On the Finite Parts of Divergent Integrals

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1. Introduction

The products of distributions have been discussed by B. Fisher [1]–[5], M. Itano and S. Hatano [6], J. Mikusiński [7], L. Schwartz [8], H. G. Tillmann [9], etc.

B. Fisher [5] gave the following result:

THEOREM A. $$x^{-r}\delta^{(r-1)}(x)=\frac{(-1)^r(r-1)!}{2(2r-1)!}\delta^{(2r-1)}(x)$$

for $r=1, 2, 3,\ldots$.

M. Itano and S. Hatano [6] gave the following result:

THEOREM B. *The product* $\mathrm{Pf}\frac{1}{x^n}\circ\delta^{(n-1)}$ *exists for any positive integer* n *and*

$$\mathrm{Pf}\frac{1}{x^n}\circ\delta^{(n-1)}=\frac{(-1)^n(n-1)!}{2(2n-1)!}\delta^{(2n-1)}.$$

The main purpose of this paper is to give the follownig two theorems:

THEOREM 1. *Let* $n\in N^{\cup}\{0\}$ *and* $\alpha\in R$. *Then*

$$\text{(1)}\qquad \mathrm{Pf.}\,(x_+^\alpha\delta^{(n)})\overset{w}{=}\frac{(-1)^{-\alpha}n!}{(n-\alpha)!}\delta^{(n-\alpha)}\qquad \text{if}\quad n-\alpha\in N^{\cup}\{0\},$$

$$\overset{w}{=}0\qquad \text{if}\quad n-\alpha\in N^{\cup}\{0\}.$$

THEOREM 2. *Let* $m\in N$ *and* $\alpha\in R$. *Then*

$$\text{(2)}\qquad \mathrm{Pf.}\left(\frac{d}{dx}(x_+^\alpha Y^m)\right)\overset{w}{=}\alpha\,\mathrm{Pf.}\,(x_+^{\alpha-1}Y)+\mathrm{Pf.}(x_+^\alpha\delta)$$

$$\overset{w}{=}\alpha\,\mathrm{Pf.}(x_+^{\alpha-1}Y)+\frac{(-1)^{-\alpha}}{(-\alpha)!}\delta^{(-\alpha)}\qquad \text{if}\quad -\alpha\in N^{\cup}\{0\},$$

$$\overset{w}{=}\alpha\,\mathrm{Pf.}(x_+^{\alpha-1}Y)\qquad \text{if}\quad -\alpha\notin N^{\cup}\{0\}.$$

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2. Preliminaries

We use the notation $(\mathscr{L}\mathscr{S})$ as follows:

$$(\mathscr{L}\mathscr{S})=\{f=(f_1, f_2,\dots, f_k,\dots); f_k\in(\mathfrak{M}) \qquad \textit{for each } k\in N\},$$

where $(\mathfrak{M})$ is the set of all real valued, locally integrable, functions defined on R.

DEFINITION 1. A function $f\in(\mathscr{L}\mathscr{S})$ is said to be class C^n if there exists a natural number k_0 such that f_k is class C^n whenever $k\geqq k_0$.

DEIFNITION 2. Let $f\in(\mathscr{L}\mathscr{S})$ and $S\in(\mathscr{D})'$. Define

$$\mathrm{Pf.}(f)\overset{w}{=}S \tag{3}$$

if there exists a function g such that

$$g(k, \varphi)=c_0(\varphi)\log k+c_1(\varphi)k^{\lambda_1(\varphi)}+\cdots+c_n(\varphi)k^{\lambda_n(\varphi)} \tag{4}$$

for each $k\in N$ and $\varphi\in(\mathscr{D})$, where the $c_i(\varphi)$, $0\leqq i\leqq n$, and the $\lambda_i(\varphi)$, $1\leqq i\leqq n$, which are real numbers, depend on φ, and $0<\lambda_1(\varphi)<\cdots<\lambda_n(\varphi)$,

$$\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx-g(k, \varphi)\right)=S(\varphi) \tag{5}$$

for each $\varphi\in(\mathscr{D})$.

