On the Solutions of the Equation $f_0 T = g_0 \delta^{(n)}$

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1. Introduction

Let f_0 and g_0 be real valued, indefinitely differentiable, functions defined on R. We assume that the function f_0 has the following property:

(1.1) There exists a neighbourhood U of 0 such that

$$f_0(x) \neq 0 \quad \text{if} \quad x \in U - \{0\}.$$

The main purpose of the present paper is to give the solutions of the following equation:

$$f_0 T = g_0 \delta^{(n)}, \tag{1.2}$$

where δ is the Dirac delta-distribution.

2. Preliminaries

We use the notations R^N and $(\mathscr{M})$ as follows:

$$R^N = \{a = (a_1, a_2, \ldots, a_k, \ldots);\ a_k \in R \quad \text{for each } k \in N\},$$

where R denotes the set of all real numbers, N denotes the set of all natural numbers.

$$(\mathscr{M}) = \{f = (f_1, f_2, \ldots, f_k, \ldots);\ f_k \in (\mathfrak{B}) \quad \text{for each } k \in N\},$$

where $(\mathfrak{B})$ is the set of all real valued, Lebesgue measurable, functions defined on R.

DEFINITION 2.1. Let $a, b \in R^N$. Define

(2.1) $a = b$ if there exists a natural number k_0 such that $a_k = b_k$ whenever $k \geqq k_0$ and

(2.2) $a \fallingdotseq b$ if it satisfies the condition:

$$\lim_{k \to \infty} |a_k - b_k| = 0.$$

REMARK 2.1. Let $a \in R$. Then we shall identify a with $(a, a, \ldots, a, \ldots)$. We have $R \subset R^N$.

DEFINITION 2.2. Let $f, g \in (\mathscr{M})$. Define

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(2.3) $f=g$ if there exists a natural number k_0 such that $f_k(x)=g_k(x)$ for a.e. $x\in R$ whenever $k\geqq k_0$.

DEFINITION 2.3. Let $f, g\in(\mathscr{M})$. Then the sum $f+g$, the product fg and the quotient $\frac{f}{g}$ are defined respectively by

$$f+g=(f_1+g_1, f_2+g_2,\dots, f_k+g_k,\dots), \tag{2.4}$$

$$fg=(f_1g_1, f_2g_2,\dots, f_kg_k,\dots), \tag{2.5}$$

$$\frac{f}{g}=\left(\left(\frac{f_1}{g_1}\right)^*, \left(\frac{f_2}{g_2}\right)^*,\dots, \left(\frac{f_k}{g_k}\right)^*,\dots\right), \tag{2.6}$$

where $\left(\frac{f_k}{g_k}\right)^*$ is defined as follows:

$$\left(\frac{f_k(x)}{g_k(x)}\right)^*=\frac{f_k(x)}{g_k(x)} \qquad \text{where } \frac{f_k(x)}{g_k(x)} \text{ is defined and}$$

$$\left(\frac{f_k(x)}{g_k(x)}\right)^*=0 \qquad \text{elsewhere.}$$

REMARK 2.2. Let $f\in(\mathfrak{B})$. Then we shall identify f with $(f, f,\dots, f,\dots)$. We have $(\mathfrak{B})\subset(\mathscr{M})$.

We immediately have the following lemma:

LEMMA 2.1. *Let $f, g\in(\mathscr{M})$. Then*

$$f+g\in(\mathscr{M}), \tag{2.7}$$

$$fg\in(\mathscr{M}). \tag{2.8}$$

DEFINITION 2.4. A function $f\in(\mathscr{M})$ is said to be class C^n if there exists a natural number k_0 such that f_k is class C^n whenever $k\geqq k_0$.

