On a Solution of the Equation $x^nT=\delta'+\delta+Y$

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(*Received April 10, 1974*)

1. Introduction

L. Schwartz [2] has given the result that

$$(\delta x)\,\mathrm{v.p.}\frac{1}{x}=0\,.$$

The notation used here is that δ is the Dirac delta-distribution.

In [1] we have given the following result:

$$(x\delta)\frac{1}{x}=x\left(\frac{\delta}{x}\right)=\left(x\cdot\frac{1}{x}\right)\delta=\delta\,.$$

In the present paper, by generalization this notion, we intend to give a solution of the following equation:

$$x^nT=\delta'+\delta+Y\,,$$

where Y is the Heaviside function.

2. Notations

We use the notation $(\mathscr{L}\mathscr{S})$ as follows:

$$(\mathscr{L}\mathscr{S})=\{f=(f_1,f_2,\dots,f_k,\dots)\,;f_k\in(\mathfrak{M})\qquad \textit{for each } k\in N\}\,,$$

where $(\mathfrak{M})$ is the set of all real valued locally integrable functions defined on R^1.

We define equality, addition and scalar product in the space $(\mathscr{L}\mathscr{S})$ as follows:

DEFINITION 1. Let

$$f=(f_1,f_2,\dots,f_k,\dots),\qquad g=(g_1,g_2,\dots,g_k,\dots)\in(\mathscr{L}\mathscr{S})\,.$$

$f=g$ if and only if, there exists an integer $k_0\geqq 1$ such that for every integer k if $k\geqq k_0$ then $f_k=g_k$.

$$f+g=(f_1+g_1,f_2+g_2,\dots,f_k+g_k,\dots)\,.$$

$$\alpha f=(\alpha f_1,\alpha f_2,\dots,\alpha f_k,\dots)\qquad(\alpha\in R)\,.$$

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By this definition, clearly the space $(\mathscr{L}\mathscr{S})$ is a linear space.

According to G. Takeuti [3], $f \overset{w}{=} g$ denotes

$$\lim_{k\to\infty}\left(\int_{-\infty}^{\infty} f_k(x)\varphi(x)dx - \int_{-\infty}^{\infty} g_k(x)\varphi(x)dx\right) = 0 \qquad \text{for each } \varphi \in (\mathscr{D}),$$

where $f = (f_1, f_2, \ldots, f_k, \ldots)$, $g = (g_1, g_2, \ldots, g_k, \ldots) \in (\mathscr{L}\mathscr{S})$.
It is clear that if $f = g$ then $f \overset{w}{=} g$.

3. On a solution of the equation $x^n T = \delta' + \delta + Y$

Let $\{\delta_k^0\}$ be a delta sequence with the following property:

1° $\qquad 0 \notin \mathrm{Car}(\delta_k^0) \qquad$ *for each* $k \in N$.

Let

$$\delta^0 = (\delta_1^0, \delta_2^0, \ldots, \delta_k^0, \ldots),$$

$$\frac{1}{x^n}(x^m\delta^0) = \left(\frac{1}{x^n}(x^m\delta_1^0), \frac{1}{x^n}(x^m\delta_2^0), \ldots, \frac{1}{x^n}(x^m\delta_k^0), \ldots\right) \qquad (m, n \in N),$$

then

$$\delta^0, \frac{1}{x^n}(x^m\delta^0) \in (\mathscr{L}\mathscr{S}),$$

$$\frac{1}{x^n}(x^m\delta^0) = \frac{x^m}{x^n}\delta^0 = x^m\left(\frac{1}{x^n}\delta^0\right) = x^m\left(\frac{\delta^0}{x^n}\right).$$

Since

$$\lim_{k\to\infty}\int_{-\infty}^{\infty}\frac{1}{x^n}(x^m\delta_k^0(x))\varphi(x)dx = 0 \qquad (m > n),$$

$$= \delta(\varphi) \qquad (m = n)$$

for each $\varphi \in (\mathscr{D})$, we have

$$\frac{1}{x^n}(x^m\delta^0) \overset{w}{=} 0 \qquad (m > n),$$

$$\overset{w}{=} \delta^0 \qquad (m = n),$$

where $0 = (0, 0, \ldots, 0, \ldots)$.

