The Chromatic E_1-Term $H^0M_n^2$ for $n \geq 2$

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Introduction

The E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups π_*X of a connected spectrum X is the group $H^*BP_*X = \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*X)$ for the derived functor $\mathrm{Ext}^*_{BP_*BP}(BP_*, \ )$ of the functor $\mathrm{Hom}_{BP_*BP}(BP_*, \ )$ where the pair of algebras $(BP_*, BP_*BP) = (\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots], BP_*[t_1, t_2, \cdots])$ is the Hopf algebroid arosen from the Brown-Peterson ring spectrum BP at a prime p. The E_2-term has some information on the abutting groups π_*X. In [1], Miller, Ravenel, and Wilson introduced the chromatic spectral sequence converging to the Adams-Novikov E_2-term for the sphere spectrum S and for the Toda-Smith spectrum $V(n)$, determined the second line of the Adams-Novikov E_2-term for an odd prime p, and gave some relations in the stable homotopy groups π_*S of the sphere spectrum. The E_1-term of the chromatic spectral sequence is the Ext group $H^*M_n^s$ for the BP_*BP-comodule M_n^s defined inductively by $N_n^0 = BP_*/(p, v_1, \cdots, v_{n-1})$, $M_n^s = v_{n+s}^{-1}N_n^s$ and $N_n^{s+1} = M_n^s/N_n^s$ as a comodule. The E_1-term $H^0M_n^s$ is determined for $s = 0, 1$ and any prime p and for $s = 2$, $n = 0$ and an odd prime p in [4] and [1], and for $s = 2$, $n = 0$ and the prime 2 in [7]. This E_1-term is also an E_2-term of the Adams-Novikov spectral sequence for some corresponding spectrum constructed by Ravenel [5]. That is to say, the computation of the E_1-term may contain some information about the localized stable homotopy. In this paper we determine $H^0M_n^2$ for $n \geq 2$ and an odd prime p (see Theorem 1.2). Now the undetermined cases for the E_1-term $H^0M_n^s$ are the following: 1) $s \geq 3$, 2) $s = 2$ for $n = 1$, and 3) $s = 2$ for the prime 2.

§1. Statement of results

Let (A, Γ) denote the Hopf algebroid obtained from the Brwon-Peterson ring spectrum BP at a prime p. That is, $A = BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$ and $\Gamma = BP_*BP = BP_*[t_1, t_2, \cdots]$, with degree $|v_i| = |t_i| = 2p^i - 2$. Then A is a Γ-comodule with the structure map η which is the right unit of the Hopf algebroid. Associating to A, we have families of comodules N_n^i and M_n^i defined by

$$N_n^0 = BP_*/(p, v_1, \cdots, v_{n-1}),\ M_n^i = v_{n+i}^{-1}N_n^i, \text{ and } N_n^{i+1} = M_n^i/N_n^i.$$

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By definition, every element x of M_n^i is a linear combination of elements of the form $y/v_n^{e_n}\cdots v_{n+i-1}^{e_{n+i-1}}$ for $y\in v_{n+i}^{-1}BP_*/(p,\cdots,v_{n-1},v_n^\infty,\dots,v_{n+i-1}^\infty)$ and $e_j>0$ $(n\le j\le n+i-1)$ with relation that $y/v_n^{e_n}\cdots v_{n+i-1}^{e_{n+i-1}}=0$ if y is a multiple of one of $v_j^{e_j}$'s in the denominator. We also denote η for the structure maps of these comodules.

Let M be one of these comodules. Then Ext group H^0M is defined to be:

$$H^0M=\mathrm{Ker}\{d=\eta-id\colon M\longrightarrow M\otimes_A\Gamma\}.$$

In order to give generators of $H^0M_{n-2}^2$, we define elements $x(i/j;n;k)$ of $v_n^{-1}BP_*/(v_{n-1}^\infty)$ for $n\ge 4$. Here $BP_*/(v_{n-1}^\infty)$ denotes the cokernel of the canonical inclusion $BP_*\to v_{n-1}^{-1}BP_*$, and every element x of $BP_*/(v_{n-1}^\infty)$ has a form y/v_{n-1}^k for $k>0$ and $y\in BP_*$ with a convention that $y/v_{n-1}^k=0$ if $k\le 0$ or y is a multiple of v_{n-1}^k. We first recall [1, p. 494] the elements $x_{n,i}$ of $v_n^{-1}BP_*$ for $n>2$:

$$\begin{aligned}
x_{n,0}&=v_n,\\
x_{n,1}&=v_n^p-v_{n-1}^pv_n^{-1}v_{n+1},\\
x_{n,i}&=x_{n,i-1}^p && \text{for } 1<i\not\equiv 1\ (n-1),\\
x_{n,i}&=x_{n,i-1}^p-v_{n-1}^{b_{n,i}}v_n^{p^i-p^{i-1}+1} && \text{for } 1<i\equiv 1\ (n-1),
\end{aligned}$$

where for $i\equiv 1$ $(n-1)$,

$$b_{n,i}=(p^{i-1}-1)(p^n-1)/(p^{n-1}-1).$$

We also recall the integers:

$$\begin{aligned}
a_{n,0}&=1,\\
a_{n,1}&=p,\\
a_{n,i}&=pa_{n,i-1} && \text{for } 1<i\not\equiv 1\ (n-1),\\
a_{n,i}&=pa_{n,i-1}+p-1 && \text{for } 1<i\equiv 1\ (n-1).
\end{aligned}$$

In [9, (4.2)], we introduced similar elements X_i of $v_n^{-1}BP_*$ such that $X_i\equiv x_{n,i}$ mod $(v_{n-1}^{1+a_{n,i}})$. The definition of X_i is slightly more complicated than that of $x_{n,i}$, and we will only take the necessary information from [9]. From here on $x_{n,i}$ sometimes denotes the element X_i but there will not be any confusion in the sequel. We further use the elements $u_{n,r}$ of $v_n^{-1}BP_*$ for $n\ge 2$ and $r\ge 0$ (*cf.* [9, (2.8)]) defined by

$$u_{n,0}=v_n^{-1},\ \text{and}\ \textstyle\sum_{i+j=r}v_{n+i}u_{n,j}^{p^i}=0\ \text{for}\ r\ge 1.$$

Let i, j, l, r and s be integers such that

$$i=sp^r\ \text{with}\ p\nmid s,\ \text{and}\ 1\le j\le a_{n,r}\ \text{with}\ p\nmid j,\qquad (1.1)$$
$$l,r\ge 0,\ r\equiv l\ (n-1)\ \text{with}\ l<n-1.$$

Now we define $x(i/j;n;k)$ for $k\ge 0$:

i) For $r=0$, that is, $i=s$ and $j=1$,

a) in the case that $i\not\equiv 1\,(p)$,

$$x(i/1;n;0)=x_{n,0}^i/v_{n-1},$$
$$x(i/1;n;1)=x(i/1;n;0)^p+v_{n-2}^p v_n x_{n,1}^i/v_{n-1}^{2p+1},$$
$$x(i/1;n;k)=x(i/1;n;k-1)^p \quad \text{for } 1<k\not\equiv 1\ (n-1), \text{ and}$$
$$x(i/1;n;k)=x(i/1;n;k-1)^p-(i-1)v_{n-2}^{p^k}v_n^{ip^{k+1}}/v_{n-1}^{2p^k+p^{k-1}-pa_{n,k-1}+1} \quad \text{for } 1<k\equiv 1\ (n-1).$$

b) in the case that $i\equiv 1\ (p)$,

$$x(i/1;n;0)=x_{n,0}^i/v_{n-1}$$

and

$$x(i/1;n;k)=x(i/1;n;k-1)^p+(-1)^{k-1}v_{n-2}^{P(k,k)}v_{n-1}^{Q(k+1,k+1)}u_{n-1,k+1}v_n^{ip^k-p^k}$$

for $k\geq 1$, where

$$P(n,r)=p^n+p^{n-1}+\cdots+p^{n-r+1} \quad \text{and}$$
$$Q(n,r)=p^n-p^{n-1}-\cdots-p^{n-r+1}.$$

ii) For the case that $r>0$, $s\not\equiv -1\ (p)$ or $s\equiv -1\ (p^2)$, or that $j=a_{n,r}$,

$$x(i/j;n;0)=x_{n,r}^s/v_{n-1}^j,$$
$$x(i/j;n;1)=x(i/j;n;0)^p+jv_{n-2}^p v_n x_{n,r}^{sp}/v_{n-1}^{jp+p+1}+v(i,j),$$
$$x(i/j;n;k)=x(i/j;n;k-1)^p+w(i,j;k) \qquad \text{for} \quad k\geq 2,$$

where

$$v(i,j)=-sv_{n-2}^p u_{n-1,2}x_{n,r-1}^{sp^2-p}/v_{n-1}^{jp-pa_{n,r}}$$
$$\text{for } l=1 \text{ and } a_{n,r}-p+2\leq j<a_{n,r},$$
$$v(i,j)=(a_{n,r}-1)sv_{n-2}^p v_n^{sp^{r+1}-p^{r+2}}/2v_{n-1}^2$$
$$\text{for } l=0 \text{ and } j=a_{n,r}-1, \text{ and}$$
$$v(i,j)=0 \qquad \text{otherwise; and}$$
$$w(i,j;k)=jsv_{n-2}^{p^k-1}v_{n-1}v_n^{sp^{r+k}-p^{r+k-1}+p^{k-1}}/v_{n-1}^{p^{k-1}}$$
$$\text{for } 2\leq k\leq n-2,\ l=n-1-k \text{ and } j=a_{n,r}-1,$$
$$w(i,j;k)=-sv_{n-2}^{p^2}v_n^{sp^{r+2}-p^{r+1}+1}x_{n,1}/v_{n-1}^{p+1}$$
$$\text{for } k=2,\ l=n-2 \text{ and } j=a_{n,r}-1,$$

$$w(i, j; k) = jv_{n-2}^{a_{n-1,k}-p} v_{n-1} x_{n,r}^{sp^k} / v_{n-1}^{jp^k + p^{k-1}}$$

for $n - 1 \leq k \equiv 1\ (n-2)$ and $j \leq a_{n,r} - 1$,

$$w(i, j; k) = -v_{n-2}^{p^k} v_n x_{n,r}^{sp^k} / v_{n-1}^{p^k a_{n,r} + p^k + p^{k-1} - a_{n,k} + p}$$

for $n \leq k \equiv 1\ (n-1)$ and $j = a_{n,r}$, and

$w(i, j; k) = 0$ otherwise.

