Note on the Right Unit Map and Some Elements of the Brown-Peterson Homology

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(*Received August 31, 1989*)

§1. Introduction

In their paper [3], Miller, Ravenel, Wilson defined elements $x_{n,k}$ to determine the E_2-term $E_2^{s,t}(n)$ $(n \geqq -1)$ of the chromatic spectral sequence which converges to the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy of the Smith-Toda spectrum $V(n)$ at a prime p. Here $V(-1)$ denotes the sphere spectrum localized at p. They gave $E_2^{s,t}(-1)$ for $s+t \leqq 2$ and $p>2$, and $E_2^{s,t}(n)$ for $(s, t)=(0, 1)$ and all primes p. We need to investigate the elements $x_{n,k}$ to proceed the computation according to the way they gave. In this paper we redefine the elements and gave similar but deeper results than theirs (see Propositions 3.1 and 4.1). The results of §3 will be used to compute the E_2-term $E_2^{1,1}(1)$ of the chromatic spectral sequence for $V(1)$ in a forthcoming paper. In this way we shall also use the results of §4 to determine the other E_2-terms of the chromatic spectral sequences.

Let BP denote the Brown-Peterson ring spectrum at a prime number p. The unit map of the ring spectrum induces the right and the left units of BP_*BP. We regard the left unit as the inclusion $BP_* \subset BP_*BP$. Quillen [5] gave the formula for the right unit in $BP_*BP \otimes \boldsymbol{Q}$, and Ravenel simplified it modulo some ideal in BP_*BP in [6] and translated it explicitly in BP_*BP in [8; §4]. In §2, we rewrite the Ravenel's formula (Proposition 2.1) and use it to compute the image of the right unit map for the Moreira's elements $u_{n,k}$ which is defined only for $k \leqq n$ in [2] and for $k>n$ in (2.8) here (Proposition 2.2). We define the elements X_k in §3, which is congruent to $x_{3,k}$ of [1] modulo some ideal, by using the Moreira's elements, and compute their differentials in a cobar complex. The manner to define X_k gives the similar elements for $x_{n,k}$ with $n>3$ and we compute their differential similarly to the case $n=3$ in §4.

§2. On the right unit of BP_*BP

Let BP denote the Brown-Peterson spectrum at a fixed prime p. Then it gives us the Hopf algebroid (BP_*, BP_*BP) (cf. [8]) and

$$BP_* = Z_{(p)}[v_1, v_2, \cdots], BP_* \otimes \boldsymbol{Q} = \boldsymbol{Q}[l_1, l_2, \cdots] \text{ and } BP_*BP = BP_*[t_1, t_2, \cdots],$$

where $|v_i| = |l_i| = |t_i| = 2p^i-2$, $l_i = [\boldsymbol{CP}^{p^i}/p^i]$ and v_i are Hazewinkel's generators given by the equality:

$$v_n = pl_n - \textstyle\sum_{i=1}^{n-1} v_{n-i}^{p^i} l_i.$$

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A summation $\sum^F$ of the formal group law F associated with BP is given by

$$\sum^F a_i = \exp(\sum \log a_i)$$

where

$$\log x = \sum_{i \geqq 0} l_i x^{p^i} \ (l_0 = 1) \text{ and } \exp(\log x) = x.$$

The right unit η: $BP_* \to BP_*BP$ satisfies the following congruence:

(2.1)[6; Th. 1] $$\sum^F_{i,j \geqq 0} t_i \eta v_j^{p^i} \equiv \sum^F_{i,j \geqq 0} v_i t_j^{p^i} \bmod (p) \ (v_0 = p).$$

In order to study this, we recall [8; §4. 3] the following: For a set S, we inductively define a polynomial $w_k(S)$ in the polynomial ring $\boldsymbol{Z}[S]$ by the equality

$$\sum_{s \in S} s^{p^n} = \sum_k p^k w_k(S)^{p^{n-k}}$$

Note here that $w_0(S) = \sum_{s \in S} s$ and $w_1(S) = (\sum_{s \in S} s^p - w_0(S)^p)/p$.

Let $J = (j_1, j_2, \cdots, j_m)$ ($m \geqq 0$, including $J = \phi$) be a sequence of positive integers. Then we use the following notation:

$$|J| = m, \ \|J\| = \sum_k j_k; \ w_J(S) = w_{|J|}(S)^{p^{\|J\| - |J|}}; \text{ and } v_J = v_j(v_{J'})^{p^j} \in BP_* \text{ for } J = (j, J').$$

For a subset $A = \{a_{i,j}\}$ of BP_*BP, we set the subset

(2.2) $$A_n = \{a_{i,j} | i + j = n\} \cup_{|J| > 0} \{v_J w_J(A_{n - \|J\|})\} \quad \textit{for } n \geqq 0$$

of BP_*BP and obtain

(2.3) [8; Lemma 4.3.11] $$\sum^F_{i,j} a^{i,j} \equiv \sum^F_{n > 0} w_0(A_n) \bmod (p) \textit{ in } BP_*BP.$$

Considering the homogeneous part of a formal sum, we find the following:

(2.4) *If* $\sum^F_{n > 0} a_n \equiv \sum^F_{n > 0} b_n \bmod (p)$ *and* $|a_n| = |b_n| = 2p^n - 2$ *for each* n, *then* $a_n \equiv b_n \bmod (p)$.

