Invariant Regular Sequences in the Brown-Peterson Homology BP_*

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§1. Introduction

The Brown-Peterson ring spectrum BP at a prime number p induces the Hopf algebroid (BP_*, BP_*BP) with the right unit $\eta: BP_* \to BP_*BP$ given by the unit $i: S \to BP$ of the ring spectrum (cf. [1]), and the coefficient ring BP_* is the polynomial ring $Z_{(p)}[v_1, v_2, \cdots]$ over Hazewinkel's generators v_k. A sequence $S: \alpha_0, \alpha_1, \cdots, \alpha_n$ of elements α_k of BP_* is an *invariant regular* sequence of length $n+1$ if $\eta\alpha_0 = \alpha_0$, $\eta\alpha_k \equiv \alpha_k$ mod J_k for $k > 0$, and α_k is a non-zero divisor of BP_*/J_k for each $k \geqq 0$, where J_k denotes the ideal $(\alpha_0, \alpha_1, \cdots, \alpha_{k-1})$ of BP_*. Let the sequence S be invariant regular, and P. S. Landweber [2] showed that $\alpha_k = v_k^{s_k} + (lower)$ for some positive integer s_k for each k. Here $(lower)$ denotes an element of the ideal $(p, v_1, v_2, \cdots, v_{k-1})$. A sequence $S: \alpha_0, \alpha_1, \cdots, \alpha_n$ is not always invariant regular even if $\alpha_k = v_k^{s_k} + (lower)$. E. Tsukada [8] investigated the case that $(lower) = 0$, and gave the necessary and sufficient condition on the integer s_k that a sequence S is invariant regular. The case that $(lower) \neq 0$ for odd prime p is studied partially by the first named author [6].

Consider the sequence

$$S: \alpha_0, \alpha_1, \cdots, \alpha_n \tag{1.1}$$

with $\alpha_0 = p^e$ and

$$\alpha_k = v_k^{s_k} + (lower), \tag{1.2}$$

where $e \geqq 1$ and $s_k = p^{i_k} e_k > 0$ with $i_k \geqq 0$ and $p \nmid e_k$ for each k. For a prime number p, an integer $a_{n,u}$ $(n \geqq 2,\ u \geqq 0)$ is defined by

$$\begin{aligned} &a_{n,u} = p^u \quad \text{if} \quad 0 \leqq u \leqq n-1, \\ &a_{n,u} = p^u + p^j(p-1)(p^{u-1-j}-1)/(p^{n-1}-1) \quad \text{if } u \geqq n, \\ &\qquad u = q(n-1)+1+j \ (q \geqq 1,\ 0 \leqq j \leqq n-2) \quad (\text{cf. } [3; (5.13)]). \end{aligned} \tag{1.3}$$

In this paper, we have the following theorems.

THEOREM E. *Let $p = 2$ and $\delta = 1$ or 2. There exists an invariant regular sequence S of* (1.1) *if*

$$2 \leqq e \leqq i_1 + \delta,\ 0 < e_k \leqq a_{k+1, u_{k+1}} \quad \textit{and} \quad e_k \leqq p^{u_{k+1}} \quad \textit{if} \quad e_{k+1} = 1, \qquad \textit{for} \quad k \geqq 1.$$

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Here $u_k = i_k - i_{k-1} - e + 1 \, (k \geqq 2)$, *and moreover we assume* $3 \leqq e = i_1 + 2$ *and* $e_1 \geqq 3$ *if* $\delta = 2$.

THEOREM O. *Let* $p > 2$. *There exists an invariant regular sequence* S *of* (1.1) *if*

$$2 \leqq e \leqq i_1 + 1, 0 < e_k \leqq a_{k+1, u_{k+1}}, \quad \text{and} \quad e_k < p^{u_{k+1}} \quad \text{if} \quad k \geqq 2 \quad \text{and}$$
$$e_{k+1} = 1, \quad \text{for} \quad k \geqq 1.$$

Here $u_k = i_k - i_{k-1} - e + 1 \quad (k \geqq 2)$.

