A Generalization of the Concept of Functions (III)

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1. Introduction

In the previous papers [2] and [3], we have introduced a concept of generalized functions. Our definition is as follows:

DEFINITION 1.1. Let $R^+=\{y \in R;\ y>0\}$, and let $F=\{(0, y);\ y \in R^+\}$. Then F has the finite intersection property. We shall denote with $\mathscr{F}$ the ultrafilter generated by F. Let $a(y), b(y) \in \prod_{y\in R^+} \mathrm{Map}(R^n, C)$. Define
$a(y) \sim b(y)$ if the following condition is satisfied:

$$\{y \in R^+;\ a(y)=b(y)\} \in \mathscr{F}.$$

It is easy to see that this relation is an equivalence relation. Define

$$^*\mathrm{Map}(R^n, C)=\prod_{y\in R^+} \mathrm{Map}(R^n, C)/\sim.$$

The equivalence class determined by a function $a(y) \in \prod_{y\in R^+} \mathrm{Map}(R^n, C)$ will be denoted by $[a(y)]$. An element of the space $^*\mathrm{Map}(R^n, C)$ is called a generalized function (*G*-function).

Similarly we can define spaces $^{**}\mathrm{Map}(R^n, C)={}^*({}^*\mathrm{Map}(R^n, C))$, $^{***}\mathrm{Map}(R^n, C)={}^*({}^{**}\mathrm{Map}(R^n, C)),\dots$.

In the present paper we intend to give an another definition of generalized functions. Using the definition we would like to show that the spaces $^{**}\mathrm{Map}(R^n, C)$, $^{***}\mathrm{Map}(R^n, C),\dots$, are given by direct generalization of the space $\mathrm{Map}(R^n, C)$.

2. Preliminaries and Several Properties

We shall first give the following definition (see Comfort and Negrepontis [1]):

DEFINITION 2.1. Let $\mathscr{F}$ be the ultrafilter defined in Definition 1.1. Define

(2.1) $\mathscr{F}\cdot\mathscr{F}=\{A \in \mathscr{P}(R^+\times R^+);\ \{y_2 \in R^+;\ \{y_1 \in R^+;\ (y_1, y_2) \in A\} \in \mathscr{F}\} \in \mathscr{F}\}$,

(2.2) $(\mathscr{F}\cdot\mathscr{F})\cdot\mathscr{F}=\{A \in \mathscr{P}(R^+\times R^+\times R^+);\ \{y_3 \in R^+;\ \{(y_1, y_2) \in R^+\times R^+;\ (y_1, y_2, y_3) \in A\} \in \mathscr{F}\cdot\mathscr{F}\} \in \mathscr{F}\}$,

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(2.3) $\mathscr{F}\cdot\mathscr{F}\cdot\mathscr{F}=\{A\in\mathscr{P}(R^+\times R^+\times R^+);\ \{y_3\in R^+;\ \{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2, y_3)\in A\}\in\mathscr{F}\}\in\mathscr{F}\}\in\mathscr{F}\}$,

(2.4) $\mathscr{F}\times\mathscr{F}=\{A\in\mathscr{P}(R^+\times R^+)$; there are $B, C\in\mathscr{F}$ such that $B\times C\subset A\}$,

(2.5) $(\mathscr{F}\times\mathscr{F})\times\mathscr{F}=\{A\in\mathscr{P}(R^+\times R^+\times R^+)$; there are $B\in\mathscr{F}\times\mathscr{F}$, $C\in\mathscr{F}$ such that $B\times C\subset A$, and

(2.6) $\mathscr{F}\times\mathscr{F}\times\mathscr{F}=\{A\in\mathscr{P}(R^+\times R^+\times R^+)$; there are $B, C, D\in\mathscr{F}$ such that $B\times C\times D\subset A\}$.

