# A Measure on an Infinite-Dimensional Space

Yukio KURIBAYASHI*

(*Received August 31, 1987*)

## 1. Introduction

In the present paper we would like to introduce a measure on the space of all sequences with real terms using non-standard analysis. Non-standard measure theories and their applications have been developed by Çutland, Keisler, Loeb, Saito and others in [1]~[7]. However, in this paper, we would like to adopt another idea on measure theory.

Now we would like to explain the basic idea of the measure. Let $A$ be a cube of the ordinary real $n$-dimensional Euclidean space with side length $a$. Then the Lebesgue measure of the cube is $a^n$. We get the number $a^n$ by calculating $a, a^2, \dots, a^n$ successively. This fact suggest us that we will be able to express a measure of a cube of an infinite-dimensional space with side length $a$, using a sequence $a, a^2, \dots, a^n, \dots$. Fortunately, we have ultra real numbers $^*R$ and we would like to express the measure by a $^*R$-valued function.

## 2. Preliminaries

In this section, we would like to give definitions and notations which will be used in this paper.

Let $N$ be the set of all positive integers and $\mathscr{F}$ be an ultra filter on $N$ which does not contain any finite subset of $N$. Let $R$ be the set of all real numbers and $R^N$ be the set of all sequences with real terms. If $a$ is an element of $R^N$, then we use notations $a=(a_1, \dots, a_n, \dots)=(a_n)_{n\in N}$.

If $a=(a_1, \dots, a_n, \dots)$, $b=(b_1, \dots, b_n, \dots)$ are elements of $R^N$ and $\lambda$ is an element of $R$, we define addition $a+b$ and scalar multiplication $\lambda a$ by

$$a + b = (a_1+b_1, \dots, a_n+b_n, \dots)$$

and

$$\lambda a = (\lambda a_1, \dots, \lambda a_n, \dots).$$

We can consider $R^N$ is a linear space over the field $R$ by the above definitions.

Let $a$ and $b$ be elements of $R^N$. The relations and operations $a\sim b$, $a<b$, $a+b$, $a-b$, and $a\cdot b$ are defined to be $\{n\in N;\ a_n=b_n\}\in\mathscr{F}$, $\{n\in N;\ a_n<b_n\}\in\mathscr{F}$, $(a_n+b_n)_{n\in N}$, $(a_n-b_n)_{n\in N}$ and $(a_nb_n)_{n\in N}$, respectively. The relation $\sim$ is an equivalence relation. $^*R$ is

---

* Department of Mathematics, Faculty of Education, Tottori University, Tottori 680, Japan.

defined to be $R^N/\sim$ which is also written as $R^N/\mathscr{F}$ and is called ultra real numbers and its element is written as $[a]$ or $[a_n]$ or $[(a_1,\dots, a_n,\dots)]$. An imbedding isomorphism $i$ from $R$ into $^*R$ is defined by

$$i(a_1) = [(a_1,\dots, a_1,\dots)].$$

Let $[a_n]$, $[b_n]$ be two ultra real numbers and $[b_n]\neq 0$. The quatient $[a_n]/[b_n]$ is defined to be $[(a_n/b_n)^*]$, where $(a_n/b_n)^* = a_n/b_n$ for $b_n\neq 0$ and $(a_n/b_n)^*=0$ for $b_n=0$. The absolute value $|[a_n]|$ of an ultra real number $[a_n]$ and the non-negative square-root $([a_n])^{1/2}$ of a non-negative ultra real number $[a_n]$ are defined to be $[|a_n|]$ and $[(|a_n|)^{1/2}]$ respectively.