In the above theorems we assume that $e \geqq 2$. For $e = 1$, we have the following

THEOREM 0. *There exists an invariant regular sequence* $S: p, \alpha_1, \cdots, \alpha_n$ *with* $\alpha_k = v_k^{s_k} + (lower)$ *if*

$$e_k \leqq a_{k+1, u_{k+1}} - 2, \text{ and}$$
$$e_k < p^{u_{k+1}} \quad \text{if} \quad e_{k+1} = 1 \quad \text{and if} \quad k \geqq 2 \text{ or } p = 2, \quad \text{for} \quad k \geqq 1.$$

Here $s_k = p^{i_k} e_k$ *with* $i_k \geqq 0$ *and* $p \not| e_k$, *and* $u_k = i_k - i_{k-1}$ $(k \geqq 2)$.

We notice that Theorem 0 is not our new result, which was studied in [6] already. In §2 we define a sequence in BP_*, which is in the above theorems, by using the element $x_{n,k} \in v_n^{-1} BP_*$ in [3]. In order to prove the theorems we study about the ideals given by the sequence in BP_* in §3.
We prove the theorems in §4 and we note the necessary and sufficient condition on invariant sequences (Theorem 5.1) in §5.

§2. Definition of sequences in BP_*

We have the Hopf algebroid (BP_*, BP_*BP) induced by the Brown-Peterson ring spectrum BP at a prime number p. We also have

(2.1) $$BP_*BP = \mathbf{Z}_{(p)}[v_1, v_2, \cdots, v_n, \cdots], \ \deg v_n = 2(p^n - 1),$$

where the v_n's are Hazewinkel's generators, and

(2.2) $$BP_*BP = BP_*[t_1, t_2, \cdots, t_n, \cdots], \ \deg t_n = 2(p^n - 1).$$

The right unit $\eta: BP_* \to BP_*BP$ of the Hopf algebroid is given by the following equalities:

(2.3) $$\eta l_k = \textstyle\sum_{i+j=k} l_i t_j^{(i)},$$

(2.4) $$v_k = p l_k - \textstyle\sum_{i=1}^{k-1} v_{k-i}^{(i)} l_i,$$

where $\eta: BP_* \otimes Q \to BP_*BP \otimes Q$, and $BP_* \otimes Q = Q[l_1, l_2, \cdots]$ (cf. [4]). Note here that

(2.5) (k) *in the exponent denotes* p^k *throughout this paper.*

We use the notation

(2.6) $$dx = \eta x - x \quad \text{for} \quad x \in BP_*.$$

In [3], Miller, Ravenel, and Wilson defined elements $x_{n,k} \in v_n^{-1}BP_*$ and integers $a_{n,k} \geqq 1$ for all primes p and $n \geqq 1$, $k \geqq 0$, in such a way that the next lemma holds.

LEMMA 2.7 *For $n = 1$ and $k \geqq 0$,*

$$dx_{1,k} \equiv 0 \mod (p^{a_{1,k}})$$

for $n \geqq 2$ and $k \geqq 0$,

$$dx_{n,k} \equiv 0 \mod (I_{n-1}, v_{n-1}^{a_{n,k}}).$$

Here I_n denotes the invariant prime ideal

(2.8) $$I_n = (p, v_1, \cdots, v_{n-1}), \quad 0 \leqq n \leqq \infty,$$

the elements $x_{n,k} \in v_n^{-1}BP_*$ are:

(2.9)
$$x_{1,0} = v_1,$$
$$x_{1,1} = v_1^2 - 4v_1^{-1}v_2 \qquad \text{for } p = 2,$$
$$x_{1,k} = x_{1,k-1}^p \qquad \text{otherwise,}$$
$$x_{2,0} = v_2,$$
$$x_{2,1} = x_{2,0}^p - v_1^p v_2^{-1} v_3,$$
$$x_{2,2} = x_{2,1}^p - v_1^{(2)-1} v_2^{(2)-p+1} - v_1^{(2)+p-1} v_2^{(2)-2p} v_3,$$
$$x_{2,k} = x_{2,k-1}^p - 2v_1^b v_2^{(k)-(k-1)+1} \ (b = b_{2,k}) \quad \text{for} \quad p \geqq 3,\ k \geqq 3,$$
$$x_{2,k} = x_{2,k-1}^2 \qquad \text{for} \quad p = 2,\ k \geqq 3,$$
$$x_{n,0} = v_n \qquad \text{for} \quad n > 2,$$
$$x_{n,1} = x_{n,0}^p - v_{n-1}^p v_n^{-1} v_{n+1},$$
$$x_{n,k} = x_{n,k-1}^p \qquad \text{for } 1 < k \not\equiv 1\ (n-1),$$
$$x_{n,k} = x_{n,k-1}^p - v_{n-1}^b v_n^{(k)-(k-1)+1} \ (b = b_{n,k}) \quad \text{for } 1 < k \equiv 1\ (n-1),$$