We now define the element

$$T_0 = 1 \textit{ and } T_i = t_i^p + \sum_I v_I^p \{w_{|I|+1}(S_{i - \|I\|}) - w_{|I|+1}(R_{i - \|I\|})\}^{p^{\|I\| - |I|}} \ (i \geqq 1)$$

for the subsets $S = \{v_i t_j^{p^i}\}$ and $R = \{t_i \eta t_j^{p^i}\}$ of BP_*BP (cf. (2.2)). Note that it satisfies the following congruence:

(2.5) $$T_i \equiv t_i^p \bmod (p, v_1, \cdots, v_{i-1}) \quad \textit{for } i \geqq 1.$$

PROPOSITION 2.1. *For* $n \geqq 1$, *we have the congruence*

$$\sum_{i=0}^n t_i \eta v_{n-i}^{p^i} \equiv \sum_{i=0}^n v_i T_{n-i}^{p^{i-1}} \bmod (p) \quad (v_0 = p, \ t_0 = 1).$$

PROOF. It follows immediately from (2. 1–3) that

$$\sum^F_{n > 0} w_0(R_n) \equiv \sum^F_{i,j} t_i \eta v_j^{p^i} \equiv \sum^F_{i,j} v_i t_j^{p^i} \equiv \sum^F_{n > 0} w_0(S_n) \bmod (p).$$

Since $|R_n| = |S_n| = 2p^n - 2$, we apply (2.4) to obtain

(2.6) $$w_0(R_n) \equiv w_0(S_n) \bmod (p).$$

On the other hand consider a sequence J with $|J| > 0$, and we have $J = (j, I)$ for some integer $j > 0$ and sequence I, and

$$|J| = |I| + 1, \ \|J\| = \|I\| + j, \text{ and } v_J = v_j(v_I)^{p^j}.$$

These equalities imply

$$w_0(\cup_{|J|>0} v_J w_J(A)) \equiv \sum_{j>0,I} v_j\{v_I^p w_{|I|+1}(A)^{p^{\|I\|-|I|}}\}^{p^{j-1}} \bmod (p)$$

for a set A, since $w_0(A \cup B) = w_0(A) + w_0(B)$. Applying this congruence to (2.6), we have the lemma by (2.2). q.e.d.

By way of example, we have the well known formulae (cf. [6]):

(2.7) $$\eta v_n \equiv v_n + v_{n-1} t_1^{p^{n-1}} - v_{n-1}^p t_1 \bmod (p, v_1, \cdots, v_{n-2}), \text{ and}$$

$$\eta v_4 \equiv v_4 + v_3 t_1^{p^3} + v_2 t_2^{p^2} - t_1 \eta v_3^p - v_2^{p^2} t_2 \bmod (p, v_1).$$

We now study the elements $u_{n,i}$ of $v_n^{-1}BP_*$ for each $n \geqq 2$ and $i \geqq 0$ defined by:

(2.8) (cf. [2; p. 1263–4]) *$u_{n,0} = v_n^{-1}$, and $\sum_{i+j=r} v_{n+i} u_{n,j}^{p^i} = 0$ for $r \geqq 1$, which also satisfy the congruence*

$$\sum_{i+j=r} u_{n,i} v_{n+j}^{p^i} \equiv 0 \bmod (p) \quad \textit{for } r \geqq 1.$$

Consider the algebra

$$C(n) = \boldsymbol{Z}/p[v_n, v_n^{-1}, v_{n+1}, v_{n+2}, \cdots; t_1, t_2, \cdots].$$

Then we have the algebra

$$B = B(n, r) = C(n) \otimes_Z \boldsymbol{Z}/p[F]/(F^{r+1}) \quad \text{for } n \geqq 2 \text{ and } r \geqq 1.$$

used in [2] whose multiplication is given by

$$(c_1 \otimes F^i)(c_2 \otimes f) = c_1 c_2^{p^i} \otimes F^i f$$

for $c_1, c_2 \in C(n)$ and $f \in \boldsymbol{Z}/p[F]/(F^{r+1})$. We make use of the following notation:

(2.9) *For an element $x \in v_n^{-1}BP_*$, $e_n(x)$ denotes an element of $C(n)$ such that*

$$\eta x \equiv e_n(x) \bmod I_n.$$

Define elements

(2.10) $$w_{n,0} = 0, \textit{ and } w_{n,r} = \sum_{j=1}^r e_n(u_{n,r-j}^{p^{j-1}}) T_j^{p^{n-2}} \ (n \geqq 2, \ r \geqq 1) \textit{ in } v_n^{-1}BP_*,$$

and we obtain the following

Proposition 2.2. *Let n and r be non-negative integers with $n \geqq 2$. Then we have*

$$\eta u_{n,r} \equiv \sum_{i+j=r} u_{n,i} t_j^{p^i} - w_{n,r}^p - v_{n-1} w_{n,r+1} \eta v_n^{-1} \bmod I_{n-1} + (v_{n-1}^p)$$

in $v_n^{-1}BP_*BP$, *where* $I_k = (p, v_1, v_2, \cdots, v_{k-1})$.

PROOF. Let J denote the ideal $I_{n-1} + (v_{n-1}^p)$. In case $r = 0$, it follows immediately from (2.7), since we have

(2.11) $$v_n \eta v_n^{-1} \equiv 1 - v_{n-1} t_1^{p^{n-1}} \eta v_n^{-1} \bmod J.$$

The congruence $\eta u_{n,i} \equiv e_n(u_{n,i}) \bmod I_n$ implies $\eta u_{n,r-i}^{p^i} \equiv e_n(u_{n,r-i}^{p^i}) \bmod J$ for $i \geqq 1$. Besides we have the congruences

$$\eta v_{n+i} \equiv e_n(v_{n+i}) + v_{n-1} T_{i+1}^{p^{n-2}} \bmod J \quad \text{for } i \geqq 0$$

by Proposition 2.1, and the equality

$$\eta u_{n,r} = -\eta v_n^{-1} \textstyle\sum_{i=1}^r \eta v_{n+i} \eta u_{n,r-i}^{p^i}$$

given by (2.8). Then we obtain $\eta u_{n,r} \equiv e_n(u_{n,r}) - v_{n-1} w_{n,r+1} \eta v_n^{-1} \bmod J$ by using (2.11).

