0$ for $n \leq k$.

For the case $n = k + 1$, we have $s \leq 1$ by Lemma 3.6, which is against to the hypothesis $s \geq 2$. Therefore

$$E_2^{s,t}(k, n) = 0.$$

We next turn to the case $n = k + 2$. It follows from Lemma 3.6 that $s \leq p + (3p-3)/(3p^{k+1}-3)$, and so

$$s \leq p.$$

We assume that $k \geq 2$.

On the other hand $E_2^{s,t} = 0$ if $t \not\equiv 0$ $(2p-2)$ by degree reason. Then in this case $t = 2p^{k+2} - 3 + s \equiv 0$ $(2p-2)$, and hence $s \equiv 1$ $(2p-2)$. These arguments lead $s = 1$, which again contradicts to $s \geq 2$. So

$$E_2^{s,t}(k, n) = 0.$$

Now we turn to the case $n = k + 3$. Similarly we see that $s \leq p^2$ by Lemma 3.6, and that $E_2^{s,t} = 0$ if $t \equiv 0$ $(2p-2)$, which imply

$$s = 2u(p-1) + 1 \quad \text{for} \quad 1 \leq u < (p+1)/2. \tag{3.9}$$

To proceed further, we prepare the following

NOTATION **3.10.** *Here we prepare the following notation: let E and F be sequences of non-negative integers $\varepsilon_{i,j}$ and $a_{i,j}$ all of which but finite are* 0. *Then we denote*

$$h^E = \textstyle\prod_{i,j} h_{i,j}^{\varepsilon_{i,j}} \quad |E| = \sum_{i,j} \varepsilon_{i,j}, \quad |F| = \sum_{i,j} a_{i,j}, \quad \textit{and} \quad b^F = \prod_{i,j} b_{i,j}^{a_{i,j}}.$$

Here $\varepsilon_{i,j} = 0$ or 1.

We study for the element $h^E b^F$ whose bidegree is (s, t). Here bideg $h_{i,j} = (1, 2p^i(p^i - 1))$ and bideg $b_{i,j} = (2, 2p^{j+1}(p^i - 1))$. Let $|h^E b^F| = t$ and $|E| + 2|F| = s$. Since s is odd, we put $|E| = 2e + 1$ for $e \geq 0$ and then

$$|F| = \frac{s - 2e - 1}{2}.$$

First we study the element of the form $x(l, q) = b_{k+1,0}^{|F|-l} b_{k+1,1}^{l-q} b_{k+2,0}^{q}$ for the equality b^F. $|h^E x(l, q)| = t$ implies $|h^E| = t - |x(l, q)|$, and we compute

$$\begin{aligned} |h^E| = {} & 2p^{k+3}(1 - l - u) + 2p^{k+2}(e + u + l) + 2p^2(l + u) \\ & - 2p(u + e + l + q) - 3 + 2q, \end{aligned} \tag{3.11}$$

by (3.9) and the equality $t = 2p^{k+3} - 3 + s$. We see $1 - l - u \geq 0$. In fact if the coefficient $1 - l - u$ of $2p^{k+3}$ is negative, then so is $|h^E|$. This is a contradiction. We get $u = 1$, $l = 0$ and $1 - l - u = 0$ by the condition $u \geq 1$. Then (3.11) turns into

$$|h^E| = 2p^{k+2}(1 + e) + 2p^2 - 2p(1 + e) - 3 + 2q,$$

and we do not have such h^E. Thus there is no $h^E x(l, q)$ whose bidegree is (s, t). If b^F is not of the form $x(l, q)$, then $|b^F| \geq |x(l, q)|$. Since $|h^E b^F| = t$, $|h^E| = t - |b^F| \leq t - |x(l, q)|$, which is negative if $l \geq 1$. Therefore we have no element with bidegree (s, t), and

$$E_2^{s,t}(k, n) = 0.$$

Lastly turn to the case $n = k + 4$. We obtain

$$s = 2u(p-1) + 1$$

for $1 \leq u < (p^2 + p + 1)/2$ from Lemma 3.6 and degree reason as we have seen above. Consider b^F similarly to the case $n = k + 3$, and we have

$$\begin{aligned} |h^E| &= (2p^{k+4} - 2) - (2p^{k+3} - 2)(u + l) \\ &+ (2p^{k+2} - 2)(e + u + l) + (2p^2 - 2)(u + l) \\ &- (2p - 2)(e - l + q) \end{aligned} \tag{3.12}$$

which corresponds to (3.11). With a routine calculation it is easy to see that there is no h^E which satisfies (3.12). In this case with an assumption that $l \leq p$, if b^F is not of the form $x(l, q)$, then $|b^F| \geq |x(l, q)|$, and $|h^E| = t - |b^F| \leq t - |x(l, q)|$. This is again negative if $l = p$. Therefore we have no element with bidegree (s, t), and

$$E_2^{s,t}(k, n) = 0.$$

q.e.d.

