Some Results on Generalized Functions

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1. Introduction

Applying non-standard anylysis we introduced a concept of generalized functions in [5]. It is very useful to apply the ideas of the theory of hyperfunctions in the theory of generalezed functions. In this paper we would like to show a few examples of the application. In particular, we would like to justify the equality

(1.1) $$\left(-\frac{1}{2}\right)\delta'(x) = \delta(x)\,\mathrm{pf.}\,\frac{1}{x}$$

2. Definitions and sseveral properties

The following definitions were given in [5].

DEFINITION 2.1. Let $R^+ = \{y \in R \mid y > 0\}$, and let $F = \{(0, y) \mid y \in R^+\}$. Then F has the finite intersection property. We denote by $\mathscr{F}$ one of the ultrafilters on R^+ containing F.

Let K be the set R, or the set C, or the set $\mathrm{Map}(R, C) = \{f \mid f: R \to C\}$, and let $a(y)$, $b(y) \in \prod_{y \in R^+} K$.

We define a relation $\sim$ as follows:

$$a(y) \sim b(y), \text{ if it satisfies the condition}$$
$$\{y \in R^+ \mid a(y) = b(y)\} \in \mathscr{F}.$$

The relation $\sim$ is an equivalence relation. We define $^*K$ to be the quotient set $\prod_{y \in R^+} K/\sim$.

An element of the set $^*R$ (resp. $^*C$) is called a hyper resl (resp. hyper complex) number, an element of the set $^*\mathrm{Map}(R, C)$ is called a generalezed function of one variable. The equivalence class determined by a function $a(y) \in \prod_{y \in R^+} K$ will be denoted by $[a(y)]$.

We can consider the set $^*R$ is a subset of the set $^*C$. The set $^*R$ and $^*C$ are made into commutative fields by defining the addition, the subtraction, the product, and the quotient in the usual way.

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DEFINITION 2.2. (i) Let $[a(y)], [b(y)] \in {}^*R$. We define $[a(y)] \leqq [b(y)]$, if it satisfies the condition

$$\{y \in R^+ \mid a(y) \leqq b(y)\} \in \mathscr{F}.$$

(ii) Let $[a(y)], [b(y)] \in {}^*C$. We define $[a(y)] \fallingdotseq [b(y)]$, if it satisfies the condition

$$\{y \in R^+ \mid |a(y) - b(y)| < \varepsilon\} \in \mathscr{F} \text{ for every } \varepsilon \in R^+$$

DEFINITION 2.3 (Interval). Let $[a(y)], [b(y)] \in {}^*R$ and $[a(y)] \leqq [b(y)]$. We define

$$[[a(y)], [b(y)]] = \{[c(y)] \in {}^*R \mid [a(y)] \leqq [c(y)] \leqq [b(y)]\}.$$

DEFINITION 2.4. We say a generalized function $[f(x, y)]$ has a property P, if it satisfies the condition

$$\{y \in R^+ \mid f(x, y) \text{ has a property } P \text{ as a function of } x\} \in \mathscr{F}.$$

DEFINITION 2.5. Let $[a(y)]$ be a hyper complex number and let $[f(x, y)]$ and $[g(x, y)]$ be generalized functions. Then the scalar product $[a(y)]\,[f(x, y)]$, the addition $[f(x, y)] + [g(x, y)]$, the subtraction $[f(x, y)] - [g(x, y)]$, the product $[f(x, y)]\,[g(x, y)]$, and the quotient $[f(x, y)]/[g(x, y)]$ are defined as follows:

$$[a(y][f(x, y)] = [a(y)f(x, y)],$$

$$[f(x, y)] + [g(x, y)] = [f(x, y) + g(x, y)],$$

$$[f(x, y)] - [g(x, y)] = [f(x, y) - g(x, y)],$$

$$[f(x, y)][g(x, y)] = [f(x, y)g(x, y)],$$

$$[f(x, y)]/[g(x, y)] = [(f(x, y)/g(x, y))^*],$$

where $(f(x, y)/g(x, y))^*$ is defined as follows:

$$(f(x, y)/g(x, y))^* = f(x, y)/g(x, y) \text{ for } g(x, y) \neq 0, \text{ and}$$
$$= 0 \quad \text{elsewhere.}$$

We have the following theorem immediately.

THEOREM 2.6. *The set* $^*\mathrm{Map}(R, C)$ *is a vector space over* $^*C$.