Now we study $e_n(u_{n,r})$ in a same way as that of [2: (7)]. Consider the elements

$$L = \textstyle\sum_{i=0}^r t_i F^i, \quad M = \sum_{i=0}^r v_{n+i} F^i, \quad N = \sum_{i=0}^r \eta v_{n+i} F^i, \quad \text{and} \quad L' = \sum_{i=0}^r T_i^{p^{n-1}} F^i$$

of $B^* (t_0 = 1)$. By (2.8), we have

$$M^{-1} = \textstyle\sum_{i=0}^r u_{n,i} F^i, \quad N^{-1} = \sum_{i=0}^r \eta u_{n,i} F^i.$$

Proposition 2.1 shows that $LN = ML'$ in B, and so

$$M^{-1} L = L' N^{-1},$$

which gives $\sum_{i+j=r} u_{n,i} t_j^{p^i} \equiv \eta u_{n,r} + w_{n,r}^p \bmod I_n$ by comparing the coefficient of F^r.

q. e. d.

The following is a corollary of Proposition 2.2.

PROPOSITION 2.3. *Let n and r be positive integers with $n \geqq 2$. Then we have*

$$d_1 w_{n,r} \equiv -\textstyle\sum_{0<i<r} w_{n,i} \otimes t_{r-i}^{p^{i-1}} + C_{n,r} \bmod I_n,$$

where $v_{n-1} C_{n,r} \equiv \Delta_n t_{r-1} - \Delta t_{r-1} \bmod I_{n-1}$ ($\Delta_n t_k \in C(n) \otimes_{BP_*} BP_*BP$: $\Delta_n t_k \equiv \Delta t_k \bmod I_n$), *especially* $C_{n,r} = 0$ *if* $r \leqq n$, *and* $C_{n,n+1} = b_{n-2}$ (*by* [8; Corollary 4.3.15]). (*See* (3.1.2) *for the definition of* d_i).

PROOF. First we notice that

(2.12) *If* $d_0 x \equiv y + v_{n-1} z \bmod J$ *and* $d_1 y \equiv w + v_{n-1} u \bmod J$ *with* $w \in C(n)$, *then* $w = 0$ *and* $d_1 z \equiv -u \bmod I_n$, *where* $J = I_{n-1} + (v_{n-1}^2)$.

In fact, we have $w + v_{n-1} u + v_{n-1} d_1 z \equiv 0 \bmod J$ by applying d_1, which implies $w \equiv 0 \bmod I_n$, and hence $w = 0$ since $w \in C(n)$. Let $d_{n,i} x \in C(n) (\otimes_{BP_*} BP_*BP$ if $i = 1)$ be an element satisfying $d_{n,i} x \equiv d_i x \bmod I_n$ for $i = 0$ or 1. Then, $d_0 u_{n,i} \equiv d_{n,0} u_{n,i} - v_{n-1} v_n^{-1} w_{n,i+1}$ and $d_1 t_{r-1} \equiv d_{n,1} t_{r-1} + v_{n-1} C_{n,n+1}$ by definition. Therefore, the hypothesis of (4.4) holds by Proposition 2.2 for

$$x = u_{n,r-1},\ y = d_{n,0}u_{n,r-1},\ z = -v_n^{-1}w_{n,r},$$
$$w = \textstyle\sum_{0<i<r}(d_{n,0}u_{n,i}\otimes t_{r-1-i}^{p^i} + u_{n,i}d_1 t_{r-1-i}^{p^i}) + u_{n,0}d_{n,1}t_{r-1} - d_1 w_{n,r-1}^p$$
$$\text{and } v_n u = -\textstyle\sum_{0<i<r} w_{n,i}\otimes t_{r-i}^{p^{i-1}} + C_{n,r},$$

and we have the proposition. q. e. d.

§3. Some elements of $v_3^{-1}BP_*$ and their differentials

In this section we assume that the prime p is odd.

For a BP_*BP-comodule M with coaction ψ, the cobar complex (Ω^*M, d_*) is

(3.1.1) $\Omega^t M = M\otimes_{BP_*}BP_*BP\otimes_{BP_*}\cdots\otimes_{BP_*}BP_*BP$ (t copies of BP_*BP)

and differential $d_t\colon \Omega^t M\to\Omega^{t+1}M$ given by

(3.1.2) $$d_0 m = \psi m - m\otimes 1,\ d_1 m\otimes x = d_0 m\otimes x + m\otimes d_1 x,$$
$$d_1 x = 1\otimes x - \Delta x + x\otimes 1 \text{ and}$$
$$d_t m\otimes x\otimes y = (d_1 m\otimes x)\otimes y - m\otimes x\otimes d_{t-1}y$$

for $m\in M$, $x\in\Omega^1 BP_*$ and $y\in\Omega^{t-1}BP_*$. The homology of the complex (Ω^*M, d_*) is denoted by

(3.1.3) $$H^*M = \mathrm{Ext}^*_{BP_*BP}(BP_*, M).$$

From here on we use the following notation:

(3.2) $M = v_3^{-1}BP_*$, *the* BP_*BP*-comodule with coaction* η, *and*

$$M\Gamma = v_3^{-1}BP_*BP = \Omega^1 M;$$

$J(k) = (p, v_1, v_2^k)$, *the invariant ideal of* M *or* $M\Gamma$ *generated by the every entry; and*

$$u_i = u_{2,i},\ U_i = u_{3,i},\ w_i = w_{2,i},\ \text{and } W_i = w_{3,i}.$$

Now define elements X_i of M by

(3.3) $$X_0 = v_3,\ X_1 = v_3^p + z(1),\ X_2 = X_1^p - z(2),$$
$$X_3 = X_2^p + y(3) + z(3),\ X_4 = X_3^p - u(4) + z(4),$$
$$X_i = X_{i-1}^p + y(i) - u(i) + z(i) \text{ for odd } i\geqq 5, \text{ and}$$
$$X_i = X_{i-1}^p - 2u(i) + z(i) \text{ for even } i\geqq 6.$$