Let $f\in(\mathscr{M})$ be a class C^n function. Define

$$\frac{d^n}{dx^n}f=\left(\left(\frac{d^n}{dx^n}f_1\right)^*, \left(\frac{d^n}{dx^n}f_2\right)^*,\dots, \left(\frac{d^n}{dx^n}f_k\right)^*,\dots\right), \tag{2.9}$$

where $\left(\frac{d^n}{dx^n}f_k\right)^*$ is defined as follows:

$$\left(\frac{d^n}{dx^n}f_k(x)\right)^*=\frac{d^n}{dx^n}f_k(x) \qquad \text{where } \frac{d^n}{dx^n}f_k(x) \text{ is defined and finite and}$$

$$\left(\frac{d^n}{dx^n}f_k(x)\right)^*=0 \qquad \text{elsewhere.}$$

As in [1], we use the notation $(\mathscr{L}\mathscr{S})$ as follows:

$$(\mathscr{L}\mathscr{S})=\{f=(f_1, f_2,\dots, f_k,\dots);\ f_k\in(\mathfrak{M}) \text{ for each } k\in N\},$$

where $(\mathfrak{M})$ is the set of all real valued, locally summable, functions defined on R.

DEFINITION 2.5. Let $f\in(\mathscr{L}\mathscr{S})$ and $\varphi\in(\mathscr{D})$. Define

$$f(\varphi)=\left(\int_{-\infty}^{\infty} f_1(x)\varphi(x)dx,\ \int_{-\infty}^{\infty} f_2(x)\varphi(x)dx,\dots,\ \int_{-\infty}^{\infty} f_k(x)\varphi(x)dx,\dots\right). \tag{2.10}$$

According to G. Takeuti [2], we use the notation $\overset{w}{=}$ as follows:

DEFINITION 2.6. Let f, $g\in(\mathscr{L}\mathscr{S})$ and $S\in(\mathscr{D})$. Define

(2.11) $f\overset{w}{=}g$ if and only if $f(\varphi)\fallingdotseq g(\varphi)$ for each $\varphi\in(\mathscr{D})$ and

(2.12) $f\overset{w}{=}S$ if and only if $f(\varphi)\fallingdotseq S(\varphi)$ for each $\varphi\in(\mathscr{D})$.

EXAMPLE 2.1. Let $\delta_0=(\delta_{01}, \delta_{02},\dots, \delta_{0k},\dots)\in(\mathscr{L}\mathscr{S})$ be a function having the following properties:

$$\delta_{0k}\in(\mathscr{E}), \tag{2.13}$$

$$\lim_{k\to\infty}\int_{-\infty}^{\infty}\delta_{0k}(x)\varphi(x)dx=\varphi(0) \qquad \text{for each} \quad \varphi\in(\mathscr{D}). \tag{2.14}$$

Let $g\in(\mathscr{E})$. Then we have

$$\begin{aligned} g(x)\delta_0^{(n)} &\overset{w}{=} \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k} g_{(0)}^{(n-k)}\delta_0^{(k)} \\ &\overset{w}{=} \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k} g_{(0)}^{(n-k)}\delta^{(k)}. \end{aligned} \tag{2.15}$$

LEMMA 2.2. *Let $s=(s_1, s_2,\dots, s_k,\dots)$, $t=(t_1, t_2,\dots, t_k,\dots)\in(\mathscr{L}\mathscr{S})$ and $f\in(\mathscr{E})$.*

$$\textit{If} \quad s\overset{w}{=}t \quad \textit{then} \quad fs\overset{w}{=}ft. \tag{2.16}$$

PROOF. Let $\varphi\in(\mathscr{D})$. Since $f\varphi\in(\mathscr{D})$, we have

$$fs(\varphi)=s(f\varphi)\fallingdotseq t(f\varphi)=ft(\varphi).$$

Therefore we have $fs\overset{w}{=}ft$.