Let $\{\delta_k\}$ be a delta sequence with the following properties:

2° $\quad \delta_k(x)$ *is a smooth function and* $\mathrm{Car}(\delta_k) \subset \left(\frac{1}{k}, \frac{1+\varepsilon(k)}{k}\right)$ *for each* $k \in N$, *where* $0 < \varepsilon(k) < \frac{1}{k^{n+2}}$,

3° $\quad \int_{-\infty}^{\infty}\delta_k(x)dx = 1 \qquad$ *for each* $k \in N$.

Let

$$\delta = (\delta_1, \delta_2, \dots, \delta_k, \dots), \quad \delta' = (\delta'_1, \delta'_1, \dots, \delta'_k, \dots)$$

and $Y = (Y_1, Y_2, \dots, Y_k, \dots)$, where

$$Y_k(x) = \int_{-\infty}^{x} \delta_k(t)dt \qquad \text{for each } k \in N.$$

It is clear that

$$\delta \overset{w}{=} \delta + c_1 x\delta \overset{w}{=} \cdots \overset{w}{=} \delta + c_1 x\delta + \cdots + c_n x^n \delta ,$$

$$\delta' \overset{w}{=} \delta' + c_1 x\delta \overset{w}{=} \cdots \overset{w}{=} \delta' + c_1 x\delta + \cdots + c_n x^n \delta ,$$

$$Y \overset{w}{=} Y + c_1 x\delta \overset{w}{=} \cdots \overset{w}{=} Y + c_1 x\delta + \cdots + c_n x^n \delta ,$$

where $c_1, c_2, \dots, c_n$ are constants and

$$c_i x^i \delta = (c_i x^i \delta_1, c_i x^i \delta_2, \dots, c_i x^i \delta_k, \dots) \qquad (1 \leqq i \leqq n) .$$

Let

$$\frac{\delta + c_1 x\delta + \cdots + c_n x^n \delta}{x^n} = \left(\frac{\delta_1 + c_1 x\delta_1 + \cdots + c_n x^n \delta_1}{x^n}, \dots, \frac{\delta_k + c_1 x\delta_k + \cdots + c_n x^n \delta_k}{x^n}, \dots \right),$$

then

$$\frac{\delta + c_1 x\delta + \cdots + c_n x^n \delta}{x^n} \in (\mathscr{L}\mathscr{S}) .$$

Since

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} \frac{\delta_k + c_1 x\delta_k + \cdots + c_n x^n \delta_k}{x^n} x^n \varphi(x) dx = \delta(\varphi) \qquad \textit{for each } \varphi \in (\mathscr{D}) ,$$

we shall call $\frac{\delta + c_1 x\delta + \cdots + c_n x^n \delta}{x^n}$ a solution of $x^n T = \delta$ in the space $(\mathscr{L}\mathscr{S})$.

Let $\varphi \in (\mathscr{D})$. Then

$$\int_{-\infty}^{\infty} \frac{\delta_k + c_1 x\delta_k + \cdots + c_n x^n \delta_k}{x^n} \varphi(x) dx$$

$$= \int_{-\infty}^{\infty} \frac{\delta_k + c_1 x\delta_k + \cdots + c_n x^n \delta_k}{x^n} \left(\varphi(0) + \varphi'(0)x + \cdots + \frac{\varphi^{(n)}(0)}{n!} x^n + \frac{\varphi^{(n+1)}(\xi)}{(n+1)!} x^{n+1} \right) dx$$