Here,

(3.3.1) $$z(1) = v_2^p v_3^p U_1,\ z(2) = v_2^{p^2+p}v_3^{p^2-p}U_2,\ \text{and } z(i) = v_2^{b(i)}v_3^{c(i)}U_3\ (i\geqq 3);$$
$$y(i) = v_2^{a(i)+1}X_{i-1}^{p-1}u_1\ (i\geqq 3);\ \text{and } u(i) = v_2^{b(i)}X_{i-2}^{c(2)}u_2\ (i\geqq 4),$$

for the integers

(3.3.2) $a(i) = p^i + (p-1)(p^{i-1}-1)/(p^2-1)$ (*odd* $i \geqq 1$), *and*

$$a(i) = pa(i-1) \text{ (even } i \geqq 2);$$

$$b(i) = p^{i-2}(p^2+p+1) \ (i \geqq 2); \text{ and } c(i) = p^{i-2}(p^2-p-1) (i \geqq 2).$$

Note that $a(i) = a_{3,i}$ and $X_i \equiv x_{3,i}$ mod $J(a(i)+1)$ for $a_{3,i}$ and $x_{3,i}$ of [1], and that $b(i) = a(i) + a(i-1) + 1$ for $i \geqq 2$.

Consider the Hopf conjugation $c: BP_*BP \to BP_*BP$, and we have

$$ct_1 = -t_1,\ ct_2 = -t_2 + t_1^{p+1}, \text{ and } ct_3 \equiv -t_3 - t_2 ct_1^{p^2} - t_1 ct_2^p \bmod (p, v_1)$$

(cf. [1:(1.4)]). In [2; Def. 6.2.1] the cycle $\zeta_3 (= T_3$ in [2]) of $M\Gamma$ is defined by

(3.4.1) $$\zeta_3 = A - B^p + C^p,$$

where

(3.4.2) $A = U_2 t_1^{p^2} + U_1 t_2^p + U_0 t_3$, $B = U_1 ct_2^p + U_0^p ct_3^p$, and $C = U_0(t_2 ct_1^{p^2} + t_3)$.

It satisfies

(3.4.3) [2; Th. 6.2.1.1] $$d_0 U_3 \equiv \zeta_3 - \zeta_3^p \bmod J(1).$$

On the other hand we have $d_0 U_3 \equiv A - W_3^p$ mod $J(1)$ by Proposition 2.2. Then these imply

(3.4.4) $$\zeta_3^p \equiv -B^p + C^p + W_3^p \bmod J(1),$$

and so $\zeta_3 \equiv -B + C + W_3$ mod $J(1)$, which gives

(3.4.5) $W_3 \equiv D - B^p + (U_2 + U_0 t_2)t_1^{p^2}$ mod $J(1)$ *for* $D = U_1 t_1^{p^2+p} - U_0^p t_1^p ct_2^{p^2}$,

since $D = U_1(t_2^p + ct_2^p) + (U_0^p ct_3^p + C^p)$. We also use the cycle

(3.4.6) $\zeta_2 = u_0 t_2 - u_0^p ct_2^p + u_1 t_1^p (= T_2$ in [2]) of $v_2^{-1}BP_*BP$,

which satisfies

(3.4.7) $$d_0 u_2 \equiv \zeta_2 - \zeta_2^p \text{ and } \zeta_2 \equiv -u_0 ct_2 + w_2 \bmod I_2.$$

PROPOSITION 3.1 (cf. [1; Prop. 5.17]). Mod $J(1 + b(i))$,

$d_0 X_i \equiv v_2 t_1^{p^2} \equiv v_2 v_3 W_1$	$i = 0$,
$\equiv v_2^p v_3^{p-1} t_1 - v_2^{p+1} v_3^{p-1} W_2$	$i = 1$,
$\equiv v_2^{a(2)} X_1^{p-1} t_1^p - v_2^{b(2)} v_3^{c(2)} (u_0 t_2 - W_3)$	$i = 2$,
$\equiv v_2^{a(3)} X_2^{p-1} t_1 - v_2^{b(3)} X_1^{c(2)} (w_2 + u_0 t_2 - C - W_3)$	$i = 3$,
$\equiv v_2^{a(i)} X_{i-1}^{p-1} t_1^p - v_2^{b(i)} v_3^{c(i)} (\zeta_2 - \zeta_3 + u_0 t_2)$	*even* $i \geqq 4$,

$$\equiv v_2^{a(i)} X_{i-1}^{p-1} t_1 - v_2^{b(i)} v_3^{c(i)} (\zeta_2 - \zeta_3 + w_2) \qquad \textit{odd } i \geqq 5.$$

Here $b(0) = 1$ *and* $b(1) = p + 1$.

PROOF. We prove this in a same manner as [1; Prop. 5.4] using

(3.5) (cf. [1; Observation 5.8]) *Let* $x \in M$, $y \in M\Gamma$, *and* $I \subset M$ *be an ideal. If* $dx \equiv y$ mod$(p, I)M\Gamma$, *then* $dx^p \equiv y^p$ mod$(p, I^p)M\Gamma$.

Computation of the differential is based on (3.1.2).

For the case $i = 0$, it follows from (2.7) for $n = 3$. Inductively we see the cases for $i = 1, 2$ by (3.5) and Proposition 2.2. For $i \geqq 3$, we use the equalities:

(3.6) $a(i) - p + 1 = pa(i-1)$ *for odd* i, $X_{i-2}^{p^2-p} \equiv X_{i-1}^{p-1}$ mod $J(a(i-1) - p)$ *and*

$$u_0^p t_2^p + e_2(u_1) t_1^p = w_2,$$

given by (3.3) and (2.8–10).