DEFINITION 2.7. Let $f\in(\mathscr{L}\mathscr{S})$ and $S\in(\mathscr{D})'$. Define

$$\text{Pf.}(f)\overset{w}{=}S, \tag{2.17}$$

if there exists a function p such that

$$p(k, \varphi)=c_0(\varphi)\log k+c_1(\varphi)k^{\lambda_1(\phi)}+\cdots+c_n(\varphi)k^{\lambda_n(\phi)} \tag{2.18}$$

for each $k \in N$ and $\varphi \in (\mathscr{D})$, the $c_i(\varphi)$, $0 \leqq i \leqq n$, and the $\lambda_i(\varphi)$, $1 \leqq i \leqq n$, which are real numbers, depend on φ, and $0 < \lambda_1(\varphi) < \cdots < \lambda_n(\varphi)$,

$$\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx - p(k, \varphi)\right) = S(\varphi) \tag{2.19}$$

for each $\varphi \in (\mathscr{D})$.

We immediately have the following lemma:

LEMMA 2.3. *Let $f \in (\mathscr{L}\mathscr{S})$ and $S \in (\mathscr{D})'$. Let p_1, p_2 be functions of the form* (2.18). *If*

$$\begin{aligned}&\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx - p_1(k, \varphi)\right)\\ &= \lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx - p_2(k, \varphi)\right)\\ &= S(\varphi) \qquad \textit{for each} \quad \varphi \in (\mathscr{D}),\end{aligned}$$

then we have $p_1 = p_2$.

LEMMA 2.4. *Let f, $g \in (\mathscr{L}\mathscr{S})$ and S, $T \in (\mathscr{D})'$.*

$$\textit{If} \quad \mathrm{Pf.}(f) \overset{w}{=} S \quad \textit{and} \quad \mathrm{Pf.}(g) \overset{w}{=} T \quad \textit{then} \quad \mathrm{Pf.}(f+g) \overset{w}{=} S+T. \tag{2.20}$$

PROOF. Let p_1, p_2 be functions of the form (2.18) with the following properties respectively:

$$\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx - p_1(k, \varphi)\right) = S(\varphi),$$

$$\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} g_k(x)\varphi(x)dx - p_2(k, \varphi)\right) = T(\varphi),$$

for each $\varphi \in (\mathscr{D})$.

Then we have

$$\begin{aligned}&\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} (f_k(x)+g_k(x))\varphi(x)dx - (p_1(k, \varphi)+p_2(k, \varphi))\right)\\ &= \lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx - p_1(k, \varphi)\right) + \lim_{k\to\infty}\left(\int_{-\infty}^{\infty} g_k(x)\varphi(x)dx - p_2(k, \varphi)\right)\\ &= S(\varphi) + T(\varphi)\\ &= (S+T)(\varphi) \qquad \text{for each} \quad \varphi \in (\mathscr{D}).\end{aligned}$$

If we define a function p_0 by

$$p_0(k, \varphi)=p_1(k, \varphi)+p_2(k, \varphi),$$

then p_0 is a function of the form (2.18).
Thus we obtain $\mathrm{Pf.}(f+g)\overset{\mathrm{w}}{=}S+T$.

LEMMA 2.5. *Let* $f\in(\mathscr{L}\mathscr{S})$, $h\in(\mathscr{E})$ *and* $S\in(\mathscr{D})'$.

(2.21) $$\textit{If}\quad \mathrm{Pf.}(f)\overset{\mathrm{w}}{=}S\quad \textit{then}\quad \mathrm{Pf.}(hf)\overset{\mathrm{w}}{=}hS.$$

PROOF. Let p be a function of the form (2.18) having the following property:

$$\lim_{k\to\infty}\left(\int_{-\infty}^{\infty}f_k(x)\varphi(x)dx-p(k, \varphi)\right)=S(\varphi)\qquad \text{for each}\quad \varphi\in(\mathscr{D}).$$

Let $\varphi\in(\mathscr{D})$. Since $h\varphi\in(\mathscr{D})$, we have

$$\begin{aligned}\lim_{k\to\infty}&\left(\int_{-\infty}^{\infty}f_k(x)h(x)\varphi(x)dx-p(k, h\varphi)\right)\\&=S(h\varphi)\\&=hS(\varphi).\end{aligned}$$

If we define a function p_0 by

$$p_0(k, \varphi)=p(k, h\varphi),$$

then p_0 is a function of the form (2.18).
Therefore we have (2.21).