DEFINITION 2.1. Let $x=(x_1,\dots, x_n,\dots)$, $y=(y_1,\dots, y_n,\dots)$ be two points in $R^N$. Then an inner product $(x|y)$ of $x$ and $y$ is defined by

$$(x|y) = [(x_1y_1,\dots, \sum_{i=1}^{n} x_iy_i,\dots)].$$

We have the following proposition:

PROPOSITION 2.2. Let $x$, $y$, $z\in R^N$ and $a\in R$. Then we have the following properties:

(2.1) $$(x+y|z) = (x|y) + (y|z),\quad (x|y+z) = (x|y) + (x|z),$$

(2.2) $$(ax|y) = a(x|y) = (x|ay),$$

(2.3) $$(x|x)\geq 0,\quad (x|x) = 0 \quad \text{if and only if} \quad x = (0,\dots, 0,\dots).$$

REMARK. If no misunderstanding isposible, we will simply write 0 instead of $[(0,\dots, 0,\dots)]$.

The norm $|x|$ of a vector $x\in R^N$ is defined to be the non-negative ultra real number

$$((x|x))^{1/2} = [((x_1^2)^{1/2},\dots, (\sum_{i=1}^{n} x_i^2)^{1/2},\dots)].$$

PROPOSITION 2.3. For all $x$, $y\in R^N$, we have the following properties:

(2.4) $$|x\pm y|^2 = |x|^2 \pm 2(x|y) + |y|^2,$$

(2.5) $$|x+y|^2 + |x-y|^2 = 2(|x|^2+|y|^2),$$

(2.6) $$|(x|y)| \leq |x|\,|y| \quad \text{(Schwarz's inequality)}.$$

PROOF. We shall only prove (2.6). For all $n\in N$ we have

$$|\sum_{i=1}^{n} x_iy_i|\leq (\sum_{i=1}^{n} x_i^2)^{1/2} (\sum_{i=1}^{n} y_i^2)^{1/2}.$$

Therefore we have

$$|(x|y)| = [(|x_1||y_1|,\ldots, |\sum_{i=1}^{n} x_i y_i|,\ldots)]$$

$$\leq [((x_1^2)^{1/2}(y_1^2)^{1/2},\ldots, (\sum_{i=1}^{n} x_i^2)^{1/2}(\sum_{i=1}^{n} y_i^2)^{1/2},\ldots)]$$

$$\leq [((x_1^2)^{1/2},\ldots, (\sum_{i=1}^{n} x_i^2)^{1/2}, \ldots)] \times$$

$$[((y_1^2)^{1/2},\ldots, (\sum_{i=1}^{n} y_i^2)^{1/2},\ldots)]$$

$$= |x||y|.$$

We define two vectors $x$, $y$ to be perpendicular or orthogonal if and only if $(x|y)=0$.

EXAMPLE 2.4. If $x$ is perpendicular to $y$, then

$$|x+y|^2 = |x|^2 + |y|^2.$$

Let $x$, $y$ be two points in $R^N$. The distance between $x$ and $y$ is defined to be the ultra real number $|x-y|$ and is written as $d(x, y)$. Clearly $d(x, y)$ is a distance function on $R^N$. The straight line passing through $x$ and $y$ is defined to be the set $\{x+t(y-x);\ t \in R\}$, and the line segment with endpoints $x$ and $y$ is defined to be the set $\{x+t(y-x);\ 0 \leq t \leq 1, t \in R\}$.

EXAMPLE 2.5. Let $x=(x_1,\ldots, x_n,\ldots)$, $y=(y_1,\ldots, y_n,\ldots)$ be two distinct points in $R^N$ and let $k$, $l$ be two real numbers and $k+l \neq 0$. Then there exists a unique point $c \in R^N$ such that

$$d(a, c)/d(c, b) = k/l. \tag{2.7}$$

PROOF. For every $n \in N$, there exists a real number $c_n$ satisfying $c_n=(lx_n+ky_n)/(k+l)$. The point $c=(c_1,\ldots, c_n,\ldots)$ satisfies the condition (2.7), and the uniqueness is clear.

