# Summation Kernels and Delta Functions (I)

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## 1. Introduction

In the previous papers [2] and [3], we have introduced a concept of generalized functions. In the present paper we intend to discuss relations between summation kernels, delta functions and the delta distribution, using generalized functions. For the sake of the purpose, we would like to introduce a new definition of summation kernels. The definition differs slightly from the traditional one. According to our definition the Gauss-Weierstrass kernel $W(x, y)$ and Poisson kernel $P(x, y)$ are summation kernels.

As an application of our method, we intend to prove the following equalities:

$$\int_{R^n} W(x, y)dx = \int_{R^n} P(x, y)dx = 1.$$

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## 2. Preliminaries

We shall first give the definition of generalized functions.

DEFINITION 2.1. Let $R^+ = \{y \in R;\ y > 0\}$, and $F = \{(0, y);\ y \in R^+\}$. Then $F$ has the finite intersection property. We shall denote with $\mathscr{F}$ the ultrafilter generated by $F$.

Let $\{X(y);\ y \in R^+\}$ be a family of non empty sets and let $a(y), b(y) \in \prod_{y \in R^+} X(y)$. Define $a(y) \sim b(y)$ if the following condition is satisfied:

$$\{y \in R^+;\ a(y) = b(y)\} \in \mathscr{F}.$$

It is easy to see this relation is an equivalence relation. Define

$$\prod_{\mathscr{F}} X(y) = \prod_{y \in R^+} X(y)/\sim.$$

The equivalence class determined by $a(y) \in \prod_{y \in R^+} X(y)$ will be denoted with $[a(y)]$.

If $X(y) = X$ for $y \in R^+$, then we denote $\prod_{\mathscr{F}} X = {}^*X$. Let $a \in X$ and let $\alpha(y) = a$ for $y \in R^+$. If we identify $[\alpha(y)]$ with $a$, then we have $X \subset {}^*X$. Throughout this paper,

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we identify $[\alpha(y)]$ with $a$. An element of the space $^*\mathrm{Map}(R^n, C)$ is called a generalized function ($G$-function).

LEMMA 2.2. *Let* $a(y), a'(y), b(y), b'(y) \in \prod_{y\in R^+} C$ *and let* $a(y) \sim a'(y)$, $b(y) \sim b'(y)$. *If* $\{y \in R^+;\ |a(y)-b(y)|<\varepsilon\} \in \mathscr{F}$ *for* $\varepsilon \in R^+$. *Then*

$$\{y \in R^+;\ |a'(y)-b'(y)|<\varepsilon\} \in \mathscr{F} \qquad \textit{for} \quad \varepsilon \in R^+.$$

PROOF. Let $\varepsilon \in R^+$. Since

$$\begin{aligned}&\{y \in R^+;\ |a'(y)-b'(y)|<\varepsilon\}\\ &\quad \supset \{y \in R^+;\ a(y)=a'(y)\} \cap \{y \in R^+;\ b(y)=b'(y)\}\\ &\quad \cap \{y \in R^+;\ |a(y)-b(y)|<\varepsilon\}, \quad \text{and}\\ &\{y \in R^+;\ a(y)=a'(y)\},\ \{y \in R^+;\ b(y)=b'(y)\},\\ &\{y \in R^+;\ |a(y)-b(y)|<\varepsilon\} \in \mathscr{F},\end{aligned}$$

we have

$$\{y \in R^+;\ |a'(y)-b'(y)|<\varepsilon\} \in \mathscr{F}.$$

DEFINITION 2.3. Let $[a(y)], [b(y)] \in {}^*C$. Define $[a(y)] \doteqdot [b(y)]$ if the following condition is satisfied:

$$\{y \in R^+;\ |a(y)-b(y)|<\varepsilon\} \in \mathscr{F} \qquad \text{for} \quad \varepsilon \in R^+.$$

According to Lemma 2.2, Definition 2.3 is well-defined.