THEOREM **3.13.** *Let the prime $p > 7$, $k \geq 2$ and $n \geq 0$ with $n \leq k + 3$. Suppose that $W_k(n)$ exists. Then there exists a non-trivial element $\xi_{(n+1)} \in \pi_* W_k(n)$ such that*

$$BP_* \xi_{(n+1)} = v_{n+1} \in BP_* W_k(n).$$

PROOF. Consider the Adams-Novikov spectral sequence $E_r^{s,t}(k, n+1)$ converging to $\pi_* W_k(n)$ (see (3.2)). Since it is known that

$$\eta_R v_{n+1} \equiv v_{n+1} \mod I_{n+1},$$

we see that

$$v_{n+1} \in E_2^{0,u}(k, n+1)$$

with $u = 2p^{n+1} - 2$, which is non-trivial. Apply now Lemma 3.7 to show that

$$d_s v_{n+1} = 0 \in E_2^{s,t}(k, n+1)$$

with $t - s = 2p^n - 3$ and $s \geq 2$. Since v_{n+1} is in the 0th line, nothing kills it. Therefore $v_{n+1} \in E_2^{0,u}(k, n+1)$ survives non-trivially to give $\xi_{(n+1)} \in \pi_* W_k(n)$. The equality $BP_* \xi_{(n+1)} = v_{n+1}$ follows from the edge homomorphism. q.e.d.

§4. Proof of Theorem A

In this section we begin with recalling that for each $n \geq 0$ there is associative commutative ring spectrum $P(n)$ with product $\bar{\mu}_n$ such that $\pi_* P(n) = BP_*/(p, v_1, \cdots, v_{n-1})$, and the canonical map $c: BP \to P(n)$ is a map of ring spectra. Let E be a ring spectrum with a map

$$u: E \longrightarrow P(n)$$

of ring spectra. We call E *good* if the map u induces the monomorphism

$$u_\# = (c \wedge u)_*: BP_*E = \pi_*(BP \wedge E) \longrightarrow \pi_*(P(n) \wedge P(n)) = P(n)_*P(n).$$

THEOREM **4.1.** *Let k and n be integers with $n \leq k+3$ and suppose that $W_k(n)$ is a good ring spectrum with structure maps $\mu_n: W_k(n) \wedge W_k(n) \to W_k(n)$ and $i_n: S \to W_k(n)$. Then there exists a map*

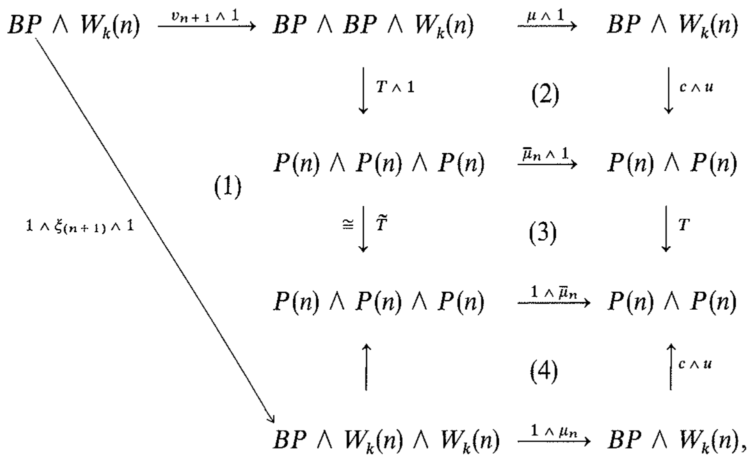
$$\xi_{n+1}: W_k(n) \longrightarrow W_k(n)$$

with $BP_*\xi_{n+1} = v_{n+1}$

PROOF. Let $i_*: \pi_* W_k(n) \to BP_* W_k(n)$ be the Hurewicz map, that is, it is induced by the unit map $i: S \to BP$. Then we have a map $\xi_{(n+1)}: S \to W_k(n)$ such that $i_*\xi_{(n+1)} = v_{n+1}$ by Theorem 3.13 in §3. Define the map ξ_{n+1} by the composition

$$W_k(n) \longrightarrow W_k(n) \wedge W_k(n) \longrightarrow W_k(n),$$

in which the first map is $\xi_{(n+1)} \wedge id$ and the second map is the product μ_n.
Then we have a commutative diagram



where 1 denotes the identity map, T interchanges the two factors and $\tilde{T} = (T \wedge 1)(1 \wedge T)(T \wedge 1)$.

Commutativity of the squares (2), (3), (4) is verified by the properties of products μ, μ_n and $\bar{\mu}_n$.

Commutativity of the squares (1) follows from the equality $i_*\xi_{(n)} = v_n = (1 \wedge i_n)v_n$. Therefore

$$\begin{aligned}(c \wedge u)_*BP_*(\xi_{n+1})(x) &= (c \wedge u)(1 \wedge \mu_n)(1 \wedge \xi_{(n+1)} \wedge 1)(x) \\ &= (c \wedge u)(\mu \wedge 1)(v_{n+1} \wedge 1)(x)\end{aligned}$$

$$= (c \wedge u)_* v_{n+1}(x).$$

Since $W_k(n)$ is good, $(c \wedge u)_*$ is a monomorphism and we have the desired equality $BP_*(\xi_{n+1})(x) = v_{n+1}(x)$.

PROOF OF THEOREM A. Let $W_k(n+1)$ be the cofiber of ξ_{n+1} in Theorem 4.1, and we can easily verified that it has the desired property. q.e.d.

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