We compute

$$d_0 y(3) \equiv v_2^{a(3)} X_2^{p-1} t_1 - v_2^{pa(2)} X_2^{p-1} t_1^{p^2} - v_2^{b(3)} X_1^{c(2)} e(u_1) t_1^p$$
$$- v_2^{b(3)} v_3^{c(3)} u_0 t_2 + v_2^{b(3)} v_3^{c(3)} U_0 t_2 c t_1^{p^2} + v_2^{b(3)} v_3^{c(3)} W_3 \text{ mod } J(1 + b(3))$$

by Proposition 2.2 and the case for $i = 2$. Proposition 2.2 also gives

$$d_0 z(3) \equiv v_2^{b(3)} v_3^{c(3)} (U_2 t_1^{p^2} + U_1 t_2^p + U_0 t_3 - W_3^p) \text{ mod } J(1 + b(3)).$$

Now we have the case for $i = 3$, noticing the congruences

$$v_2^{pa(2)} (X_1^{p^2-p} - X_2^{p-1}) t_1^{p^2} \equiv - v_2^{b(3)} v_3^{c(3)} U_2 t_1^{p^2} \text{ and}$$
$$v_2^{b(3)} v_3^{c(3)} (U_1 - u_0^p) t_2^p \equiv - v_2^{b(3)} X_1^{c(2)} u_0^p t_2^p \text{ mod } J(1 + b(3)).$$

Notice also

$$v_2^{b(4)} X_2^{c(2)} w_2^p \equiv v_2^{b(4)} (X_1^{c(3)} w_2^p - v_3^{c(4)} U_2 t_1^{p^2}) \text{ mod } J(1 + b(4)),\ v_2^{a(2)} u_2 = - u_1 X_1,$$
$$v_2^{p^2+p} w_2^p t_1^p \equiv - X_1 t_1^{p^2+p} + v_2^p v_3^p D \text{ mod } J(p^2),\ u_1 t_1^p - u_0^p t_1^{p^2+p} + u_0 t_2 = \zeta_2 - u_0^p t_2^p,$$
$$W_3 + v_2^{b(2)} U_0^{p+1} u_0 t_2 w_2^p \equiv D - B^p + U_2 t_1^{p^2} \text{ mod } J(1), \text{ and } X_1^p \equiv v_3^{p^2} \text{ mod } J(p^2),$$

which follow from (2.8–10), (3.3) and (3.4.5–6). Then we obtain

$$d_0 u(4) \equiv v_2^{a(4)} X_2^{(p-2)p} y(3) t_1^p + v_2^{b(4)} (- X_1^{c(3)} w_2^p + v_3^{c(4)} (\zeta_2 - u_0^p t_2^p + u_0 t_2 + B^p))$$

mod $J(1 + b(4))$ by the case for $i = 2$, Proposition 2.2 and (3.3). Now the case $i = 4$ follows from (3.4.4), (3.5) and the congruence

(3.7) $$d_0 z(i) \equiv v_2^{b(i)} v_3^{c(i)} (\zeta_3 - \zeta_3^p) \text{ mod } J(1 + b(i)) \ (i \geqq 4)$$

shown by (3.4.3).

For $i \geqq 5$, we use the congruence

(3.8) $$d_0u(i) \equiv v_2^{b(i)}v_3^{c(i)}(\zeta_2 - \zeta_2^p) \bmod J(1 + b(i)) \ (i \geqq 4)$$

shown by (3.4.7). If i is odd $\geqq 5$, then we see the proposition by the conguences (3.7–8) and

$$d_0y(i) \equiv v_2^{a(i)+1}X_{i-1}^{p-1}(u_0t_1 - u_0^pt_1^{p^2}) - v_2^{b(i)}v_3^{c(i)}e(u_1)t_1^p$$

mod $J(1 + b(i))$ obtained by Proposition 2.2. For even $i \geqq 6$, use

$$d_0u(i) \equiv v_2^{a(i)}X_{i-1}^{p-2}y(i-1)t_1^p + v_2^{b(i)}v_3^{c(i)}(u_0t_2 - w_2^p) \bmod J(1 + b(i)) \ (i \geqq 6)$$

given by Proposition 2.2 and (3.3), in place of $d_0y(i)$ of odd case. Thus we complete the induction. q.e.d.

§4. Some elements of $v_n^{-1}BP_*$ for $n \geqq 4$ and their differentials

In this section we give a similar result to Proposition 3.1 for $n \geqq 4$, and so we fix an integer $n \geqq 4$. Then we use the following notation as §3:

(4.1) *$M = v_n^{-1}BP_*$, the BP_*BP-comodule with coaction η, and*

$$M\Gamma = v_n^{-1}BP_*BP = \Omega^1M;$$

$J(k) = (p, v_1, \cdots, v_{n-2}, v_{n-1}^k)$ and $L(k) = J(p^k + p^{k-1} + \cdots + p^{k-n+1} + 1)$ the invariant ideals of M or $M\Gamma$; and

$$u_i = u_{n,i},\ U_i = u_{n,i},\ w_i = w_{n-1,i},\ W_i = w_{n,i},\ \text{and } e(x) = e_{n-1}(x).$$

We begin with the definition of the elements X_i:

(4.2) *$X_0 = v_n$, $X_i = X_{i-1}^p + z(i)$ for $1 \leqq i \leqq n-1$,*

$X_i = X_{i-1}^p + y(i)$ for $i = n-1+j$ with $1 \leqq j \leqq n-1$, and

$X_i = X_{i-1}^p + y(i) + (-1)^iu(i)$ for $i = k(n-1)+j$ with $1 \leqq j \leqq n-1$ and $k \geqq 2$.