We have the following lemma:

LEMMA 2.2. (i) $\mathscr{F}\cdot\mathscr{F}$ *is an ultrafilter on* $\mathscr{P}(R^+\times R^+)$,
(ii) $\mathscr{F}\times\mathscr{F}$ *is a filter on* $\mathscr{P}(R^+\times R^+)$ *and we have* $\mathscr{F}\times\mathscr{F}\subset\mathscr{F}\cdot\mathscr{F}$,
(iii) $(\mathscr{F}\cdot\mathscr{F})\cdot\mathscr{F}$ *is an ultrafilter on* $\mathscr{P}(R^+\times R^+\times R^+)$,
(iv) *In the same way as* (2.2) *we can define* $\mathscr{F}\cdot(\mathscr{F}\cdot\mathscr{F})$ *and we have*

$$(\mathscr{F}\cdot\mathscr{F})\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{F}\cdot\mathscr{F}=\mathscr{F}\cdot(\mathscr{F}\cdot\mathscr{F}),$$

(v) $(\mathscr{F}\times\mathscr{F})\times\mathscr{F}$ *is a filter on* $\mathscr{P}(R^+\times R^+\times R^+)$ *and we have*

$$(\mathscr{F}\times\mathscr{F})\times\mathscr{F}\subset(\mathscr{F}\cdot\mathscr{F})\cdot\mathscr{F},\quad \textit{and}$$

(vi) *In the same way as* (2.5) *we can define* $\mathscr{F}\times(\mathscr{F}\times\mathscr{F})$ *and we have*

$$(\mathscr{F}\times\mathscr{F})\times\mathscr{F}=\mathscr{F}\times\mathscr{F}\times\mathscr{F}=\mathscr{F}\times(\mathscr{F}\times\mathscr{F}).$$

PROOF. We shall only prove (i), (ii) and (iv).
(i) 1° It is clear that $\phi\notin\mathscr{F}\cdot\mathscr{F}$.
2° Let $A\in\mathscr{F}\cdot\mathscr{F}$ and $A\subset B$. Since

$$\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in A\}\in\mathscr{F}\}\subset\{y_2\in R^+;\ \{y_1\in R^+:\ (y_1, y_2)\in B\}\in\mathscr{F}\}\quad\text{and}$$
$$\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in A\}\in\mathscr{F}\}\in\mathscr{F},$$

we have $\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in B\}\in\mathscr{F}\}\in\mathscr{F}$ and therefore $B\in\mathscr{F}\cdot\mathscr{F}$.
3° Let $A, B\subset\mathscr{F}\cdot\mathscr{F}$. Since

$$\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in A\cap B\}\in\mathscr{F}\}$$
$$=\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in A\}\in\mathscr{F}\}\cap\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in B\}\in\mathscr{F}\},$$

we have

$$\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1, y_2)\in A\cap B\}\in\mathscr{F}\}\in\mathscr{F},$$
$$A\cap B\in\mathscr{F}\cdot\mathscr{F}.$$

4° Let $A\in\mathscr{P}(R^+\times R^+)$ and $A\notin\mathscr{F}\cdot\mathscr{F}$. Define

$$T_{y_2}=\{y_1\in R^+;\ (y_1, y_2)\in A\}\quad\text{for}\quad y_2\in R^+,\quad\text{and}$$

$$S=\{y_2\in R^+;\ T_{y_2}\in\mathscr{F}\}.$$

Then $S\notin\mathscr{F}$ and hence $R^+-S\in\mathscr{F}$. Since

$$R^+-S=\{y_2\in R^+;\ T_{y_2}\notin\mathscr{F}\},\quad\text{and}$$

$$T_{y_2}\notin\mathscr{F}\rightleftarrows\{y_1\in R^+;\ (y_1,\ y_2)\in R^+\times R^+-A\}\in\mathscr{F}.$$

we have

$$R^+-S=\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1,\ y_2)\in R^+\times R^+-A\}\in\mathscr{F}\}\in\mathscr{F},$$

and hence

$$R^+\times R^+-A=A^c\in\mathscr{F}\cdot\mathscr{F}.$$

We have therefore proved (i).
(ii) 1° It is clear that $\phi\notin\mathscr{F}\times\mathscr{F}$.
2° Let $A\in\mathscr{F}\times\mathscr{F}$ and $A\subset A_0$. There are $B_1,\ C_1\in\mathscr{F}$ such that $B_1\times C_1\subset A$. Thus we have $B_1\times C_1\subset A_0$, and therefore $A_0\in\mathscr{F}\cdot\mathscr{F}$.
3° Let $A_1,\ A_2\in\mathscr{F}\times\mathscr{F}$. There are $B_1,\ B_2,\ C_1,\ C_2\in\mathscr{F}$ such that $B_1\times C_1\subset A_1$ and $B_2\times C_2\subset A_2$. Since

