A Generalization of the Concept of Functions (I)

Yukio KURIBAYASHI*

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1. Introduction

In the present paper we intend to generalize the concept of functions, using ultrapowers.

We shall denote with $*\mathrm{Map}(R, C)$ the set of all generalized functions of one variable. Given any $S \in (\mathscr{D}')$, there is a set $U \subset *\mathrm{Map}(R, C)$ such that

$$S \overset{w}{=} [f(x, y)] \qquad \text{for each} \quad [f(x, y)] \in U.$$

We should like to use the property to solve the division problem of distributions.

2. The concept of generalized functions

We shall first give the definition of generalized functions of one variable.

DEFINITION 1. Let $R^+ = \{x \in R;\ x > 0\}$, and let $F = \{(0, a);\ a \in R^+\}$. Then F has the finite intersection property. We shall denote with $\mathscr{F}$ the ultrafilter generated by F.

Let K be the set R, or the set C, or the set $\mathrm{Map}(R, C)$, and let $A(y), B(y) \in \prod_{y \in R^+} K$. Define

$A(y) \sim B(y)$ if it satisfies the condition:

$$\{y;\ A(y) = B(y)\} \in \mathscr{F}.$$

It is easy to see that this relation $\sim$ is an equivalence relation. Define

$$*K = \Big(\prod_{y \in R^+} K\Big) / \sim.$$

The equivalence class determined by a function $A(y) \in \prod_{y \in R^+} K$ will be denoted with $[A(y)]$.

An element of the set $*\mathrm{Map}(R, C)$ is called the generalized function (G-function) of one variable.

REMARK 1. If $A(y) = A_0 \in K$, then we shall identify A_0 with $[A(y)]$. We have $K \subsetneqq *K$.

DEFINITION 2. (1) Let $[a(y)], [b(y)] \in *R$. Define

$[a(y)] \leqq [b(y)]$ if it satisfies the condition:

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* Laboratory of Mathematics, Faculty of Education, Tottori University, Tottori, Japan.

$$\{y;\ a(y) \leqq b(y)\} \in \mathscr{F}.$$

(2) Let $[a(y)], [b(y)] \in {}^*C$. Define
$[a(y)] \fallingdotseq [b(y)]$ if it satisfies the condition:

$$\lim_{y \to +0} |a(y) - b(y)| = 0.$$

(3) Let $[a(y)] \in {}^*R$. Define
$[a(y)] \fallingdotseq \pm\infty$ if it satisfies the condition:

$$\lim_{y \to +0} a(y) = \pm\infty.$$

REMARK 2. Definitions 2, 3,... and 9 are clearly well defined and proofs are omitted.

DEFINITION 3 (Interval). Let $[a(y)], [b(y)] \in {}^*R$ and $[a(y)] \leqq [b(y)]$. Define

$$[[a(y)], [b(y)]] = \{[c(y)] \in {}^*R;\ [a(y)] \leqq [c(y)] \leqq [b(y)]\}.$$

PROPOSION 1. *Let the hypotheses of Definition 3 be fulfilled. Then*

$$[[a(y)], [b(y)]] = \{[c(y)] \in {}^*R;\ \{y;\ a(y) \leqq c(y) \leqq b(y)\} \in \mathscr{F}\}.$$

DEFINITION 4. Let $[f(x, y)] \in {}^*\mathrm{Map}\,(R, C)$. We say $[f(x, y)]$ has a property P, if it satisfies the condition:

$$\{y;\ f(x, y) \text{ has a property } P \text{ as a function of } x \text{ alone}\} \in \mathscr{F}.$$

DEFINITION 5. Let $[a(y)] \in {}^*C$, and let $[f(x, y)], [g(x, y)] \in {}^*\mathrm{Map}(R, C)$. Then the scalar product $[a(y)][f(x, y)]$, the sum $[f(x, y)] + [g(x, y)]$, the difference $[f(x, y)] - [g(x, y)]$, the product $[f(x, y)][g(x, y)]$ and the quotient $[f(x, y)]/[g(x, y)]$ are defined respectively by