Here the elements $z(i)$, $y(i)$ and $u(i)$ are given by:

(4.2.1)
$$z(i) = (-1)^{i-1}v_{n-1}^{a(i,i)}v_n^{p^i-a(i-1,i-1)}U_i,$$
$$y(i) = (-1)^{j-1}v_{n-1}^{a(i,j)+1}u_jX_{i-j}^{p^j-a(j-1,j)},\ \text{and}$$
$$u(i) = (-1)^iv_{n-1}^{b(i)}u_{i-n+1}X_{n-1}^{p^{i-n+1}-a(i-n,i-n+1)},$$

and the integers $a(i, k)$ and $b(i)$ are:

(4.2.2) $a(i, k) = 0$ *if $i < k-1$ or $k = 0$,*

$a(i, k) = p^i + p^{i-1} + \cdots + p^{i-k+1}$ *for $0 \leqq i-k+1 \leqq n-1$,*

$a(i, k) = pa(i-1, k)$ *for $0 < i-k+1 \not\equiv 1\ (n-1)$,*

$a(i, k) = pa(i-1, k) + p^k - 1$ *for $1 < i-k+1 \equiv 1\ (n-1)$, and*

(4.2.3) $$b(2n-1)=p^{2n-1}+p^{2n-2}+\cdots+p^{n+1}+2p^{n}, \quad \textit{and}$$
$$b(i)=p(b(i-1)+p^{n-1}) \quad \textit{for} \quad i \geqq 2n.$$

An easy calculation brings the equalities on these integers

(4.2.4) $$a(i, j)=p^{i}+p^{i-1}+\cdots+p^{i-j+1}+(p^{j}-1)(p^{i-j}-1)/(p^{n-1}-1), \quad \textit{and}$$
$$b(i, j)=p^{i}+p^{i-1}+\cdots+p^{i-n+2}+2p^{i-n+1}+p^{i-n}+\cdots+p^{n}.$$

Here the integer j for given integer i satisfies the following:

(4.3) $$i=k(n-1)+j \ \textit{with} \ 1 \leqq j \leqq n-1,$$

for an integer k. We have some relations between the integers defined in (4.2.2–3):

(4.4) $$a(i, j)=a(i, r)+a(i-r, j-r) \quad \textit{for} \quad 0 \leqq r \leqq j,$$
$$p^{r}a(i-j, r)+p^{r}=a(i-j+r, r)+1, \quad \textit{and}$$
$$b(i)+a(n-1, r)=p^{-r}b(i+r).$$

Using the Hopf conjugation c, Moreira defined the cycle ζ by:

(4.5)[2; Def. 6.2.1] $$\zeta=\sum_{1 \leqq r \leqq s \leqq n} U_{n-s}^{p^{s-r}}\left(\sum_{q=r}^{s} t_{q} c t_{s-q}^{p^{q}}\right)^{p^{n-r}},$$

and showed

(4.6)[2; Th. 6.2.1.1] $$d_{0} U_{n} \equiv \zeta-\zeta^{p} \bmod J(1).$$

On the other hand, Proposition 2.2 says

$$d_{0} U_{n} \equiv \sum_{r=1}^{n} U_{n-r} t_{r}^{p^{n-r}}-W_{n}^{p} \bmod J(1).$$

Therefore we obtain

(4.7) $$W_{n}^{p}=\zeta^{p}-\zeta+\sum_{r=1}^{n} U_{n-r} t_{r}^{p^{n-r}}, \text{ and}$$
$$W_{n}=\zeta-\zeta^{*},$$

where,

(4.8) $$\zeta^{*}=\sum_{1 \leqq r<s \leqq n} U_{n-s}^{p^{s-r-1}}\left(\sum_{q=r}^{s} t_{q} c t_{s-q}^{p^{q}}\right)^{p^{n-r-1}}.$$

In fact we have the following by comparing with (4.4):

(4.9) $$\zeta^{*p}=\zeta-\sum_{r=1}^{n} U_{n-r} t_{r}^{p^{n-r}}$$

To state our results, we define the following elements for the integers i and j in (4.3):

(4.10) $$A(i, r)=(-1)^{r+1} v_{n-1}^{a(i, r)} X_{i-r}^{p^{r}-a(r-1, r)} t_{r}^{p^{j-r}},$$
$$Az(i, r)=(-1)^{i+1} v_{n-1}^{a(i, i)} v_{n}^{p^{i}-a(i-1, i-1)} U_{i-r} t_{r}^{p^{i-r}} \quad \textit{for} \quad 1 \leqq i \leqq n-1,$$
$$Ay(i, r)=(-1)^{j+1} v_{n-1}^{a(i, j)+1} u_{j-r} X_{i-j}^{p^{j}-a(j-1, j)} t_{r}^{p^{j-r}} \quad \textit{for} \quad i \geqq n.$$
$$B(i, r, s)=(-1)^{j+r-s} v_{n-1}^{a(i, j+1-s)+p^{s-1} a(i-j, r)+p^{s-1}} e(u_{j-s+1}^{p^{s-1}}) X_{i-j-r}^{p^{j+r}-a(j+r-1, j+1+r-s)} t_{r}^{p^{n-2-r+s}},$$

$$By(i) = v_{n-1}^{a(i,j)+1} X_{i-j}^{p^j - a(j-1,j)} w_j^p,$$

$$Bu(i) = v_{n-1}^{b(i)} X_{n-1}^{p^{i-n+1} - a(i-n,i-n+1)} w_{i-n+1}^p,$$

$$C(i,r,s) = (-1)^{i-n+1+r-s} v_{n-1}^{p^{s-r-1}b(i+1+r-s)} e(u_{i-n+2-s}^{p^{s-1}}) X_{i-j-r}^{p^{i-n+1+r} - a(i-n+r,i-n+2+r-s)} t_r^{p^{n-r+s}},$$

$$D(i) = DZ(i) + \sum_{s=2}^n \sum_{r=s}^n DU(i, r, s),$$

$$DZ(i) = (-1)^{n-1} v_{n-1}^{p(i)} v_n^{q(i)} (\zeta^{p^{i-n+1}} - \zeta^*),$$

$$DU(i, r, s) = (-1)^{n-1} v_{n-1}^{p(i)} v_n^{q(i)} U_{n-r}^{p^{s-2}} t_r^{p^{n-2-r+s}},$$

$$E(i) = (-1)^i v_{n-1}^{a(i,i+1)} v_n^{p^i - a(i-1,i)} W_{i+1} \quad for \quad 1 \leqq i \leqq n-1,$$

$$E = E(n-1) = (-1)^{n-1} v_{n-1}^{a(n-1,n)} v_n^{p^{n-1} - a(n-2,n-1)} (\zeta - \zeta^*),$$