3. Main Results

Let $\delta_*=(\delta_1, \delta_2,\dots, \delta_k,\dots)\in(\mathscr{L}\mathscr{S})$ be a function having the following properties:

(3.1) $$0<a_k<b_k,$$

(3.2) $$a_k^{-1}\quad\text{is a monomial of } k,$$

(3.3) $$b_k-a_k<e^{-k},$$

(3.4) $$\delta_k\in(\mathscr{D}),$$

(3.5) $$\delta_k(x)\geqq 0,$$

(3.6) $$\mathrm{Car}(\delta_k)\subset(a_k, b_k),$$

(3.7) $$\int_{-\infty}^{\infty}\delta_k(x)dx=1.$$

LEMMA 3.1. *Let* δ_0 *satisfy the same assumptions in Example* 2.1. *Then we have*

(3.8) $$\delta_*^{(n)} \overset{w}{=} \delta_0^{(n)} \overset{w}{=} \delta^{(n)} \qquad \textit{if} \quad n \in N.$$

PROOF. Clear from Definition 2.6.

THEOREM 3.1. *Let* $f_0(0)=f'_0(0)=\cdots=f_0^{(m-1)}(0)=0$, $f_0^{(m)}(0)\neq 0$, $m\leqq M$ *and* $c_l\in R$, $1\leqq l\leqq M$. *Let G be a function defined by*

$$G(x)=\frac{1}{\dfrac{f_0(x)}{x^m}} \qquad (x\neq 0),$$

$$G(0)=\frac{1}{\dfrac{f_0^{(m)}(0)}{m!}}.$$

Then

(3.9) $$f_0(\delta_*^{(n)}/f_0)=\delta_*^{(n)},$$

(3.10) $$f_0((\delta_*^{(n)}+\sum_{l=1}^{M} c_l x^l \delta_*)/f_0) \overset{w}{=} \delta_*^{(n)},$$

(3.11) *The distribution*

$$\mathrm{Pf.}((\delta_*^{(n)}+\sum_{l=1}^{M} c_l x^l \delta_*)/f_0)$$

$$\overset{w}{=} \sum_{k=0}^{n}(-1)^{m-k}\frac{n!}{k!(m+n-k)!}G^{(k)}(0)\delta^{(m+n-k)}+\sum_{l=1}^{m}(-1)^{m-l}c_l G(x)\delta^{(m-l)}$$

is the distributional general solution of the equation

$$f_0 T=\delta^{(n)}.$$

PROOF. We shall only prove (3.11).
Let $\varphi\in(\mathcal{D})$. Then

$$\int_{-\infty}^{\infty}\frac{\delta_k^{(n)}(x)\varphi(x)}{f_0(x)}dx$$

$$=\int_{a_k}^{b_k}\delta_k^{(n)}(x)\frac{\varphi(x)}{x^m}G(x)dx$$

$$=(-1)^n\int_{a_k}^{b_k}\delta_k(x)\left(\frac{\varphi(x)}{x^m}G(x)\right)^{(n)}dx$$

$$=(-1)^n\int_{a_k}^{b_k}\delta_k(x)\left(\left(\frac{\varphi(x)}{x^m}\right)^{(n)}G(x)\right.$$

$$\left.+\cdots+\binom{n}{k}\left(\frac{\varphi(x)}{x^m}\right)^{(n-k)}G^{(k)}(x)+\cdots+\frac{\varphi(x)}{x^m}G^{(n)}(x)\right)dx$$

$$=(-1)^n\int_{a_k}^{b_k}\delta_k(x)\Big(\Big(\cdots+\frac{n!\,\varphi^{(m+n)}(0)}{(m+n)!}+\cdots\Big)G(x)$$

$$+\cdots+\Big(\cdots+\binom{n}{k}\frac{(n-k)!\,\varphi^{(m+n-k)}(0)}{(m+n-k)!}+\cdots\Big)G^{(k)}(x)$$

$$+\cdots+\Big(\cdots+\frac{\varphi^{m}(0)}{m!}+\cdots\Big)G^{(n)}(x)\Big)\,dx.$$