DEFINITION 2.7 (Derivative). Let $[f(x, y)]$ be a defferentiable generalized function. We define

$$\frac{d}{dx}[f(x, y)] = \left[\left(\frac{\partial}{\partial x} f(x, y)\right)^*\right],$$

where $\left(\frac{\partial}{\partial x} f(x, y)\right)^*$ is defined as follows:

$$\left(\frac{\partial}{\partial x}f(x, y)\right)^* = \frac{\partial}{\partial x}f(x, y) \quad \text{if } f(x, y) \text{ is defferentiable as a function of } x, \text{ and}$$
$$= \quad 0 \qquad \text{otherwise.}$$

DEFINITION 2.8 (Integral). Let $[f(x, y)]$ be an integrable generalized function over an interval $[[a(y)], [b(y)]]$. We define

$$\int_{[a(y)]}^{[b(y)]} [f(x, y)]dx = \left[\left(\int_{a(y)}^{b(y)} f(x, y)dx\right)^*\right],$$

where $\left(\int_{a(y)}^{b(y)} f(x, y)dx\right)^*$ is defined as follows:

$$\left(\int_{a(y)}^{b(y)} f(x, y)dx\right)^* = \int_{a(y)}^{b(y)} f(x, y)dx \text{ if } f(x, y) \text{ is integrable over } [[a(y)], [b(y)]], \text{ and}$$
$$= \quad 0 \qquad \text{otherwise.}$$

We define the integral $\int_{-\infty}^{\infty} [f(x, y)]dx$, similarly.

According to G. Takeuti [12], we would like to use *a* notation $\overset{w}{=}$ as follows:

DEFINITION 2.9. Let $[f(x, y)]$ and $[g(x, y)]$ be locally integrable generalized functions and let S be a distribution.
We define

$[f(x, y)] \overset{w}{=} [g(x, y)]$, if it satisfies the condition

$$\int_{-\infty}^{\infty} [f(x, y)][\varphi(x)]dx \fallingdotseq \int_{-\infty}^{\infty} [g(x, y)][\varphi(x)]dx \text{ for every } \varphi \in (\mathscr{D}),$$

where $(\mathscr{D})$ is the set of all test functions, and

$[f(x, y)] \overset{w}{=} S$, if it satisfies the condition

$$\int_{-\infty}^{\infty} [f(x, y)][\varphi(x)]dx \fallingdotseq [S(\varphi)] \text{ for every } \varphi \in (\mathscr{D}).$$

We innediately have the following theorem.

THEOREM 2.10. (i) *Let* $[f(x, y)]$ *and* $[g(x, y)]$ *be locally integrable generalized functions.*

If $[f(x, y)] = [g(x, y)]$, *then* $[f(x, y)] \overset{w}{=} [g(x, y)]$.

(ii) *Let* $[f(x, y)]$ *be a locally integrable generalized function, and let* S *be a destribution.*

$$\text{If } [f(x, y)] \overset{w}{=} S, \text{ then } [f(x, y)y] \overset{w}{=} 0.$$

(iii) *Let* $[f(x, y)]$ *be a continuously defferentiable generalized function, and let* S *be a distribution.*

$$\text{If } [f(x, y)] \overset{w}{=} S, \text{ then } \frac{d}{dx}[f(x, y)] \overset{w}{=} S'.$$

In the theory of hyper functions, the Heaviside function $Y(x)$, the Dirac delta function $\delta(x)$, and the finite part Pf. $\frac{1}{x}$ of the function $1/x$ are defined as in the following:

$$Y(x) = -\frac{1}{2\pi}\text{Arg}(-x - i0) + \frac{1}{2\pi}\text{Arg}(x + i0),$$

$$\delta(x) = -\frac{1}{2\pi i}\left(\frac{1}{x + i0} - \frac{1}{x - i0}\right),$$

$$\text{Pf.}\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x + i0} - \frac{1}{x - i0}\right),$$

We would like to modify the above functions as follows:

$$[Y(x, y)] = \left[-\frac{1}{2\pi}\text{Arg}(-x - iy) + \frac{1}{2\pi}\text{Arg}(x + iy)\right],$$

$$[\delta(x, y)] = \left[-\frac{1}{2\pi i}\left(\frac{1}{x + iy} - \frac{1}{x - iy}\right)\right],$$

$$= \left[\frac{y}{\pi(x^2 + y^2)}\right],$$

$$[\text{Pf.}\frac{1}{x}] = \left[\frac{1}{2}\left(\frac{1}{x + iy} + \frac{1}{x - iy}\right)\right]$$

