On the Distributions in the Dirac Spaces (I)

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1. Introduction

The theory of Dirac Spaces, discovered by G. Takeuti [9], is of great importance to study the theory of distributions. Related work has been discussed by I. Amemiya [1], W. A. J. Luxemburg [6], A. Robinson [7], T. Shibata [8], etc.

The purpose of this paper is to give an approximation of distributions of finite order by class C^∞ functions.

2. The approximation of distributions of finite order by class C^∞ functions

According to the fundamental theorem of Lebesgue, the relation

$$\lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)\,dy=f(x)$$

holds for almost every x, whenever f is a locally integrable function defined on R^n. The notation here used is that $B(x, r)$ is the ball of radius r, centered at x, and $m(B(x, r))$ denotes its measure.

Let $\{\varphi_k\}$ be a sequence of real valued functions defined on R^n with the following properties:

(1) $R_k > r_k > 0$

(2) $\lim\limits_{k\to\infty} R_k = \lim\limits_{k\to\infty} r_k = 0$

(3) $\lim\limits_{k\to\infty}\dfrac{m(B(0, R_k))-m(B(0, r_k))}{m(B(0, r_k))}=0$

(4) $\varphi_k\in(\mathscr{D})$

(5) $\varphi_k(x)=\varphi_k(-x)$

(6) $\varphi_k(x)=1,\quad$ *for each* $x\in B(0, r_k)$

(7) $\mathrm{Car}(\varphi_k)\subset B(0, R_k)$

(8) $0\leq\varphi_k(x)\leq 1.$

For each integer $k\geq 1$, put

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$$\delta_k^0(x)=\frac{1}{m(B(0, R_k))}\varphi_k(x).$$

PROPOSITION 1. *Let f be a locally integrable function defined on R^n, then*

$$\lim_{k\to\infty}\int_{R^n} f(y)\delta_k^0(y-x)\,dy=f(x)$$

for almost every x.

PROOF. Since

$$\frac{1}{m(B(x, R_k))}\int_{B(x,R_k)} f(y)dy-\frac{1}{m(B(x, r_k))}\int_{B(x,r_k)} f(y)dy$$

$$=\frac{1}{m(B(x, R_k))}\int_{B(x,R_k)-B(x,r_k)} f(y)dy$$

$$+\frac{m(B(x, r_k))}{m(B(x, R_k))}\left(1-\frac{m(B(x, R_k))}{m(B(x, r_k)}\right)\frac{1}{m(B(x, r_k))}\int_{B(x,r_k)} f(y)dy$$

for each integer $k\geq 1$, we get

$$\lim_{k\to\infty}\frac{1}{m(B(x, R_k))}\int_{B(x,R_k)-B(x,r_k)} f(y)dy=0 \tag{1}$$

for almost every x.
By (1), we have

$$\left|\frac{1}{m(B(x, R_k))}\int_{B(x,R_k)} f(y)dy-\int_{B(x,R_k)} f(y)\,\delta_k^0(y-x)dy\right|$$

$$=\left|\frac{1}{m(B(x, R_k))}\int_{B(x,R_k)} f(y)dy-\frac{1}{m(B(x, R_k))}\int_{B(x,Rk)} f(y)\varphi_k(y-x)dy\right|$$

$$=\frac{1}{m(B(x, R_k))}\left|\int_{B(x,R_k)} f(y)(1-\varphi_k(y-x))dy\right|$$

$$\leq\frac{1}{m(B(x, R_k))}\int_{B(x,R_k)-B(x,r_k)} |f(y)|dy$$

$$\to 0 \qquad (k\to\infty).$$

for almost every x.

This implies that

$$\lim_{k\to\infty}\int_{R^n} f(y)\delta_k^0(y-x)dy=f(x)$$

for almost every x. Q.E.D.

Let $\delta^0=(\delta_1^0, \delta_2^0, \delta_3^0, \ldots)$.