(1) $[a(y)][f(x, y)] = [a(y)f(x, y)]$,
(2) $[f(x, y)] + [g(x, y)] = [f(x, y) + g(x, y)]$,
(3) $[f(x, y)] - [g(x, y)] = [f(x, y) - g(x, y)]$,
(4) $[f(x, y)][g(x, y)] = [f(x, y)g(x, y)]$,
(5) $[f(x, y)]/[g(x, y)] = [(f(x, y)/g(x, y))^*]$,

where $(f(x, y)/g(x, y))^*$ is defined as follows:

$$(f(x, y)/g(x, y))^* = f(x, y)/g(x, y) \quad \text{where} \quad f(x, y)/g(x, y)$$

is defined and

$$= 0 \quad \text{elsewhere.}$$

We immediately have the following proposition:

PROPOSITION 2. *The space* $^*\mathrm{Map}(R, C)$ *is a commutative ring.*

DEFINITION 6 (Derivative). Let $[f(x, y)]$ is a differentiable G-function. Define

$$\frac{d}{dx}[f(x, y)]=\left[\left(\frac{\partial}{\partial x}f(x, y)\right)^*\right],$$

where $\left(\frac{\partial}{\partial x}f(x, y)\right)^*$ is defined as follows:

$$\left(\frac{\partial}{\partial x}f(x, y)\right)^*=\frac{\partial}{\partial x}f(x, y) \qquad \text{where } \frac{\partial}{\partial x}f(x, y)$$

is defined and finite and

$$=0 \quad \text{elsewhere.}$$

DEFINITION 7 (Integral). Let $f(x, y)$ be a integrable G-function over $[[a(y)], [b(y)]]$. Define

$$\int_{[a(y)]}^{[b(y)]}[f(x, y)]dx=\left[\int_{a(y)}^{b(y)}f(x, y)dx^*\right]$$

where $\int_{a(y)}^{b(y)}f(x, y)dx^*$ is defined as follows:
$\int_{a(y)}^{b(y)}f(x, y)dx^*=\int_{a(y)}^{b(y)}f(x, y)dx$ where $f(x, y)$ is integrable over $[a(x), b(y)]$ and

$$=0 \quad \text{elsewhere.}$$

According to G. Takeuti [3], we use the notation $\overset{w}{=}$ as follows:

DEFINITION 8. Let $[f(x, y)]$, $[g(x, y)]$ be locally integrable G-functions, and let $S\in(\mathscr{D}')$. Define

(1) $[f(x, y)]\overset{w}{=}[g(x, y)]$ if and only if

$$\lim_{y\to+0}\left|\int_{-\infty}^{\infty}f(x, y)\varphi(x)dx-\int_{-\infty}^{\infty}g(x, y)\varphi(x)dx\right|=0 \quad \text{for each} \quad \varphi\in(\mathscr{D})$$

and

(2) $[f(x, y)]\overset{w}{=}S$ if and only if

$$\lim_{y\to+0}\int_{-\infty}^{\infty}f(x, y)\varphi(x)dx=S(\varphi) \qquad \text{for each} \quad \varphi\in(\mathscr{D}).$$

We immediately have the following proposition:

PROPOSITION 3. *Let $[f(x, y)]$, $[g(x, y)]$ be locally integrable G-functions.*

If $\quad [f(x, y)]=[g(x, y)]$ *then* $[f(x, y)]\overset{w}{=}[g(x, y)]$.

A. Robinson [1] gave the following result:

THEOREM 1. *Given any $S \in (\mathscr{D}')$ there is a G-function $[f(x, y)]$ such that*

$$S \overset{w}{=} [f(x, y)].$$

EXAMPLES. We have the following equalities:

(1) $\int_{[\sqrt{y}]}^{[2]} \left[\frac{1}{x}\right] dx = \left[\log 2 - \frac{1}{2} \log y\right],$

(2) $[(x+iy)-(x-iy)] = [i2y] \overset{w}{=} 0,$

(3) $[(x^2-y^2+i2xy)-(x^2-y^2-i2xy)] = [i4xy] \overset{w}{=} 0,$

(4) $\dfrac{[(x+iy)-(x-iy)]}{[(x^2-y^2+i2xy)-(x^2-y^2-i2xy)]} = \left[\left(\frac{1}{2x}\right)^*\right],$