$$(B_1\cap B_2)\times(C_1\cap C_2)\subset A_1\cap A_2\quad\text{and}\quad B_1\cap B_2,\ C_1\cap C_2\in\mathscr{F},$$

we have $A_1\cap A_2\in\mathscr{F}\times\mathscr{F}$.
4° Let $A\in\mathscr{F}\times\mathscr{F}$. There are $B,\ C\in\mathscr{F}$ such that $B\times C\subset A$. Since $\{y_2\in R^+;\ \{y_1\in R^+;\ (y_1,\ y_2)\in B\times C\}\in\mathscr{F}\}\in\mathscr{F}$ we have $B\times C\in\mathscr{F}\cdot\mathscr{F}$ and hence $A\in\mathscr{F}\cdot\mathscr{F}$. We have therefore proved (ii).
(iv) Since

$$\begin{aligned}&\{A\in\mathscr{P}(R^+\times R^+\times R^+);\ \{y_3\in R^+;\ \{(y_1,\ y_2)\in R^+\times R^+;\ (y_1,\ y_2,\ y_3)\in A\}\in\mathscr{F}\cdot\mathscr{F}\}\in\mathscr{F}\}\\ &\quad=\{A\in\mathscr{P}(R^+\times R^+\times R^+);\ \{y_3\in R^+;\ \{y_2\in R^+;\ \{y_1\in R^+;\ (y_1,\ y_2,\ y_3)\in A\}\in\mathscr{F}\}\\ &\quad\in\mathscr{F}\}\in\mathscr{F}\}\\ &\quad=\{A\in\mathscr{P}(R^+\times R^+\times R^+);\ \{(y_2,\ y_3)\in R^+\times R^+;\ \{y_1\in R^+;\ (y_1,\ y_2,\ y_3)\in A\}\in\mathscr{F}\}\\ &\quad\in\mathscr{F}\cdot\mathscr{F}\}\end{aligned}$$

we have

$$(\mathscr{F}\cdot\mathscr{F})\cdot\mathscr{F}=\mathscr{F}\cdot\mathscr{F}\cdot\mathscr{F}=\mathscr{F}\cdot(\mathscr{F}\cdot\mathscr{F}).$$

DEFINITION 2.3. Let $K\neq\phi$, and let $a(y_1,\ y_2),\ b(y_1,\ y_2)\in\prod_{(y_1,y_2)\in R^+\times R^+}K$. Define $a(y_1,\ y_2)\sim 2\,b(y_1,\ y_2)$ if the following condition is satisfied:

$$\{(y_1,\ y_2)\in R^+\times R^+;\ a(y_1,\ y_2)=b(y_1,\ y_2)\}\in\mathscr{F}\cdot\mathscr{F}.$$

It is easy to see that this relation ~ 2 is an equivalence relation. Define

$$^{(*2)}K = \prod_{(y_1, y_2) \in R^+ \times R^+} K/\sim 2.$$

The equivalence class determined by $a(y_1, y_2)$ will be denoted with $[a(y_1, y_2)]$.

THEOREM 2.4. $$^{**}K = {}^{(*2)}K.$$

PROOF. If $(a(y_1))(y_2) \in \prod_{y_2 \in R^+} (\prod_{y_1 \in R^+} K)$, then we consider $(a(y_1))(y_2) \in \prod_{(y_1, y_2) \in R^+ \times R^+} K$, and write $(a(y_1))(y_2) = a(y_1, y_2)$. Let $[[a(y_1)](y_2)], [[b(y_1)](y_2)] \in {}^{**}K$. Since

$$\begin{aligned}
&[[a(y_1)](y_2)] = [[b(y_1)](y_2)] \\
&\rightleftarrows \{y_2 \in R^+;\ [a(y_1)](y_2) = [b(y_1)](y_2)\} \in \mathscr{F} \\
&\rightleftarrows \{y_2 \in R^+;\ \{y_1 \in R^+;\ (a(y_1))(y_2) = (b(y_1))(y_2)\} \in \mathscr{F}\} \in \mathscr{F} \\
&\rightleftarrows \{(y_1, y_2\} \in R^+ \times R^+;\ a(y_1, y_2) = b(y_1, y_2)\} \in \mathscr{F} \cdot \mathscr{F} \\
&\rightleftarrows [a(y_1, y_2)] = [b(y_1, y_2)],
\end{aligned}$$

we immediately have $^{**}K = {}^{(*2)}K$.