Since

$$\int_{R^n} f(y)\delta_k^0(y-x)dy = \int_{R^n} f(y)\,\delta_k^0(x-y)\,dy$$

we have the following proposition:

PROPOSITION 2. *Let f be a locally integrable function defined on R^n, then*

$$f*\delta^0(x) \doteqdot f(x)$$

for almost every x.

REMARK. Let $x \in R^1$, and for each integer $k \geq 1$, define

$$\begin{aligned}
\delta_k(x) &= 0 \qquad \left(x \leq -\frac{1}{k}\right)\\
&= k + k^2 x \left(-\frac{1}{k} \leq x \leq 0\right)\\
&= k - k^2 x \left(0 \leq x \leq \frac{1}{k}\right)\\
&= 0 \qquad \left(\frac{1}{k} \leq x\right).
\end{aligned}$$

Let $\delta = (\delta_1, \delta_2, \delta_3, \ldots)$, then we have

$\delta \overset{w}{=} \delta^0$ and do not have $\delta \doteqdot \delta^0$.

The notations $\overset{w}{=}$ and $\doteqdot$ are due to G. Takeuti [9].

PROPOSITION 3. *Let f be a locally integrable function defined on R^n. We define*

$$F(x) = \lim_{k\to\infty} \int_{R^n} f(y)\delta_k^0(y-x)dy.$$

Then there exists a continuous function f_0 defined on R^n, such that

$$f(x) = f_0(x)$$

for almost every x if, and only if, $F(x)$ is a continuous function.

PROOF. Assume first that there exists a continuous function f_0, such that $f(x) = f_0(x)$ for almost every x.

Then

$$\begin{aligned}
F(x) &= \lim_{k\to\infty} \int_{R^n} f(y)\delta_k^0(y-x)dy\\
&= \lim_{k\to\infty} \int_{R^n} f_0(y)\delta_k^0(y-x)dy\\
&= f_0(x)
\end{aligned}$$

for every x. (*See*, [5]).

Conversely, let $F(x)$ be a continuous function.
Since

$$f(x)=F(x)$$

for almost every x, we can choose F as f_0. Q.E.D.

EXAMPLE 1. Let Y be the Heaviside function defined on R^1. Then we have

$$\begin{aligned}\lim_{k\to\infty}\int_{R^n} Y(y)\delta_k^0(y-x)dy &= 1 \qquad (x>0)\\ &= \frac{1}{2} \qquad (x=0)\\ &= 0 \qquad (x<0).\end{aligned}$$

By Proposition 3 we immediately have the following proposition:

PROPOSITION 4. *Let f be a locally integrable function defined on R^n. Then there exists a continuous function f_0 defined on R^n, such that*

$$f(x)=f_0(x)$$

for almost every x if, and only if,

$$St(f, \delta_x^0)\equiv St\left(\int_{R^n} f(y)\delta_1^0(y-x)\,dy,\ \int_{R^n} f(y)\delta_2^0(y-x)\,dy,\ \cdots\right)$$

is a continuous function.

PROPOSITION 5. *For each $S\in(\mathscr{D}^m)'$, $m<\infty$, there exist an m_0 and a continuous function f defined on R^n such that for each $\varphi\in(\mathscr{D})$,*

$$\lim_{k\to\infty}\int_{R^n}\left\{\int_{R^n} f(x)D_t^{m_0}\delta_k^0(x-t)\,dx\right\}\varphi(t)\,dt=S(\varphi).$$

Let

$$f_k(t)=\int_{R^n} f(x)D_t^{m_0}\delta_k^0(x-t)\,dt$$

then $f_k\in(\mathscr{E})$.