LEMMA 1. *Let $f\in(\mathscr{L}\mathscr{S})$ be a class C^1 function and $S\in(\mathscr{D})'$.*

$$\textit{If}\quad \mathrm{Pf.}(f)\overset{w}{=}S,\quad \textit{then}\quad \mathrm{Pf.}\left(\frac{d}{dx}f\right)\overset{w}{=}S'. \tag{6}$$

PROOF. Let $\varphi\in(\mathscr{D})$. Since $\mathrm{Pf.}(f)\overset{w}{=}S$, we have

$$\begin{aligned}&\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)(-\varphi'(x))dx-(c_0(-\varphi')\log k+c_1(-\varphi')k^{\lambda_1(-\varphi')}+\cdots+c_n(-\varphi')k^{\lambda_n(-\varphi')})\right)\\ &=S(-\varphi')\\ &=S'(\varphi).\end{aligned}$$

Since f is class C^1, there exists a natural number k_0 such that

$$\int_{-\infty}^{\infty} f_k(x)(-\varphi'(x))dx=\int_{-\infty}^{\infty} f'_k(x)\varphi(x)dx$$

whenever $k\geqq k_0$.
Hence, we have

$$\lim_{k+\infty}\left(\left(\int_{-\infty}^{\infty} f'_k(x)\varphi(x)dx-(c'_0(\varphi)\log k+c'_1(\varphi)k^{\lambda'_1(\varphi)}+\cdots+c'_n(\varphi)k^{\lambda'_n(\varphi)})\right)\right.$$
$$=S'(\varphi),$$

where $$c'_0(\varphi)=c_0(-\varphi'),\ c'_1(\varphi)=c_1(-\varphi'),\ldots,\ c'_n(\varphi)=c_n(-\varphi'),$$
$$\lambda'_1(\varphi)=\lambda_1(-\varphi'),\ldots,\ \lambda'_n(\varphi)=\lambda_n(-\varphi').$$

Therefore we have

$$\text{Pf.}\left(\frac{d}{dx}f\right)\overset{w}{=}S'.$$

Let $\delta\in(\mathscr{L}\mathscr{S})$ be a function having the following properties:

(7) $$0<a_k<b_k,$$

(8) a_k^{-1} is a monomial of k,

(9) $$b_k-a_k<e^{-k},$$

(10) $$\delta_k\in(\mathscr{D}),$$

(11) $$\delta_k(x)\geqq 0,$$

(12) $$\text{Car}(\delta_k)\subset(a_k,\ b_k),$$

(13) $$\int_{a_k}^{b_k}\delta_k(x)dx=1.$$

LEMMA 2. *Let $f,\ g\in(\mathscr{L}\mathscr{S})$ be class C^n functions.*

(14) *If* $f\overset{w}{=}g$ *then* $f^{(n)}\overset{w}{=}g^{(n)}$.

PROOF. Since $f\overset{w}{=}g$, we have

$$\lim_{h\to\infty}\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx=\lim_{k\to\infty}\int_{-\infty}^{\infty} g_k(x)\varphi(x)dx$$

for each $\varphi\in(\mathscr{D})$.
Since $f,\ g$ are class C^n functions, we have

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} f^{(n)}(x)\varphi(x)dx=\lim_{k\to\infty}(-1)^n\int_{-\infty}^{\infty} f_k(x)\varphi^{(n)}(x)dx$$
$$=\lim_{k\to\infty}(-1)^n\int_{-\infty}^{\infty} g_k(x)\varphi^{(n)}(x)dx$$
$$=\lim_{k\to\infty}\int_{-\infty}^{\infty} g_k^{(n)}(x)\varphi(x)dx$$

for each $\varphi \in (\mathscr{D})$.
Therefore we have $f^{(n)} \overset{w}{=} g^{(n)}$.

Let Y be the function defined by

$$Y(x) = (Y_1(x),\ Y_2(x), \dots,\ Y_k(x), \dots),$$

where $Y_k(x) = \int_{-\infty}^{x} \delta_k(t)dt \quad for\ k \in N.$

We have the following lemma:

LEMMA 3. *Let $m,\ n \in N$. Then*

$$\frac{d^n}{dx^n} Y^m = \sum_{p_1 + \cdots + p_m = n} \frac{n!}{p_1! \dots p_m!} \delta^{(p_1 - 1)} \dots \delta^{(p_m - 1)} \tag{15}$$

$$\overset{w}{=} \delta^{(n-1)},$$

where $\delta^{(-1)} = Y$ and $\delta^{(0)} = \delta$.