PROPOSITION 2.4. Let $a_0(y) \in [a(y)] \in {}^*C$ and let $\lim_{y\to 0} a_0(y) = a$. Then $[a(y)] \doteqdot a$.

PROOF. For every positive number $\varepsilon$, we can select a suitable positive number $\delta$ such that $y \in (0, \delta)$ implies $|a_0(y)-a|<\varepsilon$. Thus we have

$$\{y \in R^+;\ |a_0(y)-a|<\varepsilon\} \supset (0, \delta).$$

Since $(0, \delta) \in \mathscr{F}$ we have $\{y \in R^+;\ |a_0(y)-a|<\varepsilon\} \in \mathscr{F}$, and therefore $[a(y)] = [a_0(y)] \doteqdot a$.

DEFINITION 2.5. Let $[f(x, y)] = [f(x_1, \dots, x_n, y)] \in {}^*\mathrm{Map}(R^n, C)$. We say $[f(x, y)]$ has a property $P$, if it satisfies the condition:

$$A(P) = \{y \in R^+;\ f(x, y) \text{ has a property } P \text{ as a function of } x \text{ alone}\} \in \mathscr{F}.$$

REMARK. Definitions 2.5, 2.6 and 3.2 are clearly well-defined and proofs are omitted.

DEFINITION 2.6. Let $[f(x, y)]$ be an integrable $G$-function on $R^n$. Define

$$\int_{R^n} [f(x, y)]dx = \left[\int_{R^n} f(x, y)dx^*\right],$$

where $\int_{R^n} f(x, y)dx^*$ is defined as follows:

$$\int_{R^n} f(x, y)dx^* = \int_{R^n} f(x, y)dx \quad \text{where } f(x, y) \text{ is integrable on } R^n\text{, and}$$

$$= 0 \qquad\qquad \text{elsewhere.}$$

### 3. Summation Kernels and Delta Functions

We shall define the delta distribution $\delta$ as follows:

DEFINITION 3.1. Let $\delta$ be the set of all $G$-functions $[f(x, y)]$ having the following properties:

(3.1) $[f(x, y)]$ is locally integrable i.e.

$A(l) = \{y \in R^+; f(x, y) \text{ is locally integrable as a function of } x \text{ alone}\} \in \mathscr{F}$, and

(3.2) $\int_{R^n} [f(x, y)]\,[\varphi(x)]dx \fallingdotseq \varphi(0) \quad \text{for} \quad \varphi \in (\mathscr{D})$.

An element of the set $\delta$ is called a delta function.

PROPOSITION 3.2. *Let $[f(x, y)]$ be a delta function having the following property:*

(3.3) *Let $I$ be a bounded interval in $R^n$ such that $\bar{I} \not\ni 0$. Then $|f|(x, y)$ is bounded on the interval, i.e.*

$$A(M, I) = \{y \in R^+;\ |f|(x, y) \leqq M,\ x \in I\} \in \mathscr{F}$$

$$\textit{for some} \quad M \in R^+, \quad \textit{where} \quad |f|(x, y) = |f(x, y)|.$$

*We have the following properties:*

(3.4) *Let $I$ be a bounded interval in $R^n$ such that $\mathring{I} \ni 0$. Then*

$$\int_I [f(x, y)]dx \fallingdotseq 1.$$

(3.5) *Let $I$ be a bounded interval in $R^n$ such that $\bar{I} \not\ni 0$. Then*

$$\int_I [f(x, y)]dx \fallingdotseq 0.$$

PROOF. (3.4) Let $\varepsilon \in R^+$. We choose a bounded interval $I_1$ in $R^n$ having the following properties:

(i) $\mathring{I}_1 \supset I$, and (ii) $m(I_1 - I) < \varepsilon$, where $m$ denotes the Lebesgue measure. We choose a

function $\varphi \in (\mathscr{D})$ having the following properties:
(iii) $0 \leqq \varphi(x) \leqq 1$, (iv) $\varphi(x)=1$ for $x \in I$, and (v) $\mathrm{Car}(\varphi) \subset I_1$.