PROOF. Let $\varphi\in(\mathscr{D})$. Then we have

$$\begin{aligned}\lim_{k\to\infty}\int_{R^n} D_t^p\delta_k^0(x-t)\varphi(t)\,dt &= \lim_{k\to\infty}\int_{R^n}\delta_k^0(x-t)(-1)^{|p|}D_t^p\varphi(t)\,dt\\ &=(-1)^{|p|}D_x^p\varphi(x).\end{aligned}$$

Let f be a continuous function defined on R^n and let

$$\varphi_k(x)=\int_{R^n} D_t^p \delta_k^0 (x-t)\varphi(t)\,dt.$$

Since

$$\begin{aligned}|\varphi_k(x)| &= \left|\int_{R^n} D_t^p \delta_k^0 (x-t)\varphi(t)\,dt\right| \\ &= \left|\int_{R^n} \delta_k^0 (x-t)(-1)^{|p|} D_t^p \varphi(t)\,dt\right| \\ &= \sup_{t\in Car(\varphi)} |D_t^p \varphi(t)| \left|\int_{R^n} \delta_k^0 (x-t)\,dt\right| \\ &\leq \sup_{t\in Car(\varphi)} |D_t^p \varphi(t)| \\ &< \infty\end{aligned}$$

by Lebesgue's theorem, the relation

$$\lim_{k\to\infty}\iint_{R^n\times R^n} f(x) D_t^p \delta_k^0 (x-t)\varphi(t)\,dt\,dx=\int_{R^n} f(x)(-1)^{|p|} D_x^p \varphi(x)\,dx$$

holds.
By Fubini's theorem we have

$$\iint_{R^n\times R^n} f(x) D_t^p \delta_k^0 (x-t)\varphi(t)\,dt\,dx=\int_{R^n}\left\{\int_{R^n} f(x) D_t^p \delta_k^0 (x-t)\,dx\right\}\varphi(t)\,dt.$$

According to the structure theorem (T. Iwamura, Y. Kawada and K. Yosida [2]), for each $S\in(\mathscr{D}^m)'$, $m<\infty$ there exist an m_0 and a continuous function f defined on R^n such that

$$S(\varphi)=\int_{R^n} f(x)(-1)^{|m_0|} D_x^{m_0} \varphi(x)\,dx$$

for every $\varphi\in(\mathscr{D})$.
Therefore, for each $S\in(\mathscr{D}^m)'$, $m<\infty$ there exist an m_0 and a continuous function f defined on R^n; the relation

$$\lim_{k\to\infty}\int_{R^n}\left\{\int_{R^n} f(x) D_t^{m_0} \delta_k^0(x-t)\,dx\right\}\varphi(t)\,dt=S(\varphi)$$

holds for every $\varphi\in(\mathscr{D})$.
It is celar that $f_k\in(\mathscr{E})$.

We have the following proposition immediately from Proposition 5:

PROPOSITION 6. *For each* $S\in(\mathscr{D}^m)'$, $m<\infty$, *there exist an* m_0 *and a continuous function* f *defined on* R^n *such that for each* $\varphi\in(\mathscr{D})$

$$(F, \varphi) \fallingdotseq S(\varphi)$$

where

$$F(t)=\left(\int_{R^n} f(x) D_t^{m_0} \delta_1^0 (x-t)\,dx, \int_{R^n} f(x) D_t^{m_0} \delta_2^0 (x-t)\,dx, \cdots\cdots\right).$$

EXAMPLE 2. Let f be a function defined on R^1 with the following property:

$$\begin{aligned} f(x)&=x \qquad (x \geq 0)\\ &=0 \qquad (x<0). \end{aligned}$$

Let Y be the Heaviside function defined on R^1 and let δ be the Delta function defined on R^1. Then

$$\lim_{k\to\infty}\int_{R^1}\left\{\int_{R^1} f(x)\frac{d}{dt}\delta_k^0(x-t)dx\right\}\varphi(t)dt = Y(\varphi)$$

$$\lim_{k\to\infty}\int_{R^1}\left\{\int_{R^1} f(x)\frac{d^2}{dt^2}\delta_k^0(x-t)dt\right\}\varphi(t)dt = \delta(\varphi)$$

for every $\varphi \in (\mathscr{D})$.

REMARK. The characterizations of the locally integrable functions and the continuous functions in the space of distributions were given in [3], [4].

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