iii) For $s = s'p - 1$ $(p \nmid s')$, $j < a_{n,r}$ and $0 \leq l \leq n-3$, using $h = (l+1)(n-2)+1$, $x(i/j; n; k)$ is same as that in (ii) for $k < h$.

$$x(i/j; n; h) = x(i/j; n; h-1)^p + jv_{n-2}^{a_{n-1,h}-p} v_{n-1} x_{n,r}^{sp^h} / v_{n-1}^{jp^h + p^{h-1}}$$
$$+ jv_{n-2}^{a_{n-1,h}} x_{n,r+h+1}^{s'} / v_{n-1}^{c(0) - p^{n-1} - 1},$$

$x(i/j; n; h+k) = (i/j; n; h+k-1)^p$ for $1 < k \not\equiv 1\ (n-1)$, and

$$x(i/j; n; h+k) = (i/j; n; h+k-1)^p - jv_{n-2}^{a_{n-1,h}p^k + p^k} v_n x_{n,r+h+1}^{s'p^k} / v_{n-1}^{c(k) + 1 - pa_{n,n+k-2}}$$

for $1 \leq k \equiv 1\ (n-1)$, where

$$c(k) = p^k a_{n,r+h+1} + jp^{k+h} + p^{k+n-1} + p^k + p^{k+n-1}.$$

iv) For $s = s'p - 1$ $(p \nmid s')$, $j < a_{n,r}$ and $l = n-2$, putting $j = ep - f$ with $0 < f < p - 1$,

$$x(i/j; n; 0) = v_{n-1}^f x_{n,r}^s / x_{n-1,1}^e - fv_{n-2} v_{n-1}^{f-1} x_{n,r+1}^{s'} / s' x_{n-1,1}^{e'}$$

$(-sv_2^p u_{3,2} v_4^{sp^{r+1} - p^r} / v_3^{j - a_{4,r}}$ if $n = 4)$ for $e' = e + a_{n,r}$,

$x(i/j; n; k) = x(i/j; n; k-1)^p + z(i, j; k)$ for $k \geq 1$,

where

$$z(i, j; k) = j(j+1) v_{n-2}^{2p^k} v_n x_{n,r+1}^{s'p^k} / s' v_{n-1}^{c'(k) + 1 - pa_{n,n+k-2}}$$

for $f \neq 1$ and $k \equiv 1\ (n-1)$,

$$z(i, j; k) = (e+1) v_{n-2}^{pa_{n-1,k}-1} v_{n-1} x_{n,r}^{sp^k} / v_{n-1}^{jp^k + p^k}$$

for $f = 1$, $e \not\equiv -1\ (p)$ and $(n-2)^2 > k \equiv 0\ (n-2)$,

$$z(i, j; k) = (e+1) v_{n-2}^{pa_{n-1,h'}-1} v_{n-1} x_{n,r}^{sp^{h'}} / v_{n-1}^{jp^{h'} + p^{h'}}$$
$$+ (e+1) v_{n-2}^{a_{n-1,h'+1}} x_{n,r+h'+1}^{s'} / s' v_{n-1}^{jp^{h'} + p^{h'} + a_{n,r+h'+1}}$$

for $f = 1$, $e \not\equiv -1\ (p)$ and $k = h' = (n-2)^2$,

$$z(i, j; k) = -(e+1) v_{n-2}^{p^{k-h'} a_{n-1,h'+1} + p^{k-h'}} v_n x_{n,r+h'+1}^{s'p^{k-h'}} / v_{n-1}^{c''(k) + 1 - pa_{n,k-h'+n-2}}$$

for $f = 1$, $e \not\equiv -1\ (p)$ and $h' = (n-2)^2 < k \equiv 1\ (n-1)$,

$z(i, j; k) = v_{n-2}^{p^{k+1}+p^k} v_n x_{n,r+1}^{s'p^k}/s' v_{n-1}^{c'(k)+1-pa_{n,k+n-2}}$

for $f = 1$, $e \equiv -1\ (p)$ and $k \equiv 1\ (n-1)$, and

$z(i, j; k) = 0$ otherwise; and

$$c'(k) = jp^k + 2p^k + a_{n,r+1}p^k + p^{k+n-1}$$

and

$$c''(k) = jp^k + p^k + a_{n,r+h'+1}p^{k-h'} + p^{k-h'} + p^{k-h'+n-1}.$$

Consider integers $a = sp^{r+k} = ip^k$, $b = jp^k$, c, $h = (l+1)(n-2)+1$ and $h' = (n-2)^2$ such that b and c are positive, $p \nmid s$, $p \nmid j$, $r \equiv l\ (n-1)$ with $0 \le l \le n-2$, $c \le c_n(a, b)$ and $b \le a_{n,r+k}$ if $i \neq 0$, where the integers $c_n(a, b)$ is given by:

$c_n(a, b) = p^k$	either for $r = 0$ and $i \not\equiv 1\ (p)$ or for $j = a_{n,r}$,
$c_n(a, b) = P(k, k+1)$	for $r = 0$, $i \equiv 1\ (p)$ and $0 \le k \le n-2$,
$c_n(a, b) = P(k, n-1)$	for $r = 0$, $i \equiv 1\ (p)$ and $k \ge n-1$,
$c_n(a, b) = a_{n-1,k}$	either for $i = 0$, for $j < a_{n,r}$ and either $s \not\equiv -1\ (p)$ or $s \equiv -1\ (p^2)$, or for $j < a_{n,r}$, $k < h$ and $s = s'p - 1$ with $p \nmid s'$,
$c_n(a, b) = p^k a_{n-1,h} + p^k$	for $j < a_{n,r}$, $s = s'p - 1$ with $p \nmid s'$, and $k \ge h$ with $l \le n-3$,
$c_n(a, b) = 2p^k$	for $l = n-2$ and $j \not\equiv -1\ (p)$,
$c_n(a, b) = a_{n-1,k+1}$	for $l = n-2$, $j = ep - 1$ with $p \nmid (e+1)$, and $0 \le k \le (n-2)^2$,
$c_n(a, b) = p^{k-h'}a_{n-1,h'+1} + p^{k-h'}$	for $l = n-2$, $j = ep - 1$ with $p \nmid (e+1)$, and $k \ge (n-2)^2$,
$c_n(a, b) = p^{k+1} + p^k$	for $l = n-2$ and $j = ep - 1$ with $p \mid (e+1)$.

We define the element $x(a/b, c)$ of M_{n-2}^2 for the above integers by:

$$x_n(a/b, c) = x(i/j; n; k)/v_{n-2}^c.$$

Note that if $x/v_{n-2}^c \in H^0M_{n-2}^2$ for $x \in v_n^{-1}BP_*/(v_{n-1}^\infty)$ with $x \not\equiv 0 \bmod (v_{n-2})$, then x/v_{n-2}^c generates a cyclic $\boldsymbol{Z}/p[v_{n-2}]$-module isomorphic to $\boldsymbol{Z}/p[v_{n-2}]/(v_{n-2}^c)$. Then we have

THEOREM 1.2. *As a $\boldsymbol{Z}/(p)[v_{n-2}]$-module, $H^0M_{n-2}^2$ for $n \ge 4$ is the direct sum of the cyclic submodules generated by $x_n(a/b, c_n(a, b))$ for $a \in \boldsymbol{Z}$ and $b > 0$, and $b \le a_{n,v(a)}$ if $a \neq 0$. Here $v(a)$ is the maximal exponent of p that devides a.*

§ 2. Computation of d

In order to study the differential $d = \eta - id\colon M \to M \otimes_A \Gamma$ for a comodule M associated to the comodule (BP_*, η), we consider the map $d' = proj \otimes 1(\eta - id)\colon L \to (L/J) \otimes_A \Gamma$ for some comodule L related to M, an ideal J of BP_* and the canonical map $proj\colon L \to L/J$. In this situation we write

$$dx \equiv y \bmod J$$

for $x \in L$ and $y \in L \otimes_A \Gamma$ such that $d'x = y$. In practice, for the elements $x_{n,i}$ of $L = v_n^{-1}BP_*$ given in § 1, if $dx_{n,i} \equiv v_{n-1}^a y \bmod (p, v_1, \cdots, v_{n-2}, v_{n-1}^{a+1})$, then we have the equality

$$dx_{n,i}/v_{n-1}^{a+1} = y/v_{n-1}$$

in $M = M_{n-1}^1$. The explicit formulae are given as follows:

(2.1)([1, Prop. 5.17]) *For integers n, i and j such that $n > 2$, $j \equiv i - 1(n-1)$ and $1 \le j < n-1$, $dx_{n,0} \equiv v_{n-1}t_1^{p^{n-1}}$ mod $(p, v_1, \cdots, v_{n-2}, v_{n-1}^2)$ and*

$$dx_{n,i} \equiv v_{n-1}^{a_{n,i}} v_n^{p^i - p^{i-1}} t_1^{p^j}$$

mod $(p, v_1, \cdots, v_{n-2}, v_{n-1}^{1+a_{n,i}})$.

As we have said above, the element $x_{n,i}$ sometimes means the element X_i given in [9] and so in some cases it also satisfies:

(2.2)([9, Prop. 4.1]) *For integers n, i and j such that $n > 2$, $j \equiv i - 1\ (n-1)$ and $1 \le j < n-1$, $dx_{n,0} \equiv v_{n-1}t_1^{p^{n-1}}$ mod $(p, v_1, \cdots, v_{n-2}, v_{n-1}^2)$ and*

$$dx_{n,i} \equiv v_{n-1}^{a_{n,i}} v_n^{p^i - p^{i-1}} t_1^{p^j}$$

mod $(p, v_1, \cdots, v_{n-2}, v_{n-1}^{pa_{n,i-1} + a_{n,i-1}})$.

