Change of Variable in the Finite Parts of Divergent Integrals

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1. Introduction

In the present paper we intend to give a method of change of variable in the finite parts of divergent integrals. Related work has been discussed by L. Schwartz [1].

2. Change of variable in the finite parts of divergent integrals

We use the notation R^N as follows:

$$R^N=\{x=(x_1, x_2,\dots, x_k,\dots);\ x_k\in R \textit{ for all } k\in N\},$$

where R denotes the set of all real numbers, N denotes the set of all natural numbers.

DEFINITION 1. Let $a\in R^N$ and let f be a real valued function having the following property:
There exists a natural number k_0 such that $f(a_k)$ is defined whenever $k\geqq k_0$. Define

$$f(a)=(f^*(a_1), f^*(a_2),\dots, f^*(a_k),\dots),$$

where $f^*(a_k)$ is defined as follows:

$$f^*(a_k)=f(a_k) \quad \text{where } f(a_k) \text{ is defined and}$$

$$f^*(a_k)=0 \quad \text{elsewhere.}$$

DEFINITION 2. Let $a, b\in R^N$. Define
$a=b$ [resp. $a<b$, $a\leqq b$] if there exists a natural number k_0 such that $a_k=b_k$ [resp. $a_k<b_k$, $a_k\leqq b_k$] whenever $k\geqq k_0$,

DEFINITION 3 (Interval). Let $a, b\in R^N$ and $a\leqq b$. Define

$$[a, b]=([a_1, b_1]^*, [a_2, b_2]^*,\dots, [a_k, b_k]^*,\dots),$$

where $[a_k, b_k]^*$ is defined as follows:

$$[a_k, b_k]^*=[a_k, b_k] \quad \text{for } a_k\leqq b_k,$$

$$[a_k, b_k]^*=\phi \quad \text{for } a_k>b_k.$$

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DEFINITION 4. A function f is said to be summable [resp. bounded, measurable, absolutely continuous] on the interval $[a, b]$ if there exists a natural number k_0 such that f is summable [resp. bounded, measurable, absolutely continuous] on the interval $[a_k, b_k]$ whenever $k \geqq k_0$.

DEFINITION 5 (Integral). Let f be summable on the interval $[a, b]$. Define

$$\int_a^b f(x)dx = \left(\int_{a_1}^{b_1} f(x)dx^*, \int_{a_2}^{b_2} f(x)dx^*, \dots, \int_{a_k}^{b_k} f(x)dx^*, \dots\right),$$

where $\int_{a_k}^{b_k} f(x)dx^*$ is defined as follows:

$\int_{a_k}^{b_k} f(x)dx^* = \int_{a_k}^{b_k} f(x)dx$ where f is summable on the interval $[a_k, b_k]$ and

$\int_{a_k}^{b_k} f(x)dx^* = 0$ elsewhere.

DEFINITION 6 (Finite part). Let f be summable on the interval $[a, b]$. Define

$$\text{Pf.} \int_a^d f(x)dx = I,$$

if there exists a function g such that

$$g(k) = c_0 \log k + c_1 k^{\lambda_1} + \cdots + c_l k^{\lambda_l}$$

for all $k \in N$, and

$$\lim_{k\to\infty} \left(\int_{a_k}^{d_k} f(x)dx - g(k)\right) = I,$$

where c_i, $0 \leqq i \leqq l$, are real constants.

The followiog theorem is derived from the above definitions.