Now, we would like to define a topology for the space $R^N$. Let $\varepsilon_i$ be a positive real number for $i \in N$ and $\varepsilon=(\varepsilon_1,\ldots, \varepsilon_n,)$ and let $a=(a_1,\ldots, a_n,\ldots)$ be a point in $R^N$. We define sets $U_n(a, \varepsilon_n)$ for $n \in N$ and $U(a, \varepsilon)$ as follows:

$$U_n(a, \varepsilon_n) = \{x \in R^N;\ (\sum_{i=1}^{n} (x_i-a)^2)^{1/2} < \varepsilon_n,\ x_i=a_i \text{ for } i \geq n+1\},$$

$$U(a, \varepsilon) = \bigcup_{n=1}^{\infty} U_n(a, \varepsilon_n).$$

Let $S$ be a subset of $R^N$ and let $\mathfrak{O}$ be a family of subsets of $R^N$ with the following property:

(2.8) For every $a \in S$, there exists a set $U(a, \varepsilon)$ such that $U(a, \varepsilon) \subset S$.

PROPOSITION 2.6. *The set $\mathfrak{O}$ satisfies the following properties:*

(2.9) $\phi \in \mathfrak{D}$ and $R^N \in \mathfrak{D}$,

(2.10) *if* $S_i \in \mathfrak{D}$, $i = 1,\dots, k$, *then* $\bigcap_{i=1}^{k} S_i \in \mathfrak{D}$,

(2.11) *if* $S_i \in \mathfrak{D}$, $i \in \Gamma$, *then* $\bigcup_{i \in \Gamma} S_i \in \mathfrak{D}$.

*where the index set* $\Gamma$ *is not necessarily finite.*

PROOF. We shall only prove (2.10). Let $S_1$, $S_2 \in \mathfrak{D}$ and $a \in S_1 \cap S_2$. Then there exist two set $U(a, \varepsilon)$ and $U(a, \eta)$ such that $U(a, \varepsilon) \subset S_1$ and $U(a, \eta) \subset S_2$, where $\varepsilon = (\varepsilon_1,\dots, \varepsilon_n,\dots)$, $\eta = (\eta_1,\dots, \eta_n,\dots)$ and $\varepsilon_n > 0$, $\eta_n > 0$ for every $n \in N$. Let $\gamma_n = \min(\varepsilon_n, \eta_n)$ for $n \in N$, and let $\gamma = (\gamma_1,\dots, \gamma_n,\dots)$. Then $U_n(a, \gamma_n) \subset U_n(a, \varepsilon_n) \cap U_n(a, \eta_n)$ for $n \in N$, and so

$$U(a, \gamma) = \bigcup_{n=1}^{\infty} U_n(a, \gamma_n) \subset S_1 \cap S_2,$$

thus $S_1 \cap S_2 \in \mathfrak{D}$.

EXAMPLE 2.7. Let $a = (a_1,\dots, a_n,\dots)$, $b = (b_1,\dots, b_n,\dots)$ be two points in $R^N$ and $a_n < b_n$ for every $n \in N$. We define an open interval $I(a, b)$ by

$$I(a, b) = \{x = (x_1,\dots, x_n,\dots) \in R^N;\ a_n < x_n < b_n \text{ for } n \in N\}.$$

The interval $I(a, b)$ is an open set.

EXAMPLE 2.8. Let $a$ be a point in $R^N$ and let $r$ be a positive real number. We define a ball $B(a, r)$ of radius $r$ and centered at $a$ by

$$B(a, r) = \{x \in R^N;\ |x - a| < r\}.$$

Then the ball $B(a, r)$ is an open set.

PROPOSITION 2.9. *Let* $a$, $b$ *be two distinct points in* $R^N$. *Then there exist two open sets* $O_1$, $O_2$ *which satisfy the following conditions:*

$$O_1 \ni a, \quad O_2 \ni b \quad \textit{and} \quad O_1 \cap O_2 = \phi.$$

PROOF. Since $a \neq b$, we have a positive integer $k$ such that $a_k \neq b_k$. We can consider $a_k < b_k$, and so we can choose a positive real number $\varepsilon$ satisfying a condition $a_k + \varepsilon < b_k - \varepsilon$. We define two open intervals $I(a - \varepsilon, a + \varepsilon)$, $I(b - \varepsilon, b + \varepsilon)$ by

$$I(a - \varepsilon, a + \varepsilon) = \{x \in R^N;\ a_n - \varepsilon < x_n < a_n + \varepsilon \text{ for } n \in N\},$$

$$I(b - \varepsilon, b + \varepsilon) = \{x \in R^N;\ b_n - \varepsilon < x_n < b_n + \varepsilon \text{ for } n \in N\}.$$

Then $I(a - \varepsilon, a + \varepsilon)$ and $I(b - \varepsilon, b + \varepsilon)$ are open sets and satisfy the above three conditions.