Since

$$(-1)^n\Big(\frac{n!\varphi^{(m+n)}(0)}{(m+n)!}G(0)+\cdots+\binom{n}{k}\frac{(n-k)!\varphi^{(m+n-k)}(0)}{(m+n-k)!}G^{(k)}(0)$$

$$+\cdots+\frac{\varphi^{(m)}(0)}{m!}G^{(n)}(0)\Big)=(-1)^m\frac{n!}{(m+n)!}G(0)\delta^{(m+n)}(\varphi)$$

$$+\cdots+(-1)^{(m-k)}\frac{n!}{k!(m+n-k)!}G^{(k)}(0)\,\delta^{(m+n-k)}(\varphi)$$

$$+\cdots+(-1)^{m-n}\frac{1}{m!}G^{(m)}(0)\delta^{(m)}(\varphi)$$

we have

$$\mathrm{Pf.}\Big(\frac{\delta_*^{(n)}}{f_0}\Big)\overset{w}{=}\sum_{n=0}^{k}(-1)^{m-k}\frac{n!}{k!(m+n-k)!}G^{(k)}(0)\delta^{(m+n-k)}.$$

By Lemma 2.5, we see that

$$f_0\mathrm{Pf.}\Big(\frac{\delta_*^{(n)}}{f_0}\Big)\overset{w}{=}\delta^{(n)}. \tag{3.12}$$

We immediately have

$$f_0\ \mathrm{Pf.}\Big(\frac{c_lx^l\delta_*}{f_0}\Big)\overset{w}{=}\mathrm{Pf.}\ (c_lx^l\delta_*)\overset{w}{=}c_lx^l\delta_*\overset{w}{=}0,$$

$$\mathrm{Pf.}\Big(\frac{c_lx^l\delta_*}{f_0}\Big)\overset{w}{=}c_lG(x)\ \mathrm{Pf.}\Big(\frac{\delta_*}{x^{m-l}}\Big)$$

$$\overset{w}{=}c_l\frac{(-1)^{m-l}}{(m-l)!}G(x)\delta^{(m-l)}\qquad(1\leqq l\leqq m),$$

$$\overset{w}{=}0\qquad(m<l).$$

Thus we obtain (3.11).

COROLLARY 3.1. *Let the hypotheses of Theorem* 3.1 *be fulfilled and let* $d_n\in R$, $0\leqq n\leqq N$. Then

(3.13)
$$f_0((\sum_{n=0}^{N} d_n\delta_*^{(n)})/f_0)=\sum_{n=0}^{N} d_n\delta_*^{(n)},$$

(3.14)
$$f_0((\sum_{n=0}^{N} d_n\delta_*^{(n)}+\sum_{l=1}^{M} c_l x^l\delta_*)/f_0 \overset{w}{=} \sum_{n=0}^{N} d_n\delta_*^{(n)},$$

(3.15) *The distribution*

$$\text{Pf. }((\sum_{n=0}^{N} d_n\delta_*^{(n)}+\sum_{l=1}^{M} c_l x^l\delta_*)/f_0)$$
$$\overset{w}{=}\sum_{n=0}^{N}\sum_{k=0}^{n}(-1)^{m-k}\frac{d_n n!}{k!(m+n-k)!}G^{(k)}(0)\delta^{(m+n-k)}+\sum_{l=1}^{m}\frac{(-1)^{m-l}}{(m-l)!}c_l G(x)\delta^{(m-l)},$$

is the distributional general solution of the equation

$$f_0T=\sum_{n=0}^{N} d_n\delta^{(n)}.$$

PROOF. Clear from Lemma 2.4.