$$= \varphi(0) \int_{-\infty}^{\infty} \frac{\delta_k(x)}{x^n} dx + (\varphi'(0) + c_1 \varphi(0)) \int_{-\infty}^{\infty} \frac{\delta_k(x)}{x^{n-1}} dx + \cdots$$

$$+ \left(\frac{\varphi^{(n-1)}(0)}{(n-1)!} + C_1 \frac{\varphi^{(n-2)}(0)}{(n-2)!} + \cdots + C_{n-2} \varphi'(0) + C_{n-1} \varphi(0) \right) \int_{-\infty}^{\infty} \frac{\delta_k(x)}{x} dx$$

$$+\left(\frac{\varphi^{(n)}(0)}{n!}+C_1\frac{\varphi^{(n-1)}(0)}{(n-1)!}+\cdots+C_{n-1}\varphi'(0)+C_n\varphi(0)\right)\int_{-\infty}^{\infty}\delta_k(x)dx+R_k$$

$$=\varphi(0)\int_{-\infty}^{\infty}\frac{\delta_k(x)}{x^n}dx+(\varphi'(0)+C_1\varphi(0))\int_{-\infty}^{\infty}\frac{\delta_k(x)}{x^{n-1}}dx+\cdots$$

$$+\left(\frac{\varphi^{(n-1)}(0)}{(n-1)!}+C_1\frac{\varphi^{(n-2)}(0)}{(n-2)!}+\cdots+C_{n-2}\varphi'(0)+C_{n-1}\varphi(0)\right)\int_{-\infty}^{\infty}\frac{\delta_k(x)}{x}dx$$

$$+\frac{(-1)^n\delta^{(n)}}{n!}(\varphi)+C_1\frac{(-1)^{(n-1)}\delta^{(n-1)}}{(n-1)!}(\varphi)+\cdots+C_n\delta(\varphi)+R_k\,,$$

where

$$R_k=\int_{-\infty}^{\infty}x\delta_k(x)\left(\frac{\varphi^{(n+1)}(\xi)}{(n+1)!}+C_1\frac{\varphi^{(n)}(0)}{n!}+\cdots+C_n\frac{\varphi^{(n+1)}(\xi)}{(n+1)!}x^n\right)dx\,.$$

Clearly $\lim\limits_{k\to\infty}R_k=0$.
We have

$$\int_{-\infty}^{\infty}\frac{\delta_k(x)}{x^n}dx=\int_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}}\frac{\delta_k(x)}{x^n}dx=\frac{1}{\left(\frac{1+\theta\varepsilon(k)}{k}\right)^n}\int_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}}\delta_k(x)dx$$

$$=\left(\frac{k}{1+\theta\varepsilon(k)}\right)^n\qquad(0<\theta<1)\,.$$

Since $0<\varepsilon(k)<\dfrac{1}{k^{n+2}}$,

we obtain

$$\lim_{k\to\infty}\left|\left(\frac{k}{1+\theta\varepsilon(k)}\right)^n-k^n\right|=0\,.$$

Therefore we have the following result:

$$\text{Pf.}\ \frac{\delta+C_1x\delta+\cdots+C_nx^n\delta}{x^n}=\frac{(-1)^n\delta^{(n)}}{n!}+C_1^0\delta^{(n-1)}\cdots+C_n^0\delta\,,$$

where

$$C_1^0=C_1\frac{(-1)^{n-1}}{(n-1)!},\cdots,\ C_n^0=C_n\,.$$

It is clear that

$$\frac{(-1)^n\delta^{(n)}}{n!}(x^n\varphi)=\delta(\varphi)\qquad \textit{fer each } \varphi\in(\mathscr{D})\,.$$

Thus we have the following theorem:

THEOREM 1. *Let* $x^nT=\delta$ (1).

Then

(1.1) $$\frac{\delta+C_1x\delta+\cdots+C_nx^n\delta}{x^n}$$

is a solution of (1) *in the space* $(\mathscr{L}\mathscr{S})$.