Here

(4.12) $$p(i) = p^i + \cdots + p^{i-n+1} \quad and \quad q(i) = p^i - \cdots - p^{i-n+1}.$$

Before stating our results we give some relations about these elements. The definition give rise to the following relations:

(4.13) $A(i, r) \equiv A(i-1, r)^p + Az(i, r) \bmod J(1 + a(i, i+1))$ *for* $1 \leqq r < i \leqq n-1$,

$A(i, r) \equiv A(i-1, r)^p + Ay(i, r) \bmod L(i)$ *for* $i \geqq n$ *if* $i - r \not\equiv 0\ (n-1)$,

$A(i, i) \equiv Az(i, i) \bmod J(1 + a(i, i+1))$ *for* $i \leqq n-1$, *and*

$A(i, j) \equiv Ay(i, j) \bmod L(i)$ *for* $i \geqq n$ *with* $i - j \equiv 0\ (n-1)$.

(4.14) $A(i, r)^p \equiv (-1)^{j+1} B(i+1, r, 2) \bmod L(i+1)$ *for* $r \geqq 1$ *and* $i \equiv 0\ (n-1)$.

(4.15) $B(i, r, s)^p \equiv B(i+1, r, s+1) \bmod L(i+1)$ *if* $i \not\equiv 0\ (n-1)$.

(4.16) $B(i, r, s)^p \equiv C(i+1, r, s+1) \bmod L(i+1)$ *if* $i \equiv 0\ (n-1)$.

(4.17) $C(i, r, s)^p \equiv C(i+1, r, s+1) \bmod L(i+1)$.

(4.18) $DU(i, r, s)^p \equiv DU(i+1, r, s+1) \bmod L(i+1)$.

(4.19) $v_{n-1}^{a(i,j)+1} e(u_j) X_{i-j}^{p^j - a(j-1,j)-1} A(i-j, r) \equiv (-1)^j B(i, r, 1) \bmod L(i)$.

(4.20) $v_{n-1}^{b(i)} e(u_{i-n+1}) X_{n-1}^{p^{i-n+1} - a(i-n,i-n+1)-1} A(n-1, r) \equiv (-1)^j C(i, r, 1) \bmod L(i)$.

By (2.9–10), we see that

$$By(i) \equiv \sum_{r=1}^j v_{n-1}^{a(i,j)+1} X_{i-j}^{p^j - a(j-1,j)} e(u_{j-r}^{p^r}) t_r^{p^{n-1}} \bmod (p, v_1, \cdots, v_{n-2}),$$

and so we get

(4.21) $By(i) \equiv -\sum_{r=1}^j (-1)^j B(i, r, r+1) \bmod L(i)$ *for* $i \geqq 2n-1$, *and*

$By(i) \equiv -\sum_{r=1}^j (-1)^j (B(i, r, r+1) + DU(i, r, r+1)) \bmod L(i)$ *for* $n \leqq i \leqq 2n-2$,

from the definition (4.2) and the relations (4.14–7). In a similar way, we have

(4.22) $$Bu(i) \equiv \sum_{r=1}^{n-1} C(i, r, r+1) - \sum_{r=1}^{n}(-1)^i DU(i, r, r+1) \bmod L(i).$$

PROPOSITION 4.1 (cf. [1]). a) $d_0 X_0 \equiv v_{n-1} t_1^{p^{n-1}} \bmod J(p)$.

b) $d_0 X_i \equiv \sum_{r=1}^{i} A(i, r) + E(i) \bmod J(1 + a(i, i+1))$ *for* $1 \leqq i \leqq n-1$.

b)′ $d_0 X_{n-1} \equiv \sum_{r=1}^{n-1} A(n-1, r) + E \bmod J(1 + a(n-1, n))$.

c) $$d_0 X_i \equiv \sum_{r=1}^{j} A(i, r) + \sum_{s=1}^{j+1}\sum_{r=s}^{n-1} B(i, r, s) + DZ(i) + \sum_{s=2}^{j+1}\sum_{r=s}^{n} DU(n, r, s)$$
$\bmod L(i)$ *for* $n \leqq i \leqq 2n-2$, *and*

d) $$d_0 X_i \equiv \sum_{r=1}^{j} A(i, r) + \sum_{s=1}^{j+1}\sum_{r=s}^{n-1} B(i, r, s) + \sum_{s=1}^{n-1}\sum_{r=s}^{n-1} C(i, r, s) + D(i)$$
$\bmod L(i)$ *for* $i \geqq 2n-1$.

PROOF. a) is Landweber's formula (2.7).

b) We have $t_1^{p^n} \equiv v_n^p W_1^p \bmod J(p+2)$ by (2.8) and (2.10), and so a) turns into

$$d_0 X_0^p \equiv v_{n-1}^p v_n^p W_1^p \equiv E(0)^p \bmod J(p+2).$$

By Proposition 2.2, we see that

$$d_0 z(1) \equiv v_{n-1}^p v_n^p (U_0 t_1 - W_1^p - v_{n-1} v_n^{-1} W_2)$$
$$\equiv Az(1) - E(0)^p + E(1) \bmod J(p+2).$$

Add these to obtain b) for $i = 1$. To proceed by induction, we note that (3.5) is also valid for n. Since $p(1 + a(i-1, i)) > 1 + a(i, i+1)$, (3.5) and the case $i-1$ yield

$$d_0 X_{i-1}^p \equiv \sum_{r=1}^{i-1} A(i-1, r)^p + E(i-1)^p \bmod J(1 + a(i, i+1)).$$