We have similar results for the elements $x(i/j; n; k)$ defined in § 1 by setting $M = M_{n-2}^2$ and $L = v_n^{-1}BP_*/(v_{n-1}^\infty)$. To state this we recall [9, (2.10)] the elements $w_{n,r}$ of $v_n^{-1}BP_*$ for $n > 1$ defined by:

(2.3) $$w_{n,0} = 0 \text{ and } w_{n,r} = \sum_{j=1}^r e_n(u_{n,r-j}^{p^{j-1}})T_j^{p^{n-2}} \quad \text{for } r > 0,$$

where $e_n(x)$ is an element of $\boldsymbol{Z}/p[v_n, v_n^{-1}, v_{n+1}, \cdots; t_1, t_2, \cdots]$ such that $\eta x \equiv e_n(x)$ mod $(p, v_1, \cdots, v_{n-1})$ and T_i is an element such that $T_i \equiv t_i^p$ mod $(p, v_1, \cdots, v_{i-1})$(for explicit definition, see [9, above (2.5)]). The algebra $L = v_n^{-1}BP_*/(v_{n-1}^\infty)$ has an ideal

$$J(k) = (p, v_1, \cdots, v_{n-3}, v_{n-2}^k).$$

Recall above the integers i, j, l, r and s which satisfy (1.1). We now have

LEMMA 2.4. 1) *Let $r = 0$, that is, $j = 1$.*

(a) *For the case that $i \not\equiv 1\ (p)$,* mod $J(1 + p^k)$,

$$dx(i/1;n;0) \equiv (i-1)v_{n-2}v_n^i t_1^{p^{n-2}}/v_{n-1}^2 + iv_{n-2}v_{n-1}^{p-1}v_n^{i-1}w_{n-1,2}$$

$$dx(i/1;n;1) \equiv (i-1)v_{n-2}^p v_n^{ip} t_1/v_{n-1}^{p+1} - \binom{i}{2} v_{n-2}^p v_n^{ip-p-1}(2v_{n+1}t_1 - v_n^p t_1^2)/v_{n-1}$$

and

$$dx(i/1;n;k) \equiv (i-1)v_{n-2}^{p^k} v_n^{ip^k} t_1^{p^\varepsilon}/v_{n-1}^{2p^k+p^{k-1}-a_{n,k}} + v_{n-2}^{p^k} z_k/v_{n-1}^{p^{k-1}}$$

for $k > 1$, *and some* $z_k \in BP_*BP$, *where* $\varepsilon \equiv k-1\ (n-1)$ *with* $0 \le \varepsilon \le n-2$.

(b) *For the case that* $i \equiv 1\ (p)$, mod $J(1 + P(k, k+1))$,

$$dx(i/1;n;k) \equiv (-1)^k v_{n-2}^{P(k,k+1)} v_{n-1}^{Q(k+1,k+2)} v_n^{ip^k - p^k} w_{n-1,k+2}$$

for $k \ge 0$.

2) *Let* $j = a_{n,r}$ *with* $r \ge n$, *then* mod $J(1 + p^k)$,

$$dx(i/j;n;0) \equiv v_{n-2}x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{a_{n,r}+1} - sv_{n-2}x_{n,r-1}^{sp-1}(w_{n-1,2} - t_1^{1+p^{n-2}}/v_{n-1})$$

and

$$dx(i/j;n;k) \equiv v_{n-2}^{p^k} x_{n,r}^{sp^k} t_1^{p^\varepsilon}/v_{n-1}^{p^k a_{n,r}+p^k+p^{k-1}-a_{n,k}} + v_{n-2}^{p^k} z_k/v_{n-1}^{p^{k+1}+p^k}$$

for $k > 0$, *where* z_k *is an element of* BP_*BP, ε *is an integer such that* $\varepsilon \equiv k-1\ (n-1)$ *with* $0 \le \varepsilon \le n-2$.

3) *Let* $r > 0$ *and either* $s \not\equiv -1\ (p)$ *or* $s \equiv -1\ (p^2)$.

(a) *For the case that* $0 < j \le a_{n,r} - 2$,

$$dx(i/j;n;0) \equiv -jv_{n-2}x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{j+1} - \tilde{s}v_{n-2}v_{n-1}^{a_{n,r}-j}x_{n,r-1}^{sp-1}w_{n-1,2}$$

mod $J(2)$ *and*

$$dx(i/j;n;k) \equiv -jv_{n-2}^{a_{n-1,k}} x_{n,r}^{sp^k} t_1^{p^{\varepsilon'}}/v_{n-1}^{jp^k+p^{k-1}} \bmod J(e + a_{n-1,k})$$

for $k > 0$.

(b) *For* $j = a_{n,k} - 1$,

$$dx(i/j;n;0) \equiv \tilde{j}v_{n-2}x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{a_{n,r}} - \tilde{s}v_{n-2}v_{n-1}x_{n,r-1}^{sp-1}w_{n-1,2}$$

mod $J(2)$,

$$dx(i/j;n;k) \equiv \tilde{j}v_{n-2}^{p^k} x_{n,r}^{sp^k} t_1^{p^{k-1}}/v_{n-1}^{jp^k+p^{k-1}} - \tilde{j}sv_{n-2}^{p^k} v_n^{sp^{r+k}-p^{r+k-1}+p^{k-1}} t_1^{p^{k+l-1}}/v_{n-1}^{p^{k-1}}$$

mod $J(1 + p^k)$ *for* $1 \le k \le n - l - 2$ *and* $l \ne 0$, *or for* $k = 1$ *and* $l = n - 2$,

$$dx(i/j;n;k) \equiv \tilde{j}v_{n-2}^{a_{n-1,k}} x_{n,r}^{sp^k} t_1^{p^{\varepsilon'}}/v_{n-1}^{jp^k+p^{k-1}}$$

mod $J(e + a_{n-1,k})$ *for* $k \ge n - l - 1$ *with* $l \ne 0$, $n-2$, $k \ge 1$ *with* $l = 0$ *and* $k \ge 2$ *with* $l = n - 2$.

Here ε' *is an integer such that* $\varepsilon' \equiv k-1\,(n-2)$ *with* $0 \le \varepsilon' \le n-3$, *and*

$$\tilde{s} = \begin{cases} 0 & (l \neq 1) \\ s & (l = 1), \end{cases} \quad \tilde{j} = \begin{cases} 1 & (l \neq 1 \text{ or } r < n) \\ 2 & (l = 1 \text{ and } r \ge n), \end{cases}$$

$$\text{and } e = \begin{cases} 1 & (k < n-1) \\ 2 & (k \ge n-1). \end{cases}$$

4) *Let* $j < a_{n,r}$, $s \equiv -1\,(p)$ *and* $s \not\equiv -1\,(p^2)$, *that is,* $s = s'p - 1$ *with* $p \not| s'$.

(a) *For* $0 \le l \le n-3$, $dx(i/j; n; k)$ *is same as that in* 2) *if* $k < h = (l+1)(n-2) + 1$, *and*

$$dx(i/j; n; h+k) \equiv jv_{n-2}^{p^k a_{n-1,h} + p^k} x_{n,r+h+1}^{s'p^k} t_1^{p^\varepsilon} / v_{n-1}^{c(k) - a_{n,n+k-1}} + v_{n-2}^{p^k a_{n-1,h} + p^k} z_k / v_{n-1}^{jp^{h+k} + p^{h+k-1} + p^{k+1}}$$

mod $J(1 + p^k + p^k a_{n-1,h})$ *for* $k \ge h$, *where* $k \ge 0$, $z_k \in BP_*BP$, $\varepsilon \equiv k-1\,(n-1)$ *with* $0 \le \varepsilon \le n-2$, *and* $c(k) = p^k a_{n,r+h+1} + jp^{k+h} + p^{k+h-1} + p^k + p^{k+n-1}$.

(b) *For* $l = n-2$ *and* $j \not\equiv -1\,(p)$, mod $J(1 + 2p^k)$,

$$dx(i/j; n; k) \equiv -j(j+1) v_{n-2}^{2p^k} x_{n,r+1}^{s'p^k} t_1^{p^\varepsilon} / s' v_{n-1}^{c'(k) - a_{n,n+k-1}} + v_{n-2}^{2p^k} z_k / v_{n-1}^{jp^k + 2p^k},$$

where $z_k \in BP_*BP$, $\varepsilon \equiv k-1\,(n-1)$ *with* $0 \le \varepsilon \le n-2$, $h' = (n-2)^2$ *and* $c'(k) = jp^k + 2p^k + p^k a_{n,r+1} + p^{k+n-1}$.

(c) *For* $l = n-2$ *and* $j = ep - 1$ *with* $e \not\equiv -1\,(p)$,

$$dx(i/j; n; k) \equiv -(1+e) v_{n-2}^{a_{n-1,k+1}} x_{n,r}^{sp^k} t_1^{\varepsilon'} / v_{n-1}^{jp^k + p^k}$$

mod $J(2 + a_{n-1,k+1})$ *for* $0 \le k < h' = (n-2)^2$ *and*

$$dx(i/j; n; k) \equiv (1+e) v_{n-2}^{p^{k-h'} a_{n-1,h'+1} + p^{k-h'}} x_{n,r+h'+1}^{s'p^{k-h'}} t_1^{p^\varepsilon} / v_{n-1}^{c''(k) - a_{n,k-h'+n-1}} + v_{n-2}^{p^{k-h'} a_{n-1,h'+1} + p^{k-h'}} z_k / v_{n-1}^{jp^k + p^k + p^{k-h'}}$$

mod $J(1 + p^{k-h'} + p^{k-h'} a_{n-1,h'+1})$ *for* $k \ge h'$, *where* $\varepsilon' \equiv k\,(n-2)$ *with* $0 \le \varepsilon' \le n-3$, $\varepsilon \equiv k-1\,(n-1)$ *with* $0 \le \varepsilon \le n-2$, $h' = (n-2)^2$ *and* $c''(k) = jp^k + p^k + p^{k-h'} a_{n,r+h'+1} + p^{k-h'} + p^{k-h'+n-1}$.