Let $y \in A(M_1, I_1) \cap A(l)$. Then

$$\int_{R^n} f(x, y)\varphi(x)dx = \int_I f(x, y)\varphi(x)dx + \int_{I_1-I} f(x, y)\varphi(x)dx,$$

and

$$\left|\int_{I_1-I} f(x, y)\varphi(x)dx\right| \leqq \int_{I_1-I} |f(x, y)|dx \leqq M_1 m(I_1-I) < M_1\varepsilon.$$

On the other hand we have

$$1=\varphi(0) \fallingdotseq \int_{R^n} [f(x, y)][\varphi(x)]dx.$$

Therefore we have

$$\int_I [f(x, y)]dx \fallingdotseq 1.$$

(3.5) Let $\varepsilon \in R^+$. We choose $I_1$ and $\varphi$ in the same way as (3.4). For sufficiently small $\varepsilon$, we have $\bar{I}_1 \not\ni 0$. Let $y \in A(M_1, I_1) \cap A(l)$. Then

$$\int_{R^n} f(x, y)\varphi(x)dx = \int_I f(x, y)dx + \int_{I_1-I} f(x, y)\varphi(x)dx, \quad \text{and}$$

$$\left|\int_{I_1-I} f(x, y)\varphi(x)dx\right| \leqq \int_{I_1-I} |f(x, y)|dx \leqq M_1\varepsilon.$$

On the other hand we have

$$0=\varphi(0) \fallingdotseq \int_{R^n} [f(x, y)][\varphi(x)]dx.$$

Therefore we have $\int_I [f(x, y)]dx \fallingdotseq 0$.

PROPOSITION 3.3. *Let $f(x, y)$ be a function having the following properties:*

(3.6) *$f(x, y)$ is integrable for all $y \in R^+$ and $[f(x, y)] \in \delta$,*

(3.7) *We can choose a bounded interval $I$ in $R^n$ and a non negative function $g(x)$ having the following properties:*

(i) $$\int_{R^n-I} g(x)dx < \infty, \quad \textit{and}$$

(ii) $$|f|(x, y) \leqq g(x) \quad \textit{for} \quad y \in R^+,\ x \in R^n - I, \quad \textit{and}$$

(3.8) $$\lim_{y\to 0} f(x, y) = 0 \quad \textit{for a.e.} \quad x \in R^n - I.$$

*Then we have*

$$\int_{R^n} [f(x, y)]dx \risingdotseq 1 .$$

PROOF. Since

$$\int_{R^n} [f(x, y)]dx = \int_I [f(x, y)]dx + \int_{R^n - I} [f(x, y)]dx ,$$

using Proposition 3.1, we have

$$\int_I [f(x, y)]dx \risingdotseq 1 .$$

By the dominated convergence theorem, we have

$$\int_{R^n - I} [f(x, y)]dx \risingdotseq 0 .$$

Therefore we have

$$\int_{R^n} [f(x, y)]dx \risingdotseq 1 .$$

COROLLARY 3.4. *Let $f(x, y)$ be a function having the properties* (3.6), (3.7) *and* (3.8).
*If* $\int_{R^n} [f(x, y)]dx \in R$, *then* $\int_{R^n} [f(x, y)]dx = 1$.

DEFINITION 3.5. A $G$-function $[k(x, y)]$ is said a summation kernel if it satisfies the following conditions:

(3.9) $$\int_{R^n} [k(x, y)]dx = 1 ,$$

(3.10) $[\|k\|_1(y)] \leqq M$ for some $M \in R^+$, where $\|k\|_1(y) = \int_{R^n} |k(x, y)|dx^*$, and

(3.11) Let $\varepsilon \in R^+$ and $I_\varepsilon = \{x; |x| \leqq \varepsilon\}$. Then

$$\int_{R^n - I_\varepsilon} [|k|(x, y)]dx \risingdotseq 0 .$$

By Definition 3.5, Gauss-Weierstrass kernel $W(x, y) = (4\pi y)^{-(n/2)} e^{-|x|^2/4y}$ and Poisson kernel $P(x, y) = C_n(y/(y^2 + |x|^2)^{(n+1)/2})$ where $C_n = \Gamma((n+1)/2)/(\pi^{(n+1)/2})$, are summation kernels.