$$= \left[\frac{x}{x^2 + y^2}\right].$$

THEOREM 2.11. *We have the follows.*

$$\frac{d}{dx}[Y(x, y)] = [\delta(x, y)], \tag{2.1}$$

$$\left(-\frac{1}{2}\right)\frac{d}{dx}[\delta(x, y)] = [\delta(x, y)]\left[\text{Pf.}\frac{1}{x}\right], \tag{2.2}$$

$$[x]\left[\text{Pf.}\frac{1}{x}\right] = 1 - \pi[\delta(x, y)y] \tag{2.3}$$

$$\overset{w}{=} 1,$$

(2.4) $$(-1)[x][\delta'(x,\ y)] = 2[\delta(x,\ y)] - 2\pi[\delta^2(x,\ y)y].$$

PROOF. We shall only prove (2.2).

Since

$$\frac{\partial}{\partial x}\frac{y}{\pi(x^2+y^2)} = \frac{-2yx}{\pi(x^2+y^2)^2},$$

we have

$$\left(-\frac{1}{2}\right)\frac{\partial}{\partial x}\frac{y}{\pi(x^2+y^2)} = \frac{y}{\pi(x^2+y^2)}\cdot\frac{x}{x^2+y^2}.$$

Therefore we have

$$(-1/2)\frac{d}{dx}[\delta(x,\ y)] = [\delta(x,\ y)]\left[\mathrm{Pf.}\frac{1}{x}\right].$$ Q.E.D.

We can consider that the equality (2.2) justifies the equality (1.1).

We would like to comment that we can transform the equality (2.2) into the following:

$$(-1/2)\frac{d}{dx}[\delta(x,\ y)]/\left[\mathrm{Pf.}\frac{1}{x}\right] = [\delta(x,\ y)],$$

$$(-1/2)\frac{d}{dx}[\delta(x,\ y)]/[\delta(x,\ y)] = \left[\mathrm{Pf.}\frac{1}{x}\right].$$

It is well known, in the theory of hyperfunctions, that the equation $x\delta(x) = 0$ holds. We can express the equation in the form

$$[x][\delta(x,\ y)] \overset{w}{=} 0.$$

By Theorem 2.10 we have

(2.5) $$(-1)[x][\delta'(x,\ y)] \overset{w}{=} [\delta(x,\ y)].$$

On the other hand, we have $[x][\delta(x,\ y)] \neq 0$. And further, we can calculate as follows:

$$\begin{aligned}([x][\delta(x,\ y)])[\mathrm{Pf.}\frac{1}{x}] &= [x]\left([\delta(x,\ y)]\left[\mathrm{Pf.}\frac{1}{x}\right]\right)\\ &= [x]\left[\frac{xy}{\pi(x^2+y^2)^2}\right]\\ &= [x][(-1/2)\delta'(x,\ y)]\end{aligned}$$

$$= (1/2)[-x][\delta'(x, y)]$$

$$\overset{w}{=} (1/2)[\delta(x, y)].$$

Using (2.4) and (2.5) we have

$$[\delta(x, y)] \overset{w}{=} 2\pi[\delta^2(x, y)y].$$

REMARK. We can get the follows directly.

$$\int_{-\infty}^{\infty} 2\pi y \frac{y^2}{\pi^2(x^2+y^2)^2} dx = 1.$$

EXAMPLE 2.12. Let $\delta_1(x, y) = \frac{1}{2\sqrt{\pi y}} \exp\left(\frac{-x^2}{4y}\right)$ and let $\delta_2(x, y) = \frac{1}{\pi} \frac{\sin\frac{x}{y}}{x}$. Then we have

$$(-1)[x][\delta'_1(x, y)] = [x^2]\left[\frac{1}{2y}\delta_1(x, y)\right] \overset{w}{=} [\delta_1(x, y)],$$

and

$$(-1)[x][\delta'_2(x, y)] = \left[-\frac{1}{\pi y}\cos\frac{x}{y}\right] + [\delta_2(x, y)]$$

$$\overset{w}{=} [\delta_2(x, y)].$$

Hence we have

$$\left[-\frac{1}{\pi y}\cos\frac{x}{y}\right] \overset{w}{=} 0.$$

Using Theorem 2.10 we have

$$\left[\sin\frac{x}{y}\right] \overset{w}{=} 0.$$

The last equality implies that

(2.6) $$\left[\int_a^b \varphi(x)\sin\frac{x}{y}dx\right] \fallingdotseq 0$$

for all test functions φ with compact support contained in the interval (a, b).

The equation (2.6) is a restricted case of Riemann-Lebesgue theorem.

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