(d) *For* $l = n-2$ *and* $j = ep - 1$ *with* $e \equiv -1\,(p)$,

$$dx(i/j; n; k) \equiv -v_{n-2}^{p^{k+1} + p^k} x_{n,r+1}^{s'p^k} t_1^{p^{\varepsilon''}} / s' v_{n-1}^{c'(k) - a_{n,n+k-1}} + v_{n-2}^{p^{k+1} + p^k} z_k / v_{n-1}^{jp^k + 2p^k},$$

for $k \ge 0$, *where* $\varepsilon'' \equiv k\,(n-1)$ *with* $0 \le \varepsilon'' \le n-2$ *and* $c'(k)$ *is same as that of* 4)(b).

PROOF. 1) We have the formula (*cf.* [3, (12)])

$$\eta v_n \equiv v_n + v_{n-1} t_1^{p^{n-1}} - v_{n-1}^p t_1 + v_{n-2} t_2^{p^{n-2}} - v_{n-2}^p t_1^{p^{n-1}+1} \tag{2.5}$$

mod $J(p^2)$ for $n \ge 4$, and so

$$\eta v_n^i \equiv v_n^i + i v_{n-1} v_n^{i-1} t_1^{p^{n-1}} + i v_{n-2} v_n t_2^{p^{n-2}} + i(i-1) v_{n-2} v_{n-1} v_n^{i-2} t_1^{p^{n-1}} t_2^{p^{n-2}}$$

mod $J(2) + (v_{n-1}^2)$. We also see that

(2.6) $$(dxy) = (dx)y + (\eta x)dy$$

by the definition of the differential d. These give the equality

(2.7) $$dx(i/j;\, n;\, 0) \equiv -\, v_{n-2}v_n^i t_1^{p^{n-2}}/v_{n-1}^2 + iv_{n-2}v_n^{i-1}(t_2^{p^{n-2}} - t_1^{p^{n-1}+p^{n-2}})/v_{n-1}$$

mod $J(2)$. Noticing that

$$w_{n-1,2} = -\, v_{n-1}^{-p-1}v_n t_1^{p^{n-2}} + v_{n-1}^{-1}t_1^{p^{n-2}+1} + v_{n-1}^{-p}(t_2^{p^{n-2}} - t_1^{p^{n-1}+p^{n-2}}),$$

we have

(2.8) $$dx(i/1;\, n;\, 0) \equiv (i-1)v_{n-2}v_n^i t_1^{p^{n-2}}/v_{n-1}^2 + iv_{n-2}v_{n-1}^{p-1}v_n^{i-1}w_{n-1,2}$$

mod $J(2)$, which is the equality for the first case.

We generally know that

(2.9) *If* $dx \equiv y$ mod (p, a), *then* $dx^p \equiv y^p$ mod (p, a^p).

By definition of the element $x_{n,1}$,

$$\begin{aligned} v_{n-2}^p v_n x_{n,1}^i/v_{n-1}^{2p+1} \equiv{}& v_{n-2}^p v_n^{ip+1}/v_{n-1}^{2p+1} - iv_{n-2}^p v_n^{ip-p}v_{n+1}/v_{n-1}^{p+1} \\ &+ \binom{i}{2} v_{n-2}^p v_n^{ip-2p-1}v_{n+1}^2/v_{n-1} \text{ mod } J(p+1). \end{aligned}$$

Using (2.5) and (2.6) we compute

$$\begin{aligned} dv_{n-2}^p v_n^{ip+1}/v_{n-1}^{2p+1} \equiv{}& v_{n-2}^p(iv_n^{ip-p+1}t_1^{p^n}/v_{n-1}^{p+1} + \binom{i}{2} v_n^{ip-2p+1}t_1^{2p^n}/v_{n-1} \\ &+ v_n^{ip}t_1^{p^{n-1}}/v_{n-1}^{2p} + iv_n^{ip-p}t_1^{p^n+p^{n-1}}/v_{n-1}^p \\ &- v_n^{ip}t_1/v_{n-1}^{p+1} - iv_n^{ip-p}t_1^{p^n+1}/v_{n-1}), \end{aligned}$$

$$\begin{aligned} -\, div_{n-2}^p v_n^{ip-p}v_{n+1}/v_{n-1}^{p+1} \equiv{}& -\, iv_{n-2}^p(v_n^{ip-p+1}t_1^{p^n}/v_{n-1}^{p+1} - v_n^{ip}t_1/v_{n-1}^{p+1} \\ &+ v_n^{ip-p}t_2^{p^{n-1}}/v_{n-1}^p - v_n^{ip-p}t_1^{p^n+1}/v_{n-1} \\ &+ (i-1)v_n^{ip-2p}v_{n+1}t_1^{p^n}/v_{n-1} \\ &+ (i-1)v_n^{ip-2p+1}t_1^{2p^n}/v_{n-1} \\ &- (i-1)v_n^{ip-p}t_1^{p^n+1}/v_{n-1}), \end{aligned}$$

and

$$\begin{aligned} d\binom{i}{2} v_{n-2}^p v_n^{ip-2p-1}v_{n+1}^2/v_{n-1} \equiv{}& \binom{i}{2} v_{n-2}^p v_n^{ip-2p-1}(v_n^2 t_1^{2p^n} + v_n^{2p}t_1^2 \\ &+ 2v_n v_{n+1}t_1^{p^n} - 2v_n^p v_{n+1}t_1 - 2v_n^{p+1}t_1^{p^n+1})/v_{n-1} \end{aligned}$$

mod $J(p+1)$. Collecting terms shows

$$dv_{n-2}^p v_n x_{n,1}^i / v_{n-1}^{2p+1} \equiv v_{n-2}^p (v_n^{ip} t_1^{p^{n-1}} / v_{n-1}^{2p} + (i-1) v_n^{ip} t_1 / v_{n-1}^{p+1}$$
$$+ i v_n^{ip-p} (t_1^{p^n + p^{n-1}} - t_2^{p^{n-1}}) / v_{n-1}^p$$
$$+ \binom{i}{2} (v_n^{ip-1} t_1^2 - 2 v_n^{ip-p-1} v_{n+1} t_1) / v_{n-1})$$

mod $J(p+1)$. By (2.7) and (2.9),

$$dx(i/j;\, n;\, 0)^p \equiv -v_{n-2}^p v_n^{ip} t_n^{p^{n-1}} / v_{n-1}^{2p} - i v_{n-2}^p v_n^{ip-p} (t_1^{p^n + p^{n-1}} - t_2^{p^{n-1}}) / v_{n-1}^p$$

mod $J(p+1)$. Therefore we have

$$dx(i/1;\, n;\, 1) \equiv (i-1) v_{n-2}^p v_n^{ip} t_1 / v_{n-1}^{p+1}$$
$$+ \binom{i}{2} v_{n-2}^p (v_n^{ip-1} t_1^2 - 2 v_n^{ip-p-1} v_{n+1} t_1) / v_{n-1}$$

mod $J(p+1)$, and $dx(i/j;\, n;\, k)$ for $k \leq n-1$ by (2.9). The definition of the elements $x(i/j;\, n;\, k)$ and (2.9) show that if the result for $dx(i/j;\, n;\, k)$ with $k \not\equiv 0\,(n-1)$ is valid, then so is the case for $dx(i/j;\, n;\, k+1)$. Now suppose that the case for $k \equiv 0\,(n-1)$ is valid, then (2.9) shows

$$dx(i/1;\, n;\, k)^p \equiv (i-1) v_{n-2}^{p^{k+1}} v_n^{ip^{k+1}} t_1^{p^{n-1}} / v_{n-1}^{2p^{k+1} + p^k - pa_{n,k}} + v_{n-2}^{p^{k+1}} z_k^p / v_{n-1}^{p^k}$$

mod $J(p^{k+1}+p)$. Notice that $pa_{n,k} + p - 1 = a_{n,k+1}$, and we compute

$$-d(i-1) v_{n-2}^{p^{k+1}} v_n^{ip^{k+1}+1} / v_{n-1}^{2p^{k+1} + p^k - pa_{n,k} + 1}$$
$$\equiv -(i-1) v_{n-2}^{p^{k+1}} v_n^{ip^{k+1}} t_1^{p^{n-1}} / v_{n-1}^{2p^{k+1} + p^k - pa_{n,k}}$$
$$+ (i-1) v_{n-2}^{p^{k+1}} v_n^{ip^{k+1}} t_1 / v_{n-1}^{2p^{k+1} + p^k - a_{n,k+1}}$$
$$- i(i-1) v_{n-2}^{p^{k+1}} v_n^{ip^{k+1} - p^{k+1}} \eta v_n t_1^{p^{k+n}} / v_{n-1}^{p^k + p^{k+1} - pa_{n,k} + 1}$$

mod $J(p^{k+1}+1)$ by (2.5) and (2.6). Putting $z_{k+1} = z_k^p - i(i-1) v_{n-1}^{pa_{n,k} - p^{k+1} - 1} v_n^{ip^{k+1} - p^{k+1}} \eta v_n t_1^{p^{k+n}}$ and adding above congruences prove the case for $k+1$, and we have 1)(a) inductively.

(b) The congruence (2.8) implies the case $k=0$ if the condition $i \equiv 1\,(p)$ holds. Suppose that (b) is valid for $k-1 \leq n-3$, and (2.9) again shows

$$dx(i/1;\, n;\, k-1)^p \equiv (-1)^{k-1} v_{n-2}^{P(k,k)} v_{n-1}^{Q(k+1,k+1)} v_n^{ip^k - p^k} w_{n-1,k+1}^p$$

mod $J(P(k,k)+p)$. A direct calculation shows

$$(-1)^{k-1} dv_{n-2}^{P(k,k)} v_{n-1}^{Q(k+1,k+1)} u_{n-1,k+1} v_n^{ip^k - p^k}$$
$$\equiv ((-1)^k v_{n-2}^{P(k,k)} v_{n-1}^{Q(k+1,k+1)} w_{n-1,k+1}^p$$
$$+ (-1)^k v_{n-2}^{P(k,k+1)} v_{n-1}^{Q(k+1,k+2)} w_{n-1,k+2}) v_n^{ip^k - p^k}$$

mod $J(P(k,k+1)+1)$, in which we use the congruence

$$\eta u_{n-1,k+1} \equiv \sum_{r=0}^{k+1} u_{n-1,r} t_{k+1-r}^{p^r} - w_{n-1,k+1}^p - v_{n-2} v_{n-1}^{-1} w_{n-1,k+2}$$

mod $J(2)$ shown in [9, Prop. 2.2] and the equality $v_{n-1}^{Q(k+1,k+1)} u_{n-1,r} = 0$ in $v_n^{-1}BP_*/(v_{n-1}^\infty)$ for $0 \le r \le k$. These imply the case (b).