The traditional definition of summation kernels are as follows (see Igari [1]).

DEFINITION 3.6. A family of functions $\{k_\lambda(x)\}_{\lambda > 0}$ defined on $R^n$ is said a summation kernel if it satisfies the following conditions:

(3.12) $$\frac{1}{\sqrt{2\pi}^n} \int_{R^n} k_\lambda(x)dx = 1 ,$$

(3.13) $$\|k\|_1 \leqq M \quad \text{for some} \quad M \geqq 0, \quad \text{and}$$

(3.14) $$\lim_{\lambda\to\infty}\int_{|x|>\varepsilon} k_\lambda(x)dx=0 \quad \text{for all} \quad \varepsilon>0.$$

PROPOSITION 3.7. *Let* $[f(x, y)]$ *be a summation kernel. Then* $[f(x, y)]\in\delta$.

PROPOSITION 3.8. *Let* $f(x, y)$ *be an integrable function on* $R^n$ *for all* $y\in R^+$ *having the following properties*:

(3.15) $$f(x, y) \text{ is bounded on } R^n\times R^+, \text{ and}$$

(3.16) $$\lim_{y\to 0} f(x, y)=1 \quad \text{for a.e.} \quad x\in R^n.$$

*Then*

(3.17) $$[\mathscr{F}(f)(x, y)]=\int_{R^n}[f(\xi, y)]\,[e^{-2\pi i x\cdot\xi}]d\xi\in\delta.$$

PROOF. Let $\varphi\in(\mathscr{D})$. Then

$$\int_{R^n}\left\{\int_{R^n} f(\xi, y)e^{-2\pi i x\cdot\xi}d\xi\right\}\varphi(x)\,dx=\int_{R^n} f(\xi, y)\left\{\int_{R^n}\varphi(x)e^{-2\pi i x\cdot\xi}dx\right\}d\xi,$$

and

$$\lim_{y\to 0}\int_{R^n} f(\xi, y)\left\{\int_{R^n}\varphi(x)e^{-2\pi i x\cdot\xi}d\xi\right\}dx$$
$$=\int_{R^n}\int_{R^n}\varphi(x)e^{-2\pi i x\cdot\xi}dxd\xi=\varphi(0).$$

Therefore $$[\mathscr{F}(f)(x, y)]\in\delta.$$

EXAMPLE 3.9. (i) Let $f(x, y)=e^{-4\pi^2|x|^2 y}$ and $g(x, y)=e^{-2\pi|x|y}$.
Then $[\mathscr{F}(f)(x, y)]=[W(x, y)]$ and $[\mathscr{F}(g)(x, y)]=[P(x, y)]$ have the properties (3.6), (3.7), (3.8) and

$$\int_{R^n}[W(t, y)]dt=\int_{R^n}[W(t, 1)]dt, \quad \text{and}$$

$$\int_{R^n}[P(t, y)]dt=\int_{R^n}[P(t, 1)]dt.$$

Therefore

$$\int_{R^n}[W(t, y)]dt=\int_{R^n}[P(t, y)]dt=1.$$

(ii) Let $fg(x, y)=f(x, y)g(x, y)$, where $f(x, y)$ and $g(x, y)$ are functions defined in (i). Then

$$[\mathscr{F}(fg)(x, y)] \in \delta.$$

## References

[1] S. Igari, Fourier series (in Japanese), Iwanami, Tokyo, 1975.
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