PROOF. We immediately have that

$$\frac{d^n}{dx^n} Y^m \overset{w}{=} \delta^{(n-1)}.$$

Since

$$\frac{d^n}{dx^n} Y_k^m(x) = \sum_{p_1 + \cdots + p_m = n} \frac{n!}{p_1! \dots p_m!} Y_k^{(p_1)}(x) \dots Y_k^{(p_m)}(x)$$

$$= \sum_{p_1 + \cdots + p_m = n} \frac{n!}{p_1! \dots p_m!} \delta_k^{(p_1 - 1)}(x) \dots \delta_k^{(p_m - 1)}(x),$$

we have (15).

REMARK 1. Let $m = 2$ and $n = 1$. Then

$$2Y\delta \overset{w}{=} \delta.$$

Let $m = 3$ and $n = 1$. Then

$$3Y^2\delta \overset{w}{=} \delta.$$

We thus have

$$\delta(Y - Y^2) \overset{w}{=} \frac{1}{6}\delta \quad \text{and} \quad Y - Y^2 \overset{w}{=} 0. \tag{16}$$

3. Proofs of Theorems 1 and 2

Now let x_+^α, where $\alpha \in R$, be the function defined by

$$x_+^\alpha = x^\alpha \qquad \text{if} \quad x>0$$

$$=0 \qquad \text{if} \quad x \leqq 0.$$

LEMMA 4. *Let $\alpha>0$ and let f be a real valued continuous function defined on R. Then*

(17) $$f x_+^\alpha \delta \overset{w}{=} 0.$$

PROOF. Since

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} f(x)x_+^\alpha \delta_k(x)\varphi(x)dx=0 \qquad \text{for each} \quad \varphi\in(\mathscr{D}),$$

we immediately have (17).

LEMMA 5. *Let $\alpha>0$ and $\alpha\notin N$. Then*

(18) $$\mathrm{Pf.}(x_+^{-\alpha}\delta) \overset{w}{=} 0.$$

PROOF. We may take $l\in N$ and $l>\alpha$. Taylor's theorem gives

$$\int_{-\infty}^{\infty} x_+^{-\alpha}\delta_k(x)\varphi(x)dx$$

$$=\int_{a_k}^{b_k}\delta_k(x)\Big(\varphi(0)x^{-\alpha}+\frac{\varphi'(0)}{1!}x^{1-\alpha}+\cdots+\frac{\varphi^{(m)}(0)}{m!}x^{m-\alpha}+\cdots+\frac{\varphi^{(l)}(\theta x)}{l!}x^{l-\alpha}\Big)dx$$

for each $\varphi\in(\mathscr{D})$.
If $m-\alpha<0$, then

$$a_k^{m-\alpha}\geqq\int_{a_k}^{b_k}\delta_k(x)x^{m-\alpha}dx\geqq b_k^{m-\alpha}.$$

Therefore, using the Lemma 4, we have (18).

PROOF OF THEOREM 1. We may take $l\in N$ and $l+\alpha>n$. Let $\varphi\in(\mathscr{D})$. Then

$$\int_{-\infty}^{\infty} x_+^\alpha\delta_k^{(n)}(x)\varphi(x)dx$$

$$=(-1)^n\int_{a_k}^{b_k}\delta_k(x)\Big(\varphi(0)x^{\alpha}+\frac{\varphi'(0)}{1!}x^{1+\alpha}+\cdots+\frac{\varphi^{(m)}(0)}{m!}x^{m+\alpha}+\cdots+\frac{\varphi^{(l)}(\theta x)}{l!}x^{l+\alpha}\Big)^{(n)}dx.$$

If $n=m+\alpha$, then

$$\Big(\frac{\varphi^{(m)}(0)}{m!}x^{m+\alpha}\Big)^{(n)}=\frac{n!\,\varphi^{(n-\alpha)}(0)}{(n-\alpha)!}.$$

Therefore we have (1).

PROOF OF THEOREM 2. We immediately have

$$\frac{d}{dx}\left(x_+^\alpha Y^m\right)=\alpha x_+^{\alpha-1}Y^m+x_+^\alpha mY^{m-1}\delta$$

$$\overset{w}{=}\alpha x_+^{\alpha-1}Y+x_+^\alpha\delta.$$

Using the Theorem 1 we have the result that

$$\text{Pf.}\,(x_+^\alpha\delta)\overset{w}{=}\frac{(-1)^{-\alpha}}{(-\alpha)!}\delta^{(-\alpha)}\qquad \text{if}\quad -\alpha\in N^{\cup}\{0\},$$

$$\overset{w}{=}0\qquad \text{if}\quad -\alpha\notin N^{\cup}\{0\}.$$

Therefore we have (2).