2) Suppose that $j = a_{n,r}$, then we see $r \equiv 1\,(r-1)$ and $r > 1$ by the condition $p \not| j$. Here we denote

$$a = a_{n,r}.$$

Since $x_{n,r}^s \equiv x_{n,r-1}^{sp} + s v_{n-1}^{a_{n,r}+1} u_{n-1,1} x_{n,r-1}^{sp-1}$ mod $(v_{n-1}^{a_{n,r}+1})$, a computation shows

$$\eta x_{n,r}^s \equiv x_{n,r}^s + s v_{n-1}^{a_{n,r}} x_{n,r-1}^{sp-1} t_1 - s v_{n-2} v_{n-1}^{a_{n,r}} x_{n,r-1}^{sp-1} w_{n-1,2} \tag{2.10}$$

mod $J(2) + (v_{n-1}^{a_{n,r}+1})$. Here note that the ideal $J(2) + (v_{n-1}^{a_{n,r}+1})$ is invariant since $p | a_{n,r} + 1$. By (2.10) we have

$$dx(i/a;\, n;\, 0) \equiv v_{n-2} x_{n,r}^s t_1^{p^{n-2}} / v_{n-1}^{a+1} + z_0 / v_{n-1}^{p+1}$$

mod $J(2)$ letting $z_0 = -s v_{n-2} v_{n-1}^{p+1} v_n^{sp^r - p^{r-1}} w_{n-1,2} + s v_{n-2} v_{n-1}^p x_{n,r-1}^{sp-1} t_1^{p^{n-2}+1}$. We also obtain $dx(i/a;\, n;\, 1)$ from this result with (2.9) and

$$\begin{aligned} -dv_{n-2}^p v_n x_{n,r}^{sp} / v_{n-1}^{pa+p+1} \equiv &- v_{n-2}^p x_{n,r}^{sp} t_1^{p^{n-1}} / v_{n-1}^{pa+p} \\ &+ v_{n-2}^p x_{n,r}^{sp} t_1 / v_{n-1}^{pa+1} - s v_{n-2}^p v_n^{sp^{r+1} - p^r + 1} t_1^p / v_{n-1}^{p+1} \end{aligned}$$

mod $J(p+1)$. Now the case $k \not\equiv 1\,(n-1)$ follows from (2.9) and the case $k-1$. We get similarly the case $k \equiv 1\,(n-1)$ with an additional congruence

$$\begin{aligned} -dv_{n-2}^{p^k} v_n x_{n,r}^{sp^k} / v_{n-1}^{p^k a + p^k + p^{k-1} - a_{n,k} + p} \equiv &- v_{n-2}^{p^k} x_{n,r}^{sp^k} t_1^{p^{n-1}} / v_{n-1}^{p^k a + p^k + p^{k-1} - p a_{n,k-1}} \\ &+ v_{n-2}^{p^k} x_{n,r}^{sp^k} t_1 / v_{n-1}^{p^k a + p^k + p^{k-1} - a_{n,k}} \\ &- s v_{n-2}^{p^k} (\eta v_n) x_{n,r-1}^{sp^{k+1} - p^k} t_1^{p^k} / v_{n-1}^{p^k + p^{k-1} - a_{n,k} + p} \end{aligned}$$

mod $J(1 + p^k)$.

3) i) First suppose that $0 < j < a_{n,r}$ and $j \le a_{n,r} - p$ if $1 < r \equiv 1\,(n-1)$. Under this supposition, the definition of the element $x_{n,r}$ shows

$$x(i/j;\, n;\, 0) = x_{n,r}^s / v_{n-1}^j = x_{n,r-1}^{sp} / v_{n-1}^j$$

and so

$$dx(i/j;\, n;\, 0) \equiv -j v_{n-2} x_{n,r}^s t_1^{p^{n-2}} / v_{n-1}^{j+1} \text{ mod } J(2),$$

which implies

$$dx(i/j;\, n;\, 0)^p \equiv -j v_{n-2}^p x_{n,r}^{sp} t_1^{p^{n-1}} / v_{n-1}^{jp+p} \text{ mod } J(2p),$$

by (2.9), and furthermore we compute,

$$\begin{aligned} dj v_{n-2}^p v_n x_{n,r}^{sp} / v_{n-1}^{jp+p+1} &= j v_{n-2}^p x_{n,r}^{sp} t_1^{p^{n-1}} / v_{n-1}^{jp+p} - j v_{n-2}^p x_{n,r}^{sp} t_1 / v_{n-1}^{jp+1} \\ &(+ j s v_{n-2}^p v_n^{sp^{r+1} - p^r + 1} t_1^{p^{1'}} / v_{n-1} \text{ if } j = a_{n,r} - 1) \end{aligned}$$

mod $J(1+p)$ for an integer l' such that $r \equiv l'(n-1)$ and $1 \le i' \le n-1$, and

$$djsv_{n-2}^{p}v_{n}^{p^{r+1}-p^{r}+2}/2v_{n-1}^{2}$$
$$\equiv jsv_{n-2}^{p}v_{n}^{sp^{r+1}-p^{r}}(2v_{n-1}v_{n}t_{1}^{p^{n-1}})/2v_{n-1}^{2} \mod J(1+p),$$

if $j=a_{n,r}-1$ and $l=0$. These give $dx(i/j;n;1)$, and then we get $dx(i/j;n;k)$ for $k \le n-2$ and for $j=a_{n,r}-1$ and $k \le \max\{1, n-l-2\}$ by (2.9) since $w(i,j;k) = 0$. For the case $j=a_{n,r}-1$ and $l=n-2$, we see

$$dx(i/j;n;1)^{p} \equiv v_{n-2}^{p^2}x_{n,r}^{sp^2}t_{1}^{p}/v_{n-1}^{jp^2+p} - sv_{n-2}^{p^2}v_{n}^{sp^{r+2}-p^{r+1}+p}t_{1}^{p^{n-1}}/v_{n-1}^{p}$$

mod $J(p+p^2)$ and

$$dsv_{n-2}^{p^2}v_{n}^{sp^{r+2}-p^{r+1}+1}x_{n,1}/v_{n-1}^{p+1}$$
$$\equiv sv_{n-2}^{p^2}v_{n}^{sp^{r+2}-p^{r+1}+p}t_{1}^{p^{n-1}}/v_{n-1}^{p} - sv_{n-2}^{p^2}v_{n}^{sp^{r+2}-p^{r+1}+p}t_{1}/v_{n-1}$$
$$+ sv_{n-2}^{p^2}v_{n}^{sp^{r+2}-p^{r+1}+p}t_{1}/v_{n-1}$$

mod $J(1+p^2)$, and for $j=a_{n,r}-1$ and $l=n-1-k$ with $2 \le k \le n-2$,

$$dx(i/j;n;k-1)^{p} \equiv \tilde{j}v_{n-2}^{p^k}x_{n,r}^{sp^k}t_{1}^{p^{k-1}}/v_{n-1}^{jp^k+p^{k-1}}$$
$$-\tilde{j}sv_{n-2}^{p^k}v_{n}^{sp^{r+k}-p^{r+k-1}+p^{k-1}}t_{1}^{p^{n-2}}/v_{n-1}^{p^{k-1}}$$

mod $J(p^k+p)$, and

$$d\tilde{j}sv_{n-2}^{p^{k-1}}v_{n-1}v_{n}^{sp^{r+k}-p^{r+k-1}+p^{k-1}}/v_{n-1}^{p^{k-1}}$$
$$\equiv \tilde{j}sv_{n-2}^{p^k}v_{n}^{sp^{r+k}-p^{r+k-1}+p^{k-1}}t_{1}^{p^{n-2}}/v_{n-1}^{p^{k-1}}$$

mod $J(p^k+p-1)$. These give

$$dx(i/j;n;2) \equiv v_{n-2}^{p^2}x_{n,r}^{sp^2}t_{1}^{p}/v_{n-1}^{jp^2+p} \mod J(1+p^2),$$

and

$$dx(i/j;n;k) \equiv \tilde{j}v_{n-2}^{p^k}x_{n,r}^{sp^k}t_{1}^{p^{k-1}}/v_{n-1}^{jp^k+p^{k-1}} \mod J(2+p^k),$$

respectively, and we have the case for $k \le n-2$. Assume the case $k-1$ with $n-1 \le k \equiv 1\ (n-2)$, and we obtain

$$dx(i/j;n;k-1)^{p} \equiv -jv_{n-2}^{pa_{n-1,k-1}}x_{n,r}^{sp^k}t_{1}^{p^{n-2}}/v_{n-1}^{jp^k+p^{k-1}} \mod J(2p+a_{n-1,k-1}p)$$

and

$$djv_{n-2}^{a_{n-1,k}-p}v_{n-1}x_{n,r}^{sp^k}/v_{n-1}^{jp^k+p^{k-1}}$$
$$\equiv jv_{n-2}^{a_{n-1,k}-p+1}x_{n,r}^{sp^k}t_{1}^{p^{n-2}}/v_{n-1}^{jp^{k-1}+p^{k-1}} - jv_{n-2}^{a_{n-1,k}}x_{n,r}^{sp^k}t_{1}/v_{n-1}^{jp^k+p^{k-1}}$$

mod $J(2+a_{n-1,k})$, which bring the case for k with $n-1 \le k \equiv 1\ (n-2)$.