(5) $\left[\frac{1}{-2\pi i}\left(\frac{1}{x+iy} - \frac{1}{x-iy}\right)\right] = \left[\frac{y}{\pi(x^2+y^2)}\right] \overset{w}{=} \delta,$

(6) $[y]\left[\frac{y}{\pi(x^2+y^2)}\right] \overset{w}{=} 0,$

(7) $[y]\left[\frac{y}{\pi(x^2+y^2)}\right]\left[\frac{1}{y}\right] \overset{w}{=} \delta,$

(8) $\left[\frac{\pi(x^2+y^2)}{y}\right]\left[\frac{y}{\pi(x^2+y^2)}\right] = [1].$

DEFINITION 9 (Fourier transform). Let $[f(x, y)]$ be a locally integrable G-function, and let $[a(y)] \fallingdotseq -\infty$, $[b(y)] \fallingdotseq \infty$.
Define

$$[\mathscr{F}(f)(t, y; a, b)] = \left[\int_{a(y)}^{b(y)} f(x, y) e^{-2\pi i t x} dx^*\right].$$

THEOREM 2. *Let* $[b(y)] \fallingdotseq \infty$. *Then*

$$[\mathscr{F}(1)(t, y; -b, b)] = \left[\frac{1}{\pi t} \sin(2\pi t b(y))\right]$$

$$\overset{w}{=} \delta.$$

PROOF. Since

$$\int_{-b(y)}^{b(y)} e^{-2\pi i t x} dx = \frac{1}{\pi t} \sin(2\pi t b(y))$$

we have

$$[\mathscr{F}(1)(t, y; -b, b)] = \left[\frac{1}{\pi t} \sin(2\pi t b(y))\right].$$

Let $\varphi \in (\mathscr{D})$. Then

$$\int_{-b(y)}^{b(y)} \frac{\varphi(t)}{\pi t} \sin(2\pi t b(y)) dt = \frac{1}{\pi} \int_{Car(\varphi)} \varphi(t) \frac{\sin(2\pi t b(y))}{t} dt$$

for sufficiently small $y>0$.
Using Dirichlet's theorem we have the result that

$$\lim_{y\to+0} \frac{1}{\pi} \int_{Car(\varphi)} \varphi(t) \frac{\sin(2\pi t b(y))}{t} dt = \varphi(0).$$

Therefore, we have

$$[\mathscr{F}(1)(t,\, y;\, -b,\, b)] \overset{w}{=} \delta.$$

THEOREM 3. *Let* $[b(y)] \fallingdotseq \infty$, *and let*

$$\delta_0(x,\, y) = \frac{1}{-2\pi i}\left(\frac{1}{x+iy} - \frac{1}{x-iy}\right).$$

Then we have

$$[\mathscr{F}(\delta_0)(t,\, y;\, -b,\, b)] \fallingdotseq 1 \qquad \textit{for each} \quad t \in R.$$

PROOF. We immediately have

$$\frac{1}{2\pi i} \int_{-b(y)}^{b(y)} \left(\frac{1}{x-iy} - \frac{1}{x+iy}\right) e^{-2\pi i t x} dx$$

$$= \frac{1}{\pi} \int_{-2\pi t b(y)}^{2\pi t b(y)} \frac{2\pi t y}{u^2 + (2\pi t y)^2} \cos u \, du \qquad (2\pi t x = u).$$

Using the equality

$$\frac{1}{\pi} \int_0^\infty \frac{\cos x}{x^2 + a^2} dx = \frac{e^{-a}}{2a}$$

we have

$$\lim_{y\to+0} \frac{1}{\pi} \int_{-2\pi t b(y)}^{2\pi t b(y)} \frac{2\pi t y}{u^2 + (2\pi t y)^2} \cos u \, du = 1 \qquad \text{for each} \quad t \in R.$$

Therefore we have

$$[\mathscr{F}(\delta_0)(t,\, y;\, -b,\, b)] \fallingdotseq 1 \qquad \text{for each} \quad t \in R.$$

References

[1] A. ROBINSON, Non-standard analysis, North-Holland, 1966.
[2] M. SAITO, Non-standard analysis (in Japanese), Tokyotosho, 1976.
[3] G. TAKEUTI, On Dirac spaces (in Japanese), Kagaku, **32**–9 (1962), 452–456.