Proposition 2.2 also implies

$$d_0 z(i) \equiv \sum_{r=1}^{i} Az(i, r) - E(i-1)^p + E(i) \bmod J(1 + a(i, i+1)).$$

Now the definition of the element implies the case i, which shows b). Then b)′ follows from (4.6).

c) By b), (3.5) and (4.14), we obtain

$$d_0 X_{n-1}^p \equiv \sum_{r=1}^{n-1} A(n-1, r)^p + E^p \equiv \sum_{r=1}^{n-1} B(n, r, 2) + E^p \bmod L(i).$$

Here we note that the definition of the differential shows

(4.23) $$d_0 xy \equiv (d_0 x)y + e(x) d_0 y \bmod (p, v_1, \cdots, v_{n-2}).$$

Apply this to $d_0 y(n)$, and we have

$$d_0 y(n) \equiv v_{n-1}^{a(n,1)+1}(u_0 t_1 - w_1^p) X_{n-1}^{p-1} - v_{n-1}^{a(n,1)+1} e(u_1)\left(\sum_{r=1}^{n-1} A(n-1, r) + E\right) X_{n-1}^{p-2}$$
$$\equiv A(n, 1) - B(n, 1, 2) + DU(n, 1, 2) + \sum_{r=1}^{n-1} B(n, r, 1) + v_{n-1}^{p^n} v_n^{p^n - 2p^{n-1}} E$$

mod $L(n)$, by b)′, (4.10), (4.19) and (4.21). Since we see that

$$E^p + v_{n-1}^{p^n} v_n^{p^n - 2p^{n-1}} E \equiv DZ(n) + \sum_{r=1}^{n} DU(n, r, 2)$$

mod $L(n)$, by b)′, (4.10), (4.19) and (4.21). Since we see that

$$E^p + v_{n-1}^{p^n} v_n^{p^n-2p^{n-1}} E \equiv DZ(n) + \sum_{r=1}^{n} DU(n, r, 2)$$

by (4.8), we have c) for n. Similarly we have

$$d_0 X_{i-1}^p \equiv \sum_{r=1}^{j-1} A(i-1, r)^p + \sum_{s=1}^{j} \sum_{r=s}^{n-1} B(i, r, s+1) + DZ(i-1)^p + \sum_{s=2}^{j} \sum_{r=s}^{n} DU(i, r, s+1)$$

by the inductive hypothesis, (4.15) and (4.18), and

$$\begin{aligned} d_0 y(i) &\equiv (-1)^{j-1} v_{n-1}^{a(i,j)+1} \left(\sum_{r=1}^{j} u_{j-r} t_r^{p^{j-r}} - w_j^p\right) X_{n-1}^{p^j - a(j-1,j)} \\ &\quad + (-1)^j v_{n-1}^{a(i,j)+1} e(u_j) \left(\sum_{r=1}^{n-1} A(n-1, r) + E\right) X_{n-1}^{p^j - a(j-1,j)-1} \\ &\equiv \sum_{r=1}^{j} (Ay(i, r) - B(i, r, r+1) - DU(i, r, r+1)) \\ &\quad + \sum_{r=1}^{n-1} B(i, r, 1) + (-1)^{n-1} v_{n-1}^{p(i)} v_n^{q(i)} (\zeta - \zeta^*) \end{aligned}$$

by b)′, Proposition 2.2, (4.23), (4.10), (4.21) and (4.19). These congruences with (4.8) show c) for i, and c) is completed by induction.

d) For $2n-1$, we compute

$$d_0 X_{2n-2}^p \equiv \sum_{r=1}^{n-1} B(2n-1, r, 2) + \sum_{s=1}^{n-1} \sum_{r=s}^{n-1} C(2n-1, r, s+1) + D(2n-2)^p$$

by (4.14) and (4.16),

$$d_0 y(2n-1) \equiv A(2n-1, 1) - B(2n-1, 1, 2) + \sum_{r=1}^{n-1} B(2n-1, r, 1)$$

by (4.10), (4.21) and (4.19), and

$$\begin{aligned} d_0 u(2n-1) &\equiv -\sum_{r=1}^{n-1} C(2n-1, r, r+1) - \sum_{r=1}^{n} (-1)^i DU(2n-1, r, r+1) \\ &\quad + \sum_{r=1}^{n-1} C(2n-1, r, 1) + (-1)^{n-1} v_{n-1}^{p(2n-1)} v_n^{q(2n-1)} (\zeta - \zeta^*) \end{aligned}$$

by (4.10), (4,22) and (4.20). Collect terms to get d) for $2n-1$. In a similar way we can verify d) for i with $i \equiv 1\,(n-1)$ under the inductive hypothesis.

Suppose $i \not\equiv 1\,(n-1)$. By the case $i-1$ with (4.15) and (4.17),

$$d_0 X_{i-1}^p \equiv \sum_{r=1}^{j-1} A(i, r)^p + \sum_{s=1}^{j} \sum_{r=1}^{n-1} B(i, r, s+1) + \sum_{s=1}^{n-1} \sum_{r=s}^{n-1} C(i, r, s+1) + D(i-1)^p.$$

We also obtain

$$d_0 y(i) = \sum_{r=1}^{j} Ay(i, r) - \sum_{r=1}^{i} B(i, r, r+1) + \sum_{r=1}^{n-1} B(i, r, 1)$$

by (4.10), (4.21) and (4.19), and

$$\begin{aligned} d_0 u(i) &\equiv -\sum_{r=1}^{n-1} C(i, r, r+1) - \sum_{r=1}^{n} (-1)^i DU(i, r, r+1) \\ &\quad + \sum_{r=1}^{n-1} C(i, r, 1) + (-1)^{n-1} v_{n-1}^{p(i)} v_n^{q(i)} (\zeta - \zeta^*) \end{aligned}$$

by (4.10), (4.22) and (4.20). Therefore d) is proved by induction. q. e. d.

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