(ii) Next suppose that $1 < r \equiv 1\ (n-1)$ and $a_{n,r}-p+2 \le j \le a_{n,r}-1$. Since we see that $dx(i/j;n;0) \equiv (dx_{n,r}^s)/v_{n-1}^j - jv_{n-2}\eta x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{j+1}$ mod $J(2)$, we have $dx(i/j;n;0)$ by (2.10). For $dx(i/j;n;1)$ we verify

$$dx(i/j;n;0)^p \equiv -sv_{n-2}^p v_{n-1}^{pa_{n,r}-jp} x_{n,r-1}^{sp^2-p} w_{n-1,2}^p - jv_{n-2}^p x_{n,r}^{sp} t_1^{p^{n-1}}/v_{n-1}^{jp+p}$$

mod $J(2p)$,

$$djv_{n-2}^p v_n x_{n,r}^{sp}/v_{n-1}^{jp+p+1} \equiv jv_{n-2}^p x_{n,r}^{sp} t_1^{p^{n-1}}/v_{n-1}^{jp+p} - jv_{n-2}^p x_{n,r}^{sp} t_1/v_{n-1}^{jp+1}$$
$$(-2sv_{n-2}^p v_n^{sp^{r+1}-p^{r+1}} t_1^p/v_{n-1} \text{ if } j = a_{n,r}-1)$$

mod $J(1+p)$ and

$$-dsv_{n-2}^p u_{n-1,2} x_{n,r-1}^{sp^2-p}/v_{n-1}^{jp-pa_{n,r}}$$
$$\equiv -sv_{n-2}^p(u_{n-1,1}t_1^p + u_{n-1,0}t_2 - w_{n-1,2}^p)x_{n,r-1}^{sp^2-p}/v_{n-1}^{jp-pa_{n,r}}.$$

Unless $j = a_{n,r}-1$, we have the case ii) and 3)(a) and (b) in a same way as i). For the case $j = a_{n,r}-1$, we have

$$dx(i/j;n;1)^{p^{n-3}} \equiv -jv_{n-2}^{p^{n-2}} x_{n,r}^{sp^{n-2}} t_1^{p^{n-3}}/v_{n-1}^{jp^{n-2}+p^{n-3}}$$
$$- sv_{n-2}^{p^{n-2}} v_n^{sp^{r+n-2}-p^{r+n-3}+p^{n-3}} t_1^{p^{n-2}}/v_{n-1}^{p^{n-3}}$$

mod $J(p^{n-2}+p^{n-3})$ and

$$dsv_{n-2}^{p^{n-2}-1} v_{n-1} v_n^{sp^{r+n-2}-p^{r+n-3}+p^{n-3}}/v_{n-1}^{p^{n-3}}$$
$$\equiv sv_{n-2}^{p^{n-2}} v_n^{sp^{r+n-2}-p^{r+n-3}+p^{n-3}} t_1^{p^{n-2}}/v_{n-1}^{p^{n-3}}$$

mod $J(p^{n-2}+p-1)$, which give us the case for $k = n-2$: $dx(i/j;n;n-2) \equiv \tilde{j}v_{n-2}^{p^{n-2}} x_{n,r}^{sp^{n-2}} t_1^{p^{n-3}}/v_{n-1}^{jp^{n-2}+p^{n-3}}$, and we obtain further congruences in a same manner as i).

4) (a) For $0 \le l \le n-3$, put $h = (l+1)(n-2)+1$. The result of 3) shows

$$dx(i/j;n;h-1)^p \equiv -jv_{n-2}^{pa_{n-1,h-1}} x_{n,r}^{sp^h} t_1^{p^{n-2}}/v_{n-1}^{jp^h+p^{h-1}}$$

mod $J(2p+pa_{n,h-1})$. We compute

$$djv_{n-2}^{a_{n-1,h}-p} v_{n-1} x_{n,r}^{sp^h}/v_{n-1}^{jp^h+p^{h-1}} \equiv jv_{n-2}^{a_{n-1,h}-p+1} x_{n,r}^{sp^h} t_1^{p^{n-2}}/v_{n-1}^{jp^h+p^{h-1}}$$
$$- jv_{n-2}^{a_{n-1,h}} x_{n,r}^{sp^k} t_1/v_{n-1}^{jp^h+p^{h-1}}$$

mod $J(2+a_{n-1,h})$ and

$$djv_{n-2}^{a_{n-1,h}} x_{n,r+h+1}^{s'}/v_{n-1}^{c(0)-p^{n-1}-1} \equiv jv_{n-2}^{a_{n-1,h}} x_{n,r+h}^s t_1/v_{n-1}^{jp^h+p^{h-1}}$$
$$+ jv_{n-2}^{a_{n-1,h}+1} x_{n,r+h+1}^{s'} t_1^{p^{n-2}}/v_{n-1}^{c(0)-p^{n-1}}$$

mod $J(2+a_{n-1,h})$. Here we note that $r+j+1 \equiv 1\ (n-1)$ and so $p^{n-2}|c(0)$. Notice further that $x_{n,r+h}^s \equiv x_{n,r}^{sp^h}$ mod $(v_{n-1}^{jp^h+p^{h-1}})$. In fact we have $x_{n,r+\varepsilon} \equiv x_{n,r}^{p^\varepsilon}$ mod $(v_{n-1}^{a_{n,r+\varepsilon}-p})$ for $r+\varepsilon \equiv 1\ (n-1)$ with $1 \le \varepsilon \le n-1$. Thus $x_{n,r+h} \equiv x_{n,r}^{p^h}$ mod $(v_{n-1}^{p^{h-\varepsilon}a_{n,r+\varepsilon}-p^{h-\varepsilon+1}})$. Furthermore we see that $p^{h-\varepsilon}a_{n,r+\varepsilon} - p^{h-\varepsilon+1} - (jp^h + p^{h-1}) \ge p^h a_{n,r} + p^{h-\varepsilon+1} - p^{h-\varepsilon}$

$-p^{h-\varepsilon+1}-p^h a_{n,r}+p^h-p^{h-1}\geq 0$. Therefore we have

$$dx(i/j;\,n;\,h)\equiv jv_{n-2}^{a_{n-1,h}+1}x_{n,r+h+1}^{s'}t_1^{p^{n-2}}/v_{n-1}^{c(0)-p^{n-1}}+v_{n-2}^{a_{n-1,h}+1}z_0/v_{n-1}^{jp^h+p^{h-1}+p}$$

mod $J(2+a_{n,h})$ for some $z_0\in v_n^{-1}BP_*$.

Next suppose that $k\equiv 1\ (n-1)$ and the case for $k-1$ holds. Then the case for k follows from the congruences

$$dx(i/j;\,n;\,h+k-1)^p\equiv jv_{n-2}^{p^k a_{n-1,h}+p^k}x_{n,r+h+1}^{s'p^k}t_1^{p^{n-1}}/v_{n-1}^{c(k)-pa_{n,n+k-2}}$$
$$+\,v_{n-2}^{p^k a_{n-1,h}+p^k}z_{k-1}^p/v_{n-1}^{jp^{k+h}+p^{k+n-1}+p^{k+1}}$$

mod $J(p+p^k+p^k a_{n,h})$, and

$$-\,djv_{n-2}^{p^k a_{n-1,k}+p^k}v_n x_{n,r+h+1}^{s'p^k}/v_{n-1}^{c(k)+1-pa_{n,n+k-2}}$$
$$\equiv\,-\,jv_{n-2}^{p^k a_{n-1,h}+p^k}x_{n,r+h+1}^{s'p^k}t_1^{p^{n-1}}/v_{n-1}^{c(k)-pa_{n,n+k-2}}$$
$$+\,jv_{n-2}^{p^k a_{n-1,h}+p^k}x_{n,r+h+1}^{s'p^k}t_1/v_{n-1}^{c(k)-a_{n,n+k-1}}$$
$$-\,jv_{n-2}^{p^k a_{n-1,h}+p^k}(\eta v_n)x_{n,r+h}^{s}t_1/v_{n-1}^{jp^{k+h}+p^{k+h-1}+p^k+p^{k+n-1}-a_{n,n+k-1}}$$

mod $J(1+p^k a_{n-1,h}+p^k)$. The other cases follow immediately from (2.9).

(b) Here we assume that $l=n-2$, $j=ep-f$ and $f\neq 1$. Since

$$x(i/j;\,n;\,0)\equiv x_{n,r}^s/v_{n-1}^j-fv_{n-2}x_{n,r+1}^{s'}/s'v_{n-1}^{j+1+a_{n,r+1}}\ \mathrm{mod}\ (v_{n-2}^3),$$

we compute

$$dx_{n,r}^s/v_{n-1}^j\equiv(dx_{n,r-1}^{sp})/v_{n-1}^j-jv_{n-2}(\eta x_{n,r}^s)t_1^{p^{n-2}}/v_{n-1}^{j+1}$$
$$+\binom{j+1}{2}v_{n-2}^2(\eta x_{n,r}^s)t_1^{2p^{n-2}}/v_{n-1}^{j+2}$$

mod $J(3)$ and

$$-\,fdv_{n-2}x_{n,r+1}^{s'}/s'v_{n-1}^{j+1+a_{n,r+1}}\equiv-\,fv_{n-2}x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{j+1}$$
$$+\,f(j+1)v_{n-2}^2x_{n,r+1}^{s'}t_1^{p^{n-2}}/s'v_{n-1}^{j+2+a_{n,r+1}}$$

mod $J(3)$ to get

$$dx(i/j;\,n;\,0)\equiv f(1-f)v_{n-2}^2x_{n,r+1}^{s'}t_1^{p^{n-2}}/s'v_{n-1}^{j+2+a_{n,r+1}}+v_{n-2}^2z_0/v_{n-1}^{j+2}$$

mod $J(3)$. Assuming the case for $k-1$, we get the case for $k\not\equiv 1\ (n-1)$ by (2.9) and the case for $k\equiv 1\ (n-1)$ by (2.9) and a congruence

$$dv_{n-2}^{2p^k}v_n x_{n,r+1}^{s'p^k}/s'v_{n-1}^{c'(k)+1-pa_{n,n+k-2}}\equiv v_{n-2}^{2p^k}x_{n,r+1}^{s'p^k}t_1^{p^{n-1}}/s'v_{n-1}^{c'(k)-pa_{n,n+k-2}}$$
$$-\,v_{n-2}^{2p^k}x_{n,r+1}^{s'p^k}t_1/s'v_{n-1}^{c'(k)-a_{n,n+k-1}}$$

mod $J(1+2p^k)$.