COROLLARY 2.5. $$^{**}\mathrm{Map}(R^n, C) = {}^{(*2)}\mathrm{Map}(R^n, C).$$

DEFINITION 2.6. Let $K \neq \phi$, and let $a(y_1, y_2, y_3), b(y_1, y_2, y_3) \in \prod_{(y_1, y_2, y_3) \in R^+ \times R^+ \times R^+} K$. Define $a(y_1, y_2, y_3) \sim 3\, b(y_1, y_2, y_3)$ if the following condition is satisfied:

$$\{(y_1, y_2, y_3) \in R^+ \times R^+ \times R^+;\ a(y_1, y_2, y_3) = b(y_1, y_2, y_3)\} \in \mathscr{F} \cdot \mathscr{F} \cdot \mathscr{F}.$$

It is easy to see that this relation ~ 3 is an equivalence relation. Define

$$^{(*3)}K = \prod_{(y_1, y_2, y_3) \in R^+ \times R^+ \times R^+} K/\sim 3.$$

The equivalence class determined by a function $a(y_1, y_2, y_3)$ will be denoted by $[a(y_1, y_2, y_3)]$.

THEOREM 2.7. *Let* $K \neq \phi$. *Then*

$$^{(*3)}K = {}^{*(*2)}K = {}^{(*2)*}K = {}^{***}K. \tag{2.7}$$

PROOF. Since

$$\begin{aligned}
\prod_{(y_1, y_2, y_3) \in R^+ \times R^+ \times R^+} K/\sim 3 &= \prod_{y_3 \in N^+} \Big(\prod_{(y_1, y_2) \in R^+ \times R^+} K/\sim 2\Big)/\sim \\
&= \prod_{(y_2, y_3) \in R^+ \times R^+} \Big(\prod_{y_1 \in R^+} K/\sim\Big)/\sim 2 \\
&= \prod_{y_3 \in R^+} \Big(\prod_{y_2 \in R^+} \Big(\prod_{y_1 \in R^+} K/\sim\Big)/\sim\Big)/\sim,
\end{aligned}$$

we immediately have (2.7).

COROLLARY 2.8. $^{(*3)}\mathrm{Map}(R^n, C) = {}^{(*2)*}\mathrm{Map}(R^n, C)$

$$= {}^{(*2)*}\mathrm{Map}(R^n, C) = {}^{***}\mathrm{Map}(R^n, C).$$

We can generalize Theorem 2.7 and Corollary 2.8 as follows:

THEOREM 2.9. *Let $K \neq \phi$. Then*

$$^{(*d)}K = {}^{*(*(d-1))}K = \cdots = {}^{*\cdots*}K.$$

COROLLARY 2.10. $^{(*d)}\mathrm{Map}(R^n, C) = {}^{*(*(d-1))}\mathrm{Map}(R^n, C) = \cdots = {}^{*\cdots*}\mathrm{Map}(R^n, C)$.

EXAMPLE 2.11. Let

$$[\delta(x_1,\ldots, x_n, y_1,\ldots, y_n)]$$

$$= \left[\frac{1}{(2\pi i)^n}\left(\frac{1}{x_1 - iy_1} - \frac{1}{x_1 + iy_1}\right)\cdots\left(\frac{1}{x_n - iy_n} - \frac{1}{x_n + iy_n}\right)\right]$$

$$= \left[\frac{1}{\pi^n}\cdot\frac{y_1}{x_1^2 + y_1^2}\cdots\frac{y_n}{x_n^2 + y_n^2}\right] \quad \text{for} \quad x_1,\ldots, x_n \in R \text{ and } y_1,\ldots, y_n \in R^+.$$

Then $[\delta(x_1,\ldots, x_n, y_1,\ldots, y_n)]$ is a delta function of n-variables and $[\delta(x_1,\ldots, x_n, y_1,\ldots, y_n)] \in {}^{(*n)}\mathrm{Map}(R^n, C)$.

References

[1] W. W. Comfort and S. Negrepontis, The theory of ultrafilters, Springer, Berlin-Heiderberg-New York. 1974.

[2] Y. Kuribayashi, A generalization of the concept of functions (I), J. Fac. Educ. Tottori Univ., Nat. Sci., 27-2 (1977), 27–31.

[3] ———, A generalization of the concept of functions (II), J. Fac. Educ. Tottori Univ., Nat. Sci., 28-1 (1978), 1–8.