COROLLARY 1 (L. Schwartz [8]). *Let $l\in R$. Then*

$$\frac{d}{dx}\left[\text{Pf.}\,\frac{Y}{x^l}\right]\overset{w}{=}\text{Pf.}\left(\frac{-l}{x^{l+1}}\right)+(-1)^l\frac{\delta^{(l)}}{l!}\qquad \textit{if}\quad l\in N^{\cup}\{0\},\tag{19}$$

$$\frac{d}{dx}\left[\text{Pf.}\,\frac{Y}{x^l}\right]\overset{w}{=}\text{Pf.}\left(\frac{-l}{x^{l+1}}\right)\qquad \textit{if}\quad l\notin N^{\cup}\{0\}.\tag{20}$$

PROOF. This result is an immediate consequence of Lemma 1 and Theorem 1.

Let x_-^α, where $\alpha\in R$, be the function defined by

$$x_-^\alpha=e^{i\alpha\pi}(-x)\qquad \text{if}\quad x<0,$$

$$=0\qquad \text{if}\quad x\geqq 0,$$

and let δ_-, Y_- be the functions defined by the following equations:

$$\delta_-(x)=\delta(-x),\quad Y_-(x)=Y(-x).$$

We immediately have the following results:

THEOREM 3. *Let $n\in N^{\cup}\{0\}$ and $\alpha\in R$. Then*

$$\text{Pf.}\,(x_-^\alpha\delta_-^{(m)})\overset{w}{=}e^{i\alpha\pi}\frac{n!}{(n-\alpha)!}\delta^{(n-\alpha)}\qquad \textit{if}\quad n-\alpha\in N^{\cup}\{0\},\tag{21}$$

$$\overset{w}{=}0\qquad \textit{if}\quad n-\alpha\notin N^{\cup}\{0\}.$$

COROLLARY 2. *Let $n\in N^{\cup}\{0\}$ and $\alpha\in R$, and let A, B be non-negative real numbers such that $A+B=1$. Then*

$$A\delta+B\delta\overset{w}{=}\delta,\tag{22}$$

$$\text{Pf.}\,(x_+^\alpha A\delta^{(n)})+\text{Pf.}\,(x_-^\alpha B\delta_-^{(n)})\tag{23}$$

$$\overset{w}{=} (-1)^{-\alpha} \frac{n!}{(n-\alpha)!} \delta^{(n-\alpha)} \qquad if \quad n-\alpha \in N\{0\},$$

$$\overset{w}{=} 0 \qquad if \quad n-\alpha \in N \cup \{0\}.$$

THEOREM 4. *Let $m \in N$ and $\alpha \in R$. Then*

(24)
$$\mathrm{Pf.}\left(\frac{d}{dx}(x_-^\alpha Y_-^m)\right) \overset{w}{=} \alpha\, \mathrm{Pf.}\,(x_-^{\alpha-1} Y_-^m) - \mathrm{Pf.}\,(x_-^\alpha m Y_-^{m-1} \delta_-)$$

$$\overset{w}{=} \alpha \mathrm{Pf.}\,(x_-^{\alpha-1} Y_-) - \mathrm{Pf.}\,(x_-^\alpha \delta_-)$$

$$\overset{w}{=} \alpha\, \mathrm{Pf.}\,(x_-^{\alpha-1} Y_-) - \frac{(-1)^{-\alpha}}{(-\alpha)!} \delta^{(-\alpha)} \qquad if \quad -\alpha \in N \cup \{0\},$$

$$\overset{w}{=} \alpha\, \mathrm{Pf.}\,(x_-^{-\alpha} Y_-) \qquad if \quad -\alpha \notin N \cup \{0\}.$$

COROLLARY 3 (L. Schwartz [8]). *Let $l \in R$. Then*

(25)
$$\frac{d}{dx}\left[\mathrm{Pf.}\left(\frac{Y_-}{x^l}\right)\right] \overset{w}{=} \mathrm{Pf.}\left(\frac{-lY_-}{x^{l+1}}\right) - (-1)^l \frac{\delta^{(l)}}{l!} \qquad if \quad l \in N \cup \{0\},$$

$$\overset{w}{=} \mathrm{Pf.}\left(\frac{-lY_-}{x^{l+1}}\right) \qquad if \quad l \in N \cup \{0\}.$$

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