THEOREM. *Let f be bounded and measurable on the interval $[a, b]$. Let φ be absolutely continuous on the interval $[\alpha, \beta]$ having the following property: There exists a natural number k_0 such that $a_k \leqq \varphi(t) \leqq b_k$ for all $t \in [\alpha_k, \beta_k]$ whenever $k \geqq k_0$.*
Then we have

$$\text{Pf.} \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \text{Pf.} \int_\alpha^\beta f(\varphi(t))\dot{\varphi}(t)dt,$$

where $\dot{\varphi}$ is defined as follows:

$\dot{\varphi}(t) = \varphi'(t)$ *where $\varphi'(t)$ is defined and finite and*

$\dot{\varphi}(t) = 0$ *elsewhere.*

PROOF. There exists a natural number k_1 such that

(1) f is bounded and measurable on the interval $[a_k, b_k]$ and

(2) φ is absolutely continuous on the interval $[\alpha_k, \beta_k]$ and $a_k \leqq \varphi(t) \leqq l_k$ for all $t \in [\alpha_k, \beta_k]$,

whenever $k \geqq k_1$.

By (1) and (2), we have

$$\int_{\varphi(\alpha_k)}^{\varphi(\beta_k)} f(x)dx = \int_{\alpha_k}^{\beta_k} f(\varphi(t))\dot{\varphi}(t)dt,$$

whenever $k \geqq k_1$.

Hence, we have

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \left(\int_{\varphi(\alpha_1)}^{\varphi(\beta_1)} f(x)dx^*, \dots, \int_{\varphi(\alpha_k)}^{\varphi(\beta_k)} f(x)dx^*, \dots\right)$$

$$= \left(\int_{\alpha_1}^{\beta_1} f(\varphi(t))\dot{\varphi}(t)dt^*, \dots, \int_{\alpha_k}^{\beta_k} f(\varphi(t))\dot{\varphi}(t)dt^*, \dots\right) = \int_{\alpha}^{\beta} f(\varphi(t))\dot{\varphi}(t)dt.$$

Using the Definition 5 we have the result that

$$\text{Pf.} \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \text{Pf.} \int_{\alpha}^{\beta} f(\varphi(t))\dot{\varphi}(t)\, dt.$$

EXAMPLE 1. Let $f(x) = \frac{1}{x}$, $\varphi(t) = 2t$, $\alpha = \left(\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2k}, \dots\right)$, $\beta = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \dots\right)$.

Then

$$\int_{\frac{1}{k}}^{1} \frac{1}{x}\, dx = \int_{\frac{1}{2k}}^{\frac{1}{2}} \frac{1}{t} dt = \log k$$

for all $k \in N$. Therefore

$$\text{Pf.} \int_{\varphi(\alpha)}^{\varphi(\beta)} \frac{1}{x}\, dx = \text{Pf.} \int_{\alpha}^{\beta} \frac{1}{t}\, dt = 0.$$

EXAMPLE 2. Let $\alpha = \left(1, \frac{1}{2}, \dots, \frac{1}{k}, \dots\right)$, $\beta = (1, 2, \dots, k, \dots)$

and let ϕ be a indefinitely differentiable function with compact support.

Let H be a function of R onto R having the following properties:

(1) $H(0) = 0$,

(2) H is continuously differentiable,

(3) $H'(x) > 0$ for all $x \in R$.

Since

$$\int_{\frac{1}{k}}^{k} \phi(H^{-1}(y)) \frac{dy}{y} = \int_{H^{-1}(\frac{1}{k})}^{H^{-1}(k)} \phi(x) \frac{H'(x)}{H(x)} dx$$

for all $k \in N$, we have

$$\mathrm{Pf.}\int_{\alpha}^{\beta}\phi(H^{-1}(y))\frac{dy}{y}$$

$$=\lim_{k\to\infty}\left(\int_{\frac{1}{k}}^{k}\phi(H^{-1}(y))\frac{dy}{y}-\phi(0)\log k\right)$$

$$=\lim_{k\to\infty}\left(\int_{H^{-1}(\frac{1}{k})}^{H^{-1}(k)}\phi(x)\frac{H'(x)}{H(x)}dx-\phi(0)\log k\right)$$

$$=\mathrm{Pf.}\int_{H^{-1}(\alpha)}^{H^{-1}(\beta)}\phi(x)\frac{H'(x)}{H(x)}\,dx\,.$$

Reference

[1] L. SCHWARTZ, Théorie des distributions, Hermann, Paris (1973), 41, 383.