(1.2) $$\text{Pf. } \frac{\delta+Cx\delta+\cdots+C_nx^n\delta}{x^n}=\frac{(-1)^n\delta^{(n)}}{n\,!}+C_1^0\delta^{(n-1)}+\cdots+C_n^0\delta$$

is the distributional general solution of (1).

Let $\varphi\in(\mathscr{D})$, then

$$\int_{-\infty}^{\infty}\frac{\delta'_k(x)}{x^n}\varphi(x)dx$$

$$=\int_{-\infty}^{\infty}\frac{\delta'_k(x)}{x^n}\left(\varphi(0)+\varphi'(0)x+\cdots+\frac{\varphi^{(n+1)}(0)}{(n+1)\,!}x^{n+1}+\frac{\varphi^{(n+2)}(\eta)}{(n+2)\,!}x^{n+2}\right)dx$$

$$=\varphi(0)\int_{-\infty}^{\infty}\frac{\delta_k^1(x)}{x^n}dx+\cdots+\frac{\varphi^{(n-1)}(0)}{(n-1)\,!}\int_{-\infty}^{\infty}\frac{\delta'_k(x)}{x}dx$$

$$+\frac{\varphi^{(n)}(0)}{n!}\int_{-\infty}^{\infty}\delta'_k(x)dx+\frac{\varphi^{(n+1)}(0)}{(n+1)\,!}\int_{-\infty}^{\infty}x\delta'_k(x)dx+R'_k,$$

where

$$R'_k=\frac{1}{(n+2)\,!}\int_{-\infty}^{\infty}\varphi^{(n+2)}(\eta)x^2\delta'_k(x)dx\,.$$

We have

$$\int_{-\infty}^{\infty}\frac{\delta'_k(x)}{x^m}dx=\int_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}}\frac{\delta'_k(x)}{x^m}dx$$

$$=\left[\frac{\delta_k(x)}{x^m}\right]_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}}+m\int_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}}\frac{\delta_k(x)}{x^{m+1}}dx$$

$$=m\int_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}}\frac{\delta_k(x)}{x^{m+1}}dx\qquad(0<m\leqq n)\,.$$

Since

$$\int_{-\infty}^{\infty}\delta'_k(x)dx=0,\quad\int_{-\infty}^{\infty}x\delta'_k(x)dx=-1,\quad\lim_{k\to\infty}R'_k=0,$$

$$0<\varepsilon(k)<\frac{1}{k^{n+2}},$$

we obtain

$$\text{Pf.}\ \frac{\delta'}{x^n}=\frac{(-1)^{n+2}\delta^{(n+1)}}{(n+1)!}.$$

It is clear that

$$\frac{(-1)^{n+2}\delta^{(n+1)}}{(n+1)!}(x^n\varphi)=\frac{(-1)^{n+2}}{(n+1)!}\delta\Big((-1)^{n+1}\frac{d^{n+1}}{dx^{n+1}}(x^n\varphi)\Big)$$

$$=\frac{-1}{(n+1)!}\delta\Big({}_{n+1}C_1\Big(\frac{d^n}{dx^n}x^n\Big)\frac{d}{dx}\varphi+\cdots\Big)$$

$$=\delta'(\varphi)\qquad \textit{for each } \varphi\in(\mathscr{D}).$$

Thus we have the following two theorems:

THEOREM 2. *Let* $x^nT=\delta'$ (2).
Then

(2.1) $$\frac{\delta'+C_1x\delta+\cdots+C_nx^n\delta}{x^n}$$

is a solution of (2) *in the space* $(\mathscr{L}\mathscr{S})$.

(2.2) $$\text{Pf.}\ \frac{\delta'+C_1x\delta+\cdots+C_nx^n\delta}{x^n}=\frac{(-1)^{n+2}}{(n+1)!}\delta^{(n+1)}+C_1^0\delta^{(n-1)}+\cdots+C_n^0\delta$$

is the distributional general solution of (2).