where $b_{n,k}$ is an integer given by

(2.10) $$b_{n,k} = (p^{k-1} - 1)(p^n - 1)/(p^{n-1} - 1) \quad \text{for } 1 < k \equiv 1\ (n-1),\ n \geqq 2,$$

and the integers $a_{n,k} \geqq 1$ are:

(2.11)
$$a_{1,0} = 1,$$
$$a_{1,k} = k + 2 \qquad \text{for } p = 2,\ k \geqq 1,$$
$$a_{1,k} = k + 1 \qquad \text{for } p > 2,\ k \geqq 1,$$

$$a_{2,0} = 1,$$
$$a_{2,1} = p$$
$$a_{2,k} = p^k + p^{k-1} - 1 \qquad \text{for } p > 2,\ k \geqq 1,$$
$$a_{2,k} = 3{\cdot}2^{k-1} \qquad \text{for } p = 2,\ k \geqq 2,$$
$$a_{n,0} = 1,$$
$$a_{n,1} = p,$$
$$a_{n,k} = pa_{n,k-1} \qquad \text{for } 1 < k \not\equiv 1\,(n-1),$$
$$a_{n,k} = pa_{n,k-1} + p - 1 \qquad \text{for } 1 < k \equiv 1\,(n-1).$$

Let n be a fixed integer greater than 1. Put $\langle k \rangle = 1 + k(n-1)$ for an integer $k \geqq 0$. Then we see easily that

(2.12) $$a_{n,k} < p^k + p^{k-n+1},$$

expect for $n = 2$ and $p = 2$, and that

(2.13) $$a_{n,\langle k \rangle} = p^{\langle k \rangle} + (p-1)(p^{\langle k \rangle - 1} - 1)/(p^{n-1} - 1) \quad \text{for } n \geqq 2,\ k \geqq 0,$$
$$a_{n,\langle k \rangle} = b_{n,\langle k \rangle} + p \qquad \text{for } n \geqq 2,\ k \geqq 1.$$

Note that $a_{n,k} = p^k$ if $n > k$ (cf.(2. 12)).

The definition (2.9) gives

(2.14) $$x_{n,\langle k \rangle} \equiv v_n^{(\langle k \rangle)} - v_{n-1}^{(\langle k \rangle) - (\langle k \rangle - n)} v_n^{-(\langle k \rangle - 1)} y_k \bmod (p) \quad \text{for} \quad n \geqq 2,\ k \geqq 1$$

for a certain element $y_k \in BP_*$.

Consider a sequence of positive integers

$$E\colon e, s_1, \cdots, s_k, \cdots$$

which satisfies

(2.15) $$s_k = e_k p^{i_k},\ p \nmid e_k \quad \text{for} \quad k \geqq 1,$$

$0 < e \leqq i_1 + \delta, 0 < e_k \leqq a_{k+1}$ and $e_k \leqq p^{u_{k+1}}$ if $e_{k+1} = 1$ and if $k \geqq 2$ or $p = 2$,

and moreover

$$e = i_1 + 2 \geqq 3 \quad \text{and} \quad e_1 \geqq 3 \text{ if } \delta = 2,$$

where

(2.16) $$u_k = i_k - i_{k-1} - e + 1 \geqq 0 \quad \text{for} \quad k \geqq 2,$$
$$a_k = a_{k,u_k},$$

and δ is an integer such that

(2.17) $\delta = 1$ if p is odd, and $\delta = 1, 2$ if $p = 2$.