### 3. A measure on the space $R^N$

DEFINITION 3.1. Let $E \subset R^N$. We define a set $E(x_{n+1},\ldots)$ by

$$E(x_{n+1},\ldots) = \{(x_1,\ldots, x_n) \in R^n;\ (x_1,\ldots, x_n, x_{n+1},\ldots) \in E\}.$$

The set $E$ is said to be measurable, if it satisfies the following condition:
(3.1) $E(x_{n+1},\ldots)$ is Lebesgue measurable in $R^n$ for every $n \in N$ and $(x_{n+1},\ldots)$.

PROPOSITION 3.2. *Let $S$ be an open set of $R^N$. Then $S$ is measurable.*

PROOF. Let $(x_1,\ldots, x_n) \in S(x_{n+1},\ldots)$. Then

$$x = (x_1,\ldots, x_n, x_{n+1},\ldots) \in S.$$

Since $S$ is an open set, there is a positive real number $\varepsilon_i$ for every $i \in N$, satisfying the the condition

$$\bigcup_{n=1}^{\infty} U_n(x, \varepsilon_n) = U(x, \varepsilon) \subset S,$$

where $\varepsilon = (\varepsilon_1,\ldots, \varepsilon_n,\ldots)$.
Therefore $U_n(x, \varepsilon_n) \subset S$, for every $n \in N$.
Hence we have

$$\{(y_1,\ldots, y_n) \in R^n;\ (\sum_{i=1}^{n} (y_i - x_i)^2)^{1/2} < \varepsilon_n\} \subset S(x_{n+1},\ldots).$$

Thus the set $S(x_{n+1},\ldots)$ is an open set of $R^n$. This relation holds for every $n \in N$ and $(x_{n+1},\ldots)$. Therefore $S$ is measurable in $R^N$.

COROLLARY 3.3. *An open interval $I(a, b)$ is measurable.*

PROPOSITION 3.4. *Let $E$ be a measurable set, then the set $E^c$ is measurable.*

PROOF. We have

$$\begin{aligned} E^c(x_{n+1},\ldots) &= \{(x_1,\ldots, x_n) \in R^n;\ (x_1,\ldots, x_n, x_{n+1},\ldots) \in E^c\} \\ &= \{(x_1,\ldots, x_n) \in R^n;\ (x_1,\ldots, x_n, x_{n+1},\ldots) \in E\}^c \\ &= (E(x_{n+1},\ldots))^c. \end{aligned}$$

Since $E(x_{n+1},\ldots)$ is measurable in $R^n$, $E^c(x_{n+1},\ldots)$ is measurable in $R^n$. This property holds for every $n \in N$ and $(x_{n+1},\ldots)$. Therefore $E^c$ is measurable in $R^N$.

We have the following proposition:

PROPOSITION 3.5. *Every closed set of $R^N$ is measurable.*

PROPOSITION 3.6. *Let $E_1,\dots, E_i,\dots$ be measurable sets in $R^N$. Then $\bigcup_{i=1}^{\infty} E_i$ is a measurable set in $R^N$.*