THEOREM 3. *Let* $x^nT=\delta'+\delta$ (3).
Then

(3.1) $$\frac{\delta'+\delta+C_1x\delta+\cdots+C_nx^n\delta}{x^n}$$

is a solution of (3) *in the space* $(\mathscr{L}\mathscr{S})$.

(3.2) $$\text{Pf.}\ \frac{\delta'+\delta\ C_1x\delta+\cdots+C_nx^n\delta}{x^n}$$

$$=\frac{(-1)^{n+2}}{(n+1)!}\delta^{(n+1)}+\frac{(-1)^n}{n!}\delta^{(n)}+C_1^0\delta^{(n-1)}+\cdots+C_1^0\delta$$

is the distributional general solution of (3).

We define a function $\frac{1}{x_k}$ by

$$\frac{1}{x_k}=\frac{1}{x}\qquad\Big(\frac{1}{k}\leqq x\Big)$$

$$=0\qquad\Big(x<\frac{1}{k}\Big).$$

for each $k\in N$, and let

$$\left(\frac{1}{x^n}\right)_{x>0}=\left(\frac{1}{x_1^n}, \frac{1}{x_2^n}, \cdots, \frac{1}{x_k^n}, \cdots\right).$$

Then $\left(\frac{1}{x^n}\right)_{x>0} \in (\mathscr{L}\mathscr{S})$.
Since

$$\left|\int_{-\infty}^{\infty} \frac{Y_k(x)}{x^n}\varphi(x)-\int_{-\infty}^{\infty}\frac{1}{x_k^n}\varphi(x)dx\right|$$

$$\leqq \int_{\frac{1}{k}}^{\frac{1+\varepsilon(k)}{k}} \frac{|Y_k(x)-1|}{x^n}|\varphi(x)|dx$$

$$\leqq Mk^{n-1}\varepsilon(k) \qquad \textit{for each } k\in N,$$

where $|\varphi(x)|\leqq M$, and $0<\varepsilon(k)<\frac{1}{k^{n+2}}$,
we have $\frac{Y}{x^n} \overset{w}{=} \left(\frac{1}{x^n}\right)_{x>0}$, where

$$\frac{Y}{x^n}=\left(\frac{Y_1}{x^n}, \frac{Y_2}{x^n}, \cdots, \frac{Y_k}{x^n}, \cdots\right).$$

We also have

$$\lim_{k\to\infty}\int_{-\infty}^{\infty}\frac{Y_k(x)}{x^n}x^n\varphi(x)dx=\int_0^{\infty}\varphi(x)dx \qquad \textit{for each } \varphi\in(\mathscr{D}).$$

In a sence of distribution, it is clear that

$$x^n\cdot \mathrm{Pf.}\left(\frac{1}{x^n}\right)_{x>0}=Y.$$

Thus we immediately have the following theorem:

THEOREM 4. *Let* $x^nT=\delta'+\delta+Y$ (4).
Then

(4.1)
$$\frac{\delta'+\delta+Y+C_1x\delta+\cdots+C_nx^n\delta}{x^n}$$

is a solution of (4) *in the space* $(\mathscr{L}\mathscr{S})$.

(4.2)
$$\mathrm{Pf.}\frac{\delta'+\delta+Y+C_1x\delta+\cdots+C_nx^n\delta}{x^n}$$

$$=\frac{(-1)^{n+2}}{(n+1)!}\delta^{(n+1)}+\frac{(-1)^n}{n!}\delta^{(n)}+\mathrm{Pf.}\left(\frac{1}{x^n}\right)_{x>0}+C_1^0\delta^{(n-1)}+\cdots+C_n^0\delta$$

is the distributional general solution of (4).

I wish to express my hearty thanks to Professor T. Shibata of Science University of Tokyo who has given me much kind advice.

A part of this paper was presented at the regular annual meeting of the Mathematical Society of Japan, held in Tokyo, April 3, 1974.

References

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