For the sequence E, define the sequence

(2.18) $$V^E\colon \alpha_0, \alpha_1, \cdots, \alpha_k, \cdots$$

with $\alpha_k \in BP_*$ by

(2.19)
$$\begin{aligned}
&\alpha_0 = p^e\\
&\alpha_1 = v_1^{s_1} \text{ if } p > 2 \quad\text{or}\quad i_1 = 0, \text{ and}\\
&\alpha_1 = x_{1,1}^g \quad\text{otherwise,}\\
&\alpha_k = v_k^{s_k} \text{ for } k > 1 \quad\text{if}\quad e_k = 1, \text{ and}\\
&\alpha_k = x_{k,u_k}^f \quad\text{otherwise,}
\end{aligned}$$

where $g = s_1/2$, $f = e_k p^u$ and $u = i_{k-1} + e - 1$. We denote V_k^E for the subsequence

(2.20) $$V_k^E\colon \alpha_0, \alpha_1, \cdots, \alpha_{k-1}$$

of V^E.

On the above equalities, $\alpha_k = x$ means

(2.21) $$\alpha_k = x^* \text{ if } x = x^* + \Delta x \text{ and } \Delta x \in (V_k^E),$$

where (V_k^E) denotes the ideal of BP_* generated by V_k^E.
The next proposition shows that this definition is well-defined.

PROPOSITION 2.22. *The element α_k in* (2.18) *belongs to BP_* for each k.*

PROOF. We proceed by induction on k. The case $k = 0$ is clear since $\alpha_0 = p^e$. For $k = 1$, we compute

$$\alpha_1 \equiv x_{1,1}^g \equiv v_1^{s_1} - 2s_1 v_1^{s_1-3} v_2 \mod (2^e)$$

if $p = 2$ and $i_1 \geqq 1$, and

$$\alpha_1 = v_1^{s_1}$$

if $p > 2$, or $i_1 = 0$. Therefore, we see that $\alpha_1 \in BP_*$. Besides, we have

(2.23)
$$\begin{aligned}
&v_1^{s_1} \equiv 2s_1 v_1^{s_1-3} v_2 \mod(V_2^E) \quad\text{if}\quad p = 2 \quad\text{and}\quad i_1 \geqq 1, \text{ and}\\
&v_1^{s_1} \equiv 0 \mod(V_2^E) \quad\text{otherwise,}
\end{aligned}$$

and so,

(2.24) $$v_1^s \equiv 0 \mod(V_2^E) \quad\text{if}\quad s \geqq s_1\ (+3 \quad\text{if}\quad p = 2, \delta = 2 \quad\text{and}\quad i_1 \geqq 1).$$

For $k \geqq 2$ we note the following consequence of the binomial theorem.

OBSERVATION 2.25. In BP_*, if $x \equiv y \mod(p, \alpha)$, then

$$x^{(n)} \equiv y^{(n)} \bmod(p^{n+1},\ p\alpha,\ \alpha^p) \text{ for } n \geqq 1.$$

Suppose that $\alpha_j \in BP_*$ for $0 \leqq j \leqq k-1$, and that

(2.26) $\quad pv_n^s \equiv 0 \bmod(V_{n+1}^E)$ if $s \geqq s_n$, and

$$v_n^s \equiv 0 \bmod(V_{n+1}^E) \text{ if } s \geqq s_n(+ p^{i_n-e+1} + p^{i_n-e} \text{ if } e_n > 1),$$

for $n \leqq k-1$.

If $u_k = 0$, then $\alpha_k = v_k^f$ by (2.19), and so $\alpha_k \in BP_*$ and further (2.26) holds. For $1 \leqq u_k < k$, we obtain the congruence

$$x_{k,u_k}^{f'} \equiv v_k^{s'_k} - e_k v_{k-1}^{(l)} v_k^{s'_k-(l)-(l-1)} v_{k+1}^{(l-1)} \bmod (p,\ \alpha)$$

by (2.9) where $s'_k = p^l e_k$, $l = i_k - e + 1$, $f' = e_k p^{i_k-1}$, and $\alpha = v_{k-1}^{s_{k-1}}$. Note that $f = p^{e-1} f'$ and apply Observation 2.25 to this, and we get

$$x_{k,u_k}^{f} \equiv v_k^{s_k} - e_k p^{e-1} v_{k-1}^{(l)} v_k^{s_k-(l)-(l-1)} v_{k+1}^{(l-1)} \bmod (p^e,\ p\alpha,\ \alpha^p).$$