REMARK 3.1. We directly have (3.12) as follows: Let $\varphi\in(\mathscr{D})$. Then

$$(-1)^m\frac{n!}{(m+n)!}G(0)\delta^{(m+n)}(f_0\varphi)+\cdots+(-1)^{m-k}\frac{n!}{k!(m+n-k)!}G^{(k)}(0)\delta^{(m+n-k)}(f_0\varphi)$$
$$+\cdots+(-1)^{m-n}\frac{1}{m!}G^{(n)}(0)\delta^{(m)}(f_0\varphi)$$
$$=(-1)^m(-1)^{m+n}\frac{n!}{(m+n)!}G(0)\delta((f_0\varphi)^{(m+n)})$$
$$+\cdots+(-1)^{m-k}(-1)^{m+n-k}\frac{n!}{k!(m+n-k)!}G^{(k)}(0)\delta((f_0\varphi)^{(m+n-k)})$$
$$+\cdots+(-1)^{m-n}(-1)^{m+n-n}\frac{1}{m!}\delta((f_0\varphi)^{(m)})$$
$$=(-1)^n\frac{n!}{(m+n)!}G(0)\left(\binom{m+n}{m}f_0^{(m)}(x)\varphi^{(n)}(x)+\binom{m+n}{m+1}f_0{}^{(m+1)}(x)\varphi^{(n-1)}(x)\right.$$
$$\left.+\cdots+\binom{m+n}{m+n}f_0^{(m+n)}(x)\right)_{x=0}$$
$$+\cdots+(-1)^n\frac{n!}{k!(m+n-k)!}G^{(k)}(0)\left(\binom{m+n-k}{m}f_0^{(m)}(x)\varphi^{(n-k)}(x)\right.$$
$$\left.+\cdots+\binom{m+n-k}{m+n-k}f_0^{(m+n-k)}(x)\varphi(x)\right)_{x=0}+\cdots+(-1)^n\frac{1}{m!}G^{(n)}(0)\left(f_0^{(m)}(x)\varphi(x)\right)_{x=0}$$

$$=(-1)^n\frac{1}{m!}G(0)f_0^{(m)}(0)\varphi^{(n)}(0)+(-1)^n\frac{n!}{(n-1)!(m+1)!}G(0)f_0^{(m+1)}(0)\varphi^{(n-1)}(0)$$

$$+\cdots+(-1)^n\frac{n!}{(m+n)!}G(0)f_0^{(m+n)}(0)\varphi(0)$$

$$+\cdots+(-1)^n\frac{n!}{k!(n-k)!m!}G^{(k)}(0)f_0^{(m)}(0)\varphi^{(n-k)}(0)$$

$$+\cdots+(-1)^n\frac{1}{m!}G^{(m)}(0)f_0^{(m)}(0)\varphi(0) \qquad \text{(A)}.$$

Now, if we select the terms with $\varphi^{(n-k)}(0)$, then

$$(-1)^n\frac{n!}{(n-k)!(m+k)!}G(0)f_0^{(m+k)}(0)\varphi^{(n-k)}(0)$$

$$+\cdots+(-1)^n\frac{n!}{k!(n-k)!m!}G^{(k)}(0)f_0^{(m)}(0)\varphi^{(n-k)}(0)$$

$$=(-1)^n\varphi^{(n-k)}(0)\frac{n!}{(n-k)!(m+k)!}\left(G(0)f_0^{(m+k)}(0)+\cdots+\frac{(m+k)!}{k!m!}G^{(k)}(0)f_0^{(m)}(0)\right)$$

$$=(-1)^n\varphi^{(n-k)}(0)\frac{n!}{(n-k)!(m+k)!}\left((G(x)f_0(x))^{(m+k)}\right)_{x=0}.$$

Since $G(x)f_0(x)=x^m$, we have

$$\left(G(x)f_0(x)\right)_{x=0}^{(m)}=m!,$$

$$\left((G(x)f_0(x)\right)_{x=0}^{(m+k)}=0 \qquad (1\leqq k\leqq n).$$

Therefore we have

$$\text{(A)}=(-1)^n\varphi^{(n)}(0)\frac{n!}{m!n!}m!$$

$$=(-1)^n\varphi^{(n)}(0)$$

$$=\delta^{(n)}(\varphi).$$

THEOREM 3.2. *Let $f_0^{(m)}(0)=0$ for each $m\in N$, $m\leqq M$ and $c_l\in R$, $1\leqq l\leqq M$. Then*