PROOF. We have

$$\bigcup_{i=1}^{\infty} E_i(x_{n+1},\dots) = \{(x_1,\dots, x_n)\in R^n;\ (x_1,\dots, x_n, x_{n+1},\dots)\in \bigcup_{i=1}^{\infty} E_i\}$$

$$= \bigcup_{i=1}^{\infty} \{(x_1,\dots, x_n)\in R^n;\ (x_1,\dots, x_n, x_{n+1},\dots)\in E_i\}$$

$$= \bigcup_{i=1}^{\infty} E_i(x_{n+1},\dots).$$

Since $E(x_{n+1},\dots)$ is measurable in $R^n$ for each $i\in N$, the set $\bigcup_{i=1}^{\infty} E_i(x_{n+1},\dots)$ is measurable in $R^n$. The property holds for every $n\in N$ and $(x_{n+1},\dots)$. Therefore $\bigcup_{i=1}^{\infty} E_i$ is measurable in $R^N$. Hence we have the following theorem:

THEOREM 3.7. *The set of all measurable sets in $R^N$ is a $\sigma$-algebra.*

DEFINITION 3.8. Let $E$ be a measurable set, contained in an interval $I(a, b)$, in $R^N$ and let $m_n$ be the Lebesgue measure on $R^n$ for each $n\in N$. We define functions $m(E)$ and $M(E)$ by

$$m(E)(x_2,\dots, x_n,\dots) = (m_1(E(x_2,\dots)),\dots, m_n(x_{n+1},\dots)),\dots),$$

$$M(E)(x_2,\dots, x_n,\dots) = [m(E)(x_2,\dots, x_n,\dots)].$$

The function $M(E)$ is said to be a measure on the space $R^N$. We have the following proposition immediately:

PROPOSITION 3.9. *Let $E_i$ be measurable in $R^N$ for $i=1,\dots, k$ satisfying the conditions*

$$E_i \cap E_j = \phi \qquad \textit{for} \quad i\neq j,$$

*and*

$$M(E_i)(x_2,\dots, x_n,\dots) < r_i \qquad \textit{for} \quad i = 1,\dots, k,$$

*where $r_i$ is a positive ultra real number for each $i=1,\dots, k$. Then we have*

$$M(\bigcup_{i=1}^{k} E_i)(x_2,\dots, x_n,\dots) = \sum_{i=1}^{k} M(E_i)(x_2,\dots, x_n,\dots).$$

DEFINITION 3.10. Let $E$ be a measurable set contained in an interval $I(a, b)$. We define pseudo measure $PM$ by

$$PM(E) = [(\sup_{(x_2,\dots)} m_1(E)(x_2,\dots),\dots, \sup_{(x_{n+1},\dots)} m_n(E)(x_{n+1},\dots),\dots)].$$

EXAMPLE 3.11. Let $I(a, b)$ be an open interval of $R^N$. Then we have

$$PM(I(a, b)) = [(b_1 - a_1, \ldots, \prod_{i=1}^{n} (b_i - a_i), \ldots)].$$

EXAMPLE 3.12. Let $B(a, r)$ be a ball of radius $r$, a real number, centered at $a$. Then we have

$$PM(B(a, r)) = [(2\pi r, \pi r^2, (4/3)\pi r^3, \ldots, v_n r^n, \ldots)],$$

where $v_n$ is the volume of the $n$-dimensional unit ball.

## References


[1] S. Albeverio, J. E. Fenstad, R. Høegh-Kron and T. Lindstrøm, Nonstandard methods in stochastic analysis and mathematical physics, Academic Press (1986).

[2] R. M. Anderson, Star-finite probability theory, Dissertation, Yale University (1977).

[3] N. J. Cutland, Nonstandard measure theory and its applications, Bull. London Math. Soc., 15 (1983), 529–589.

[4] A. E. Hurd and P. A. Loeb, An introduction to nonstandard real analysis, Academic Press (1985).

[5] H. J. Keisler, An infinitesimal approach to stochastic analysis, Memoirs AMS 297 (1984).

[6] P. A. Loeb, An introduction to nonstndard analysis and hyperfinite probability theory, Probabilistic analysis and related topics, Vol. 2, Academic Press (1979).

[7] M. Saito, Ultraproducts and non-standard analysis (in Japanese), An enlarged edition, Tokyo Tosho (1987).