In case $u_k \geqq k$, similar calculations with (2.14) give

$$\alpha_k \equiv v_k^{s_k} - p^{e-1} v_{k-1}^{(l)-(l-k)} v_k^{s_k-(l)-(l-1)} y \bmod (p^e,\ p\alpha,\ \alpha^p)$$

for some $y \in BP_*$. Since $(p^e,\ p\alpha,\ \alpha^p) \subset (V_k^E)$ by (2.26), the above congruences imply

$$\alpha_k \in BP_*$$

(see (2.21)), and (2.26) for $n = k$, which shows the proposition and (2.26) inductively.
q. e. d.

§3. Invariant sequences

We recall [2] the definitions of invariant and regular sequences. Let $\alpha_0, \cdots, \alpha_n$ be a sequence of elements of BP_*. We call the sequence *invariant* if $d\alpha_0 = 0$ and $d\alpha_k \in (\alpha_0, \cdots, \alpha_{k-1})$ for $k > 0$, and *regular* if $(\alpha_0, \cdots, \alpha_n)$ is a proper ideal, $a_0 \neq 0$ and α_k is not a zero-divisor on $BP_*/(\alpha_0, \cdots, \alpha_{k-1})$ for $k > 0$ (see (2.6) for d).

We prepare some lemmas to prove Theorems **E**, **O**, and 0. To state lemmas, we use the following notations.

For a given ideal $J = (p,\ \alpha_1, \cdots, \alpha_k,\ \alpha)$, we denote an ideal $J^{i,n}$ by

(3.1) $$J^{i,n} = (p^{i+1},\ \alpha_1^{(n)}, \cdots, \alpha_k^{(n)},\ p\alpha^{(n)},\ \alpha^{(n+v)}),$$

where $v = \min\{i,\ 1\}$.

For u_n in (2.16), we provide the integer a_n by

(3.2) $$a_n = a_{n,u_n}.$$

LEMMA 3.3. *Let* $K = (p,\ v_1, \cdots, v_{n-2},\ v_{n-1}^{a_n})$. *Then*

$$K^{e-1,i_{n-1}} \subset (V_n^E),$$

if $n \geqq 3,\ e \geqq 2$.

PROOF. We shall show that every generator of $K^{e-1,i_{n-1}}$ belongs to (V_k^E). It is clear that $p^e \in (V_n^E)$. By using (2.12), (2.16) and (2.15), we get the inequality

$$p^j > p^{i_k}a_{k+1} + p^{i_k-e+1} + p^{i_k-e}\ (j = i_{n-1}) \text{ if } e-1 \geqq 2 \quad \text{or} \quad n \geqq 4$$

$$p^{i_2} > s_1\ (+3 \text{ if } \delta = 2 \text{ or } u_2 \geqq 2) \text{ if } e-1 = 1 \quad \text{and} \quad n = 3,$$

for $1 \leqq k \leqq n-2$, which implies that

$$v_k^{(j)} \in (V_n^E)$$

by (2.26) and (2.24).

We also have inequalities

$$p^j a_n \geqq p^j a_n \text{ and}$$

$$p^{j+1}a_n > p^j a_n + p^{j-e+1} + p^{j-e}$$

by (2.12) and (2.15). Use (2.26) again, and both generators $p\alpha$ and $\alpha^p (\alpha = v_{n-1}^{p^j a_n})$ belong to (V_n^E).

q. e. d.

To prove that the sequences are invariant, we use the following lemma related on d in (2.6).

LEMMA 3.4. *Let* $p \geqq 2$ *and* $J = (p, \alpha_1, \cdots, \alpha_k, \alpha)$. *In* BP_*BP, *if* $dx \equiv 0 \bmod J$ *for* $x \in BP_*$, then $dx^{(n+i)} \equiv 0 \bmod J^{i,n}$.