(3.16) $$f_0(\delta_*^{(n)}/f_0)=\delta_*^{(n)},$$

(3.17) $$f_0((\delta_*^{(n)}+\sum_{l=1}^{M}c_lx^l\delta_*)/f_0)\overset{w}{=}\delta_*^{(n)},$$

(3.18) *The equation*

$$f_0T=\delta^{(n)}$$

does not have any distributional solution.

PROOF. We shall only prove (3.18).
Since

$$f_0\delta^{(m)} \overset{w}{=} 0 \qquad \text{if} \quad m \in N,$$

we immediately have (3.18).

THEOREM 3.3. *Let the hypotheses of Theorem* 3.1 *be fulfilled. Then*

(3.19) $$f_0((g_0\delta_*^{(n)})/f_0) = g_0\delta_*^{(n)},$$

(3.20) $$f_0((g_0\delta_*^{(n)} + \sum_{l=1}^{M} c_l x^l \delta_*)/f_0) \overset{w}{=} g_0\delta_*^{(n)},$$

(3.21) *The distribution*

$$\text{Pf.}\,((g_0\delta_*^{(n)} + \sum_{l=1}^{M} c_l x^l \delta_*)/f_0)$$

$$\overset{w}{=} \sum_{k=0}^{n} (-1)^{m-k} \frac{n!}{k!(m+n-k)!} G^{(k)}(0) g_0(x) \delta^{(m+n-k)} + \sum_{l=1}^{m} \frac{(-1)^{m-l}}{(m-l)!} c_l G(x) \delta^{(m-l)}$$

is the distributional general solution of (1.2).

PROOF. We shall only prove (3.21).
We have, by Lemma 2.5,

$$\text{Pf.}\,((g_0\delta_*^{(n)})/f_0) \overset{w}{=} g_0 \text{Pf.}\,(\delta_*^{(n)}/f_0)$$

$$\overset{w}{=} g_0(x)\Big(\sum_{k=0}^{n} (-1)^{m-k} \frac{n!}{k!(m+n-k)!} G^{(k)}(0) \delta^{(m+n-k)}\Big)$$

$$\overset{w}{=} \sum_{k=0}^{n} (-1)^{m-k} \frac{n!}{k!(m+n-k)!} G^{(k)}(0) g_0(x) \delta^{(m+n-k)}.$$

Hence we have (3.21).

COROLFORY 3.2. *Let the hypotheses of Theorem* 3.1 *be fulfilled and let* $g_n \in (\mathscr{E})$, $0 \leqq n \leqq N$. *then*

(3.22) $$f_0((\sum_{n=0}^{N} g_n\delta_*^{(n)})/f_0) = \delta_*^{(n)},$$

(3.23) $$f_0((\sum_{n=0}^{N} g_n\delta_*^{(n)} + \sum_{l=1}^{M} c_l{}^l \delta_*)/f_0) \overset{w}{=} \sum_{n=0}^{N} g_n\delta_*^{(n)},$$

(3.24) *The distribution*

$$\text{Pf.}\,((\sum_{n=0}^{N} g_n\delta_*^{(n)} + \sum_{l=1}^{M} c_l x^l \delta_*)/f_0)$$

$$\overset{w}{=}\sum_{n=0}^{N}\sum_{k=0}^{n}(-1)^{m-k}\frac{n!}{k!(m+n-k)!}G^{(k)}(0)g_n(x)\delta^{(m+n-k)}$$

$$+\sum_{l=1}^{m}\frac{(-1)^{m-l}}{(m-l)!}c_lG(x)\delta^{(m-l)}$$

is the distributional general solution of the equation

$$f_0T=\sum_{n=0}^{N}g_n\delta^{(n)}.$$

PROOF. Clear from Lemma 2.4 and Theorem 3.3.

References

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