(c) In the case $l=n-2$ and $j=ep-1$ with $e\not\equiv -1\ (p)$, we compute $dx(i/j;\,n;\,0)$

similarly to (b). So first have we $x(i/j;n;0) \equiv v_{n-1}x_{n,r}^s/x_{n-1,1}^e - v_{n-2}x_{n,r+1}^{s'}/s'v_{n-1}^{j+1+a_{n,r+1}} \mod (v_{n-2}^{p+1})$ and

$$dv_{n-1}x_{n,r}^s/x_{n-1,1}^e \equiv v_{n-2}x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{ep} - v_{n-2}^p x_{n,r}^s t_1/v_{n-1}^{ep}$$
$$+ v_{n-2}^p x_{n,r-1}^{sp-1}(\eta v_n)t_1^{p^{n-3}}/v_{n-1}^{j+p+1-a_{n,r}} - ev_{n-2}^p x_{n,r}^s t_1/v_{n-1}^{j+1}$$
$$(+\, sx_{4,r-2}^{sp^2-p}t_1^p/v_3^{ep-1-a_{4,r}} - sv_2^p x_{4,r-2}^{sp^2-p}w_{3,2}^p/v_3^{ep-1-a_{4,r}} \text{ if } n=4 \text{ by (2.10))}$$

mod $J(p+1)$. We note that $ep+p=j+1+p \le a_{4,r}$ since $j \le a_{4,r}$ and $e \not\equiv -1\ (p)$. Then the third term of the above congruence is 0 and so is the first term in the parenthesis if $n=4$. The inequality $j+1+p \le a_{4,r}$ is also used to show

$$dv_2^p u_{3,2}v_4^{sp^{r+1}-p^r}/v_3^{j-a_{4,r}} \equiv -v_2^p v_4^{sp^{r+1}-p^r}w_{3,2}^p/v_3^{j-a_{4,r}} \mod J(p+1).$$

Now we have

$$dv_{n-2}x_{n,r+1}^{s'}/s'v_{n-1}^{j+1+a_{n,r+1}} \equiv v_{n-2}x_{n,r}^s t_1^{p^{n-2}}/v_{n-1}^{j+1} \mod J(p+1)$$

to gain the desired congruence

$$dx(i/j;n;0) \equiv -(e+1)v_{n-2}^p x_{n,r}^s t_1/v_{n-1}^{ep}$$

$(+\, v_{n-2}^p x_{n,r-1}^{sp-1}(\eta v_n)t_1^{p^{n-3}}/v_{n-1}^p$ if $j=a_{n,r}-1)$ mod $J(p+1)$. This implies the cases for $k \not\equiv 0\ (n-2)$ and $k<(n-2)^2$. For the case $k \equiv 0\ (n-2)$ and $k<(n-2)^2$, we need more calculation:

$$dx(i/j;n;k-1)^p \equiv -(e+1)v_{n-2}^{pa_{n-1,k}}x_{n,r}^{sp^k}t_1^{p^{n-2}}/v_{n-1}^{jp^k+p^k}$$

and

$$dv_{n-2}^{pa_{n-1,k}-1}v_{n-1}x_{n,r}^{sp^k}/v_{n-1}^{jp^k+p^k} \equiv v_{n-2}^{pa_{n-1,k}}x_{n,r}^{sp^k}t_1^{p^{n-2}}/v_{n-1}^{jp^k+p^k} - v_{n-2}^{a_{n-1,k+1}}x_{n,r}^{sp^k}t_1/v_{n-1}^{jp^k+p^k}$$

both mod $J(2p+pa_{n-1,k})$. In case $k=h'=(n-2)^2$, add further

$$dv_{n-2}^{a_{n-1,h'+1}}x_{n,r+h'+1}^{s'}/s'v_{n-1}^{jp^{h'}+p^{h'}+a_{n,r+h'+1}} \equiv v_{n-2}^{a_{n-1,h'+1}}x_{n,r+h'}^s t_1/v_{n-1}^{jp^{h'}+p^{h'}}$$
$$+\, v_{n-2}^{a_{n-1,h'+1}+1}x_{n,r+h'+1}^{s'}t_1^{p^{n-1}}/v_{n-1}^{c''(h')-p^{n-1}} + v_{n-2}^{a_{n-1,h'+1}+1}z_0/v_{n-1}^{jp^{h'}+p^{h'}+1}$$

mod $J(2+a_{n-1,h'+1})$ for $z_0=-s'v_n^{sp^{r+h'+1}-p^{r+h'}}w_{n-1,2}$, and we obtain the case for $x(i/j;n;h')$. We have the case for $h'<k \not\equiv 1\ (n-1)$ by (2.9) and the case $h'<k \equiv 1\ (n-1)$ follows from the congruences:

$$dx(i/j;n;k-1)^p \equiv (e+1)v_{n-2}^{p^{k-h'}a_{n-1,h'+1}+p^{k-h'}}x_{n,r+h'+1}^{s'p^{k-h'}}t_1^{p^{n-1}}/v_{n-1}^{c''(k)-pa_{n,k-h'+n-2}}$$
$$+\, v_{n-2}^{p^{k-h'}a_{n-1,h'+1}+p^{k-h'}}z_{k-h'-1}^p/v_{n-1}^{jp^k+p^k+p^{k-h'}}$$

and

$$(e+1)dv_{n-2}^{p^{k-h'}a_{n-1,h'+1}+p^{k-h'}}v_n x_{n,r+h'+1}^{s'p^{k-h'}}/v_{n-1}^{c''(k)+1-pa_{n,k-h'+n-2}}$$
$$\equiv (e+1)v_{n-2}^{p^{k-h'}a_{n-1,h'+1}+p^{k-h'}}x_{n,r+h'+1}^{s'p^{k-h'}}t_1^{p^{n-1}}/v_{n-1}^{c''(k)-pa_{n,k-h'+n-2}}$$
$$-(e+1)v_{n-2}^{p^{k-h'}a_{n-1,h'+1}+p^{k-h'}}x_{n,r+h'+1}^{s'p^{k-h'}}t_1/v_{n-1}^{c''(k)-a_{n,k-h'+n-1}}$$

$$+ v_{n-2}^{p^{k-h'}a_{n-1,h'+1}+p^{k-h'}} \tilde{z}_k / v_{n-1}^{jp^k+p^k+p^{k-h'}}$$

mod $J(1 + p^{k-h'}a_{n-1,h'+1} + p^{k-h'})$.

(d) Lastly we assume $l = n-2$ and $j = ep-1$ with $e \equiv -1\ (p)$. The following congruences give the case for $k=0$ by noticing the equality $e'p+1 = ep + a_{n,r+1} + 1 = j + 2 + a_{n,r+1} = c'(0) - a_{n,n-1}$:

$$dv_{n-1}x_{n,r}^s/x_{n-1,1}^e \equiv v_{n-2}x_{n,r}^s t_1^{p^{n-2}}/x_{n-1,1}^e - v_{n-2}^p x_{n,r}^s t_1/x_{n-1,1}^e + v_{n-2}^p(\eta v_{n-1}x_{n,r}^s)t_1/v_{n-1}^{j+2}$$

$(-sv_2^p x_{4,r-2}^{sp^2-p} w_{3,2}^p(\eta v_3)/v_3^{j-a_{4,r}+1}$ if $n=4)$ $(-sv_2^p x_{4,r-1}^{sp-1} v_4(\eta v_3)t_1^p/v_3^{p+1}$ if $n=4$ and $j = a_{4,r}-1$ with $r \geq 4)$

mod $J(p+2)$, $dv_{n-2}x_{n,r+1}^{s'}/s'x_{n-1,1}^{e'} \equiv v_{n-2}x_{n,r}^s t_1^{p^{n-2}}/x_{n-1,1}^e + v_{n-2}^{p+1}x_{n,r+1}^{s'}t_1/s'v_{n-1}^{e'p+1}$

mod $J(p+2)$, and

$$dv_2^p u_{3,2} v_3 v_4^{sp^{r+1}-p^r}/v_3^{j+1-a_{4,r}}$$
$$\equiv v_2^p v_4^{sp^{r+1}-p^r} w_{3,2}^p(\eta v_3)/v_3^{j+1-a_{4,r}} - v_2^{p+1}y/v_3^{j+p+1-a_{4,r}}$$

$(-v_2^p v_4^{sp^r-p^{r-1}+1}t_1^p/v_3^p$ if $j = a_{4,r}-1$ with $r \geq 4)$ mod $J(p+2)$ for the case $n=4$. For the case $k \equiv 1\ (n-1)$, we have the congruences

$$dv_{n-2}^{p^{k+1}+p^k} v_n x_{n,r+1}^{s'p^k}/s'v_{n-1}^{c'(k)+1-pa_{n,k+n-2}} \equiv v_{n-2}^{p^{k+1}+p^k} x_{n,r+1}^{s'p^k} t_1^{p^{n-1}}/s'v_{n-1}^{c'(k)-pa_{n,k+n-2}}$$
$$- v_{n-2}^{p^{k+1}+p^k} x_{n,r+1}^{s'p^k} t_1/s'v_{n-1}^{c'(k)-a_{n,k+n-1}} + v_{n-2}^{p^{k+1}+p^k} y/v_{n-1}^{jp^k+2p^k+p^{k+n-1}+1-pa_{n,k+n-2}}$$

mod $J(1+p^{k+1}+p^k)$, which and (2.9) bring the results inductively. q.e.d.