PROOF. Since $dx \equiv 0 \bmod J$, $dx = a\alpha + pb + \sum c_j\alpha_j$ for some $a, b, c_j \in BP_*BP$. Then

$$\begin{aligned} dx^{(n)} &\equiv (x + a\alpha + \textstyle\sum c_j\alpha_j)^{(n)} - x^{(n)} \\ &\equiv 0 \bmod J^{0,n}, \end{aligned}$$

which shows

$$dx^{(n)} = a'\alpha^{(n)} + pb' + \sum c_j'\alpha_j^{(n)} \qquad (a', b', c_j' \in BP_*BP).$$

Hence

$$\begin{aligned} dx^{(n+i)} &\equiv (x^{(n)} + a'\alpha^{(n)} + pb')^{(i)} - x^{(n+i)} \\ &\equiv (x^{(n)} + a'\alpha^{(n)})^{(i)} - x^{(n+i)} \\ &\equiv 0 \mod J^{i,n} \end{aligned}$$

q. e. d.

§4. Proofs of Theorems

In this section we shall prove Theorems stated in §1.

PROOF OF THEOREM **E**. For $n = 1$, it is trivial. For $n = 2$, see [2; pp. 503–504]. We easily read off the case $n = 3$ from [6; Lemma 2.5] and [5; Th. 1.5].

Now proceed by induction. Let $n \geqq 4$. The definition (2.19) indicates

$$\alpha_n = \beta_n^{(u)}$$

for $u = i_{n-1} + e - 1$ and $\beta_n = x_{n,u_n}^{e_n}$. Lemma 2.7 shows

$$d\beta_n \equiv 0 \bmod K$$

where $K = (p, v_1, \cdots, v_{n-2}, v_{n-1}^{a_n})$ (see (3.2) for a_n). Therefore it follows from Lemma 3.4 that

$$d\alpha_n \equiv 0 \bmod K^{e-1, i_{n-1}}.$$

On the other hand, we have

$$K^{e-1, i_{n-1}} \subset (V_n^E)$$

by Lemma 3.3. Hence

$$d\alpha_n \equiv 0 \bmod (V_n^E),$$

and V_{n+1}^E is invariant regular. Thus we complete the induction. q. e. d.

PROOF OF THEOREM **O**. For $n \leqq 3$, we can read off the result from [2], [6; Lemma 2.5], and [3; Th. 6.1] as the proof of Theroem **E**.

For $n \geqq 4$, we also use (2.19), Lemmas 2.7, 3.4, and 3.3 as the proof of Theorem **E** to complete the induction.

q. e. d.

PROOF OF THEOREM 0. For odd prime p, it is proved in [6; Prop. 3.8]. We have the case $p = 2$ by noticing that the proof for odd p *ibid.* is also valid for $p = 2$.

q. e. d.

§5. Concluding remarks

We studied about a pre-MRW sequence in [6], and added some more conditions in order to prove that the sequence of BP_* arisen from the pre-MRW sequence is invariant regular. We here call a sequence of integers *pre-MRW* if it satisfies (2.5). In this paper we have proved this invariantness without any conditions to be added if $e \geqq 2$. Therefore we can rewrite [6; Prop. 3.9] together with our theorems in §1 as follows:

THEOREM 5.1. *Let E be a sequence of integers e, s_1, $s_2, \cdots$, and suppose $e \geqq 2$. Then the sequence V_n^E is invariant regular if and only if the sequence E satisfies* (2.15).

We note that the case $p = 2$ is also valid though [6; Prop. 3.9] treats only the case $p > 2$.

As an application of invariant regular sequence, we know that we can construct a spectrum YJ with $BP_* YJ = v_n^{-1} BP_*/(J)$ for an invariant regular sequence J of length n and an odd prime p such that $n^2 + n < 2p$ ([7; Th. 5.7]). Thus Theorem **O** shows us some examples of spectra YJ.

We also know that an invariant regular sequence gives rise to an element of the E_2-term of the chromatic spectral sequence (cf. [6; Lemma 2.5]) which converges to the E_2-term of the Adams-Novikov spectral sequence converging to the p-component of the stable homotopy groups of spheres (cf. [3]). This means that an invariant regular sequence in the theorems in §1 may survive to the stable homotopy and give a new element in it.

We lastly note that the detail computations of this text will appear in [9].

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