§ 3. Proof of the theorem

By the definition of the comodules M_n^s, we have the short exact sequence

$$0 \longrightarrow M_{n-r+1}^{r-1} \xrightarrow{i} M_{n-r}^r \xrightarrow{v_{n-r}} M_{n-r}^r \longrightarrow 0,$$

for i defined by $ix = x/v_{n-r}$. Define $d_1 \colon M \otimes_A \Gamma \to M \otimes_A \Gamma \otimes_A \Gamma$ by

$$d_1 m \otimes \gamma = \eta m \otimes \gamma - m\Delta\gamma + m \otimes \gamma \otimes 1,$$

for a Γ-comodule M with structure map η, where (A, Γ) denotes the Hopf algebroid (BP_*, BP_*BP) associated to the Brown-Peterson spectrum BP. Then we can verify that $d_1 d = 0$, and define

$$H^1M = \mathrm{Ker}\, d_1/\mathrm{Im}\, d \subset \mathrm{Coker}\, d.$$

Now the snake lemma implies the exact sequence

(3.1) $$0 \longrightarrow H^0M_{n-r+1}^{r-1} \xrightarrow{i_*} H^0M_{n-r}^r \xrightarrow{(v_{n-r})_*} H^0M_{n-r}^r \xrightarrow{\delta_r} H^1M_{n-r+1}^{r-1},$$

in which δ_r is defined to be:

$$\delta_r x = i^{-1}\{dv_{n-r}^{-1}x\}.$$

In the exact sequence (3.1) for $r=1$, the module $H^1M_n^0$ is given in [4, Th. 2.2]:

$H^1M_n^0$ is the $Z/p[v_n, v_n^{-1}]$-vector space with basis ζ_n and $h_j=\{t_1^{p^j}\}$ for $0\le j<n$,

and Im δ_1 is in [1, (5.18)]:

Im δ_1 *is the Z/p-vector space with basis $v_n^{s-1}h_{n-1}$, $v_n^{sp^i-p^{i-1}}h_j$ for j with $j\equiv i-1$ $(n-1)$ and $0\le j<n-1$ and for any s prime to p.*

Since the sequence

$$H^0M_{n-r}^r \xrightarrow{\delta_r} H^1M_{n-r-1}^{r-1} \xrightarrow{i_*} H^1M_{n-r}^r \xrightarrow{(v_{n-r})_*} H^1M_{n-r}^r$$

is also exact, we have an isomorphism $i_*\colon H^1M_n^0/\mathrm{Im}\,\delta_1 \approx \mathrm{Ker}\,(v_{n-1})_*$, and

Lemma 3.2. *Let p be an odd prime. In the above exact sequence for $r=1$,* Ker$(v_{n-1})_*$ *is the Z/p-vector space spanned by:*

(a) $v_n^m h_{n-1}/v_{n-1}$ *for $m\in Z$ such that $p|(m+1)$,*
(b) $v_n^m h_j/v_{n-1}$ *for $0\le j<n-1$ and $m=lp^k\in Z$ with $p\not| l$ such that*

$$\text{either } k\not\equiv j\,(n-1),\ p\not\equiv -1\,(p) \text{ or } k\equiv -1\,(p^2), \text{ and}$$

(c) $v_n^m\zeta_n/v_{n-1}$ *for any $m\in Z$.*

Consider now the exact sequence (3.1) for $r=2$. Then the first module of the sequence is known to be:

(3.3)[1, Th. 5.10] *$H^0M_{n-1}^1$ is the $Z/p[v_{n-1}]$-module generated by $x_{n,i}^s/v_{n-1}^{a_{n,i}}$ for $i\ge 0$ and $s\not\equiv 0\,(p)$, and Z/p-module generated by $1/v_{n-1}^j$ for $j\ge 1$.*

We will determine the module $H^0M_{n-2}^2$ by the fact that

(3.4)[1, Remark 3.11] *If a submodule B of $H^0M_{n-2}^2$ satisfies that $(v_{n-2})_*B\subset B$ and the sequence $B\xrightarrow{(v_{n-2})_*} B\xrightarrow{\delta_2} H^0M_{n-2}^2$ is exact, then $B=H^0M_{n-2}^2$.*

Now we can prove the theorem.

Proof of Theorem. Let B denote the direct sum of the cyclic submodules of $H^0M_{n-2}^2$ generated by the element $x_n(a/b, c_n(a,b))$ for a and $b\in Z$ such that $b>0$ and $b\le a_{n,\nu(a)}$ if $a\ne 0$. Then it is trivial by definition that $B\subset H^0M_{n-2}^2$, $(v_{n-2})_*B\subset B$ and $\delta_2(v_{n-2})_*=0$. It is sufficient to show that Ker $\delta_2\subset \mathrm{Im}\,(v_{n-2})_*$. Suppose that $x=\sum\lambda_g g\in B$ satisfies that $x\notin \mathrm{Im}\,(v_{n-2})_*$ and $\delta_2 x=0$, where g is one of the generators and $\lambda_g\in Z/p[v_{n-2}]$. By Lemma 2.4, for $b=jp^k$, $a=sp^{r+k}$ and $i=sp^r$ with $p\not| j$ and $p\not| s$, we obtain the following

$$\delta_2 x_n(a/b, c_n(a, b)) = (i-1)v_n^i t_1^{p^{n-2}}/v_{n-1}^2 + \cdots$$

for $r = 0$, $i \not\equiv 1\,(p)$ and $k = 0$,

$$\delta_2 x_n(a/b, c_n(a, b)) = (i-1)v_n^{ip^k} t_1^{p^\varepsilon}/v_{n-1}^{2p^k + p^{k-1} - a_{n,k}} + \cdots$$

for $r = 0$, $i \not\equiv 1\,(p)$ and $k > 0$,

$$\delta_2 x_n(a/b, c_n(a, b)) = (-1)^k v_{n-1}^{Q(k+1,k+2)} v_n^{ip^k - p^k} w_{n-1,k+2} + \cdots$$

for $r = 0$, $i \equiv 1\,(p)$ and $k \le n-2$,

$$\delta_2 x_n(a/b, c_n(a, b)) = x_{n,r}^{sp^k} t_1^{p^\varepsilon}/v_{n-1}^{p^k a_{n,r} + p^k + p^{k-1} - a_{n,k}} + \cdots$$

for $r > 0$ and $j = a_{n,r}$,

$$\delta_2 x_n(a/b, c_n(a, b)) = -j x_{n,r}^s t_1^{p-2}/v_{n-1}^{j+1} + \cdots$$

for $r > 0$, $j < a_{n,r}$ and $k = 0$,

$$\delta_2 x_n(a/b, c_n(a, b)) = -j x_{n,r}^{sp^k} t_1^{p^{\varepsilon'}}/v_{n-1}^{jp^k + p^{k-1}} + \cdots$$

for $r > 0$, $j < a_{n,r}$, $k > 0$ and either $s \not\equiv -1\,(p)$ or $s \equiv -1\,(p^2)$; or

for $r > 0$, $j < a_{n,r}$, $0 < k < h$, $s = s'p - 1$ with $p \nmid s'$, and $l \ne n-2$,

$$\delta_2 x_n(a/b, c_n(a, b)) = x_{n,r+h+1}^{s'p^{k-h}} t_1^{p^\varepsilon}/v_{n-1}^{p^{k-h}a_{n,r+h+1} + jp^k + p^{k-1} + p^{k-h} + p^{k-h+n-1} - a_{n,n+k-h-1}} + \cdots$$

for $r > 0$, $j < a_{n,r}$, $k \ge h$, $s = s'p - 1$ with $p \nmid s'$, and $l \ne n-2$,

$$\delta_2 x_n(a/b, c_n(a, b)) = -j(j+1) x_{n,r+1}^{s'p^k} t_1^{p^\varepsilon}/v_{n-1}^{p^k a_{n,r+1} + jp^k + 2p^k + p^{k+n-1} - a_{n,n+k-1}} + \cdots$$

for $r > 0$, $j < a_{n,r}$ with $j \not\equiv -1\,(p)$, $k \ge 0$, $s'p - 1$ and $l = n-2$,

$$\delta_2 x_n(a/b, c_n(a, b)) = -(1+e) x_{n,r}^{sp^k} t_1^{p^{\varepsilon'}}/v_{n-1}^{jp^k + p^k} + \cdots$$

for $r > 0$, $j < a_{n,r}$ with $j \equiv -1\,(p)$ and $j \not\equiv -p-1\,(p^2)$,
$0 \le k < h'$, $s = s'p - 1$ and $l = n-2$,

$$\delta_2 x_n(a/b, c_n(a, b))$$
$$\doteq (1+e) x_{n,r+h'+1}^{s'p^{k-h'}} t_1^{p^\varepsilon}/v_{n-1}^{jp^k + p^k + p^{k-h'}a_{n,r+h'+1} + p^{k-h'} + p^{k-h'+n-1} - a_{n,n+k-h'-1}} + \cdots$$

for $r > 0$, $j < a_{n,r}$ with $j \equiv -1\,(p)$ and $j \not\equiv -p-1(p^2)$,
$k \ge h'$, $s = s'p - 1$ and $l = n-2$,

$$\delta_2 x_n(a/b, c_n(a, b)) = -x_{n,r+1}^{s'p^k} t_1^{p^{\varepsilon''}}/s' v_{n-1}^{jp^k + 2p^k + p^k a_{n,r+1} + p^{k+n-1} - a_{n,n+k-1}} + \cdots$$

for $r > 0$, $j < a_{n,r}$ with $j \equiv -p-1\,(p^2)$,
$0 \le k$, $s = s'p - 1$ and $l = n-2$,

These show that there exists a generator $g(x)$ of B and a nonnegative integer $e(x)$ such that $v_{n-1}^{e(x)}\delta_2(x) = \lambda_{g(x)} v_{n-1}^{e(x)} g(x)$. Here notice that $x_{n,r} \equiv v_n^{p^r} \mod (v_{n-1})$. Since $v_{n-1}^{e(x)} g(x)$ is one of the elements in (a) and (b) of Lemma 3.2, they are independent and so $\lambda_{g(x)}$

$= 0$. Inductively we deduce that $\lambda_g = 0$. Therefore $x = 0$, which contradicts to the assumption $x \notin \mathrm{Im}\, v_{n-2}$. Hence we have that $\mathrm{Ker}\, \delta_2 \subset \mathrm{Im}\, v_{n-2}$. q.e.d.

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