On the Distributions in the Dirac Spaces (II)

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1. Introduction

J. Mikusiński [2] defined the product $\delta\cdot\frac{1}{x}$ of the Dirac delta-distribution δ by the function $\frac{1}{x}$ as the distributional limit of $\delta_k\left(\frac{1}{x}*\delta_k\right)$, and gave the result that

$$\delta\cdot\frac{1}{x}=-\frac{1}{2}\delta'. \tag{1}$$

The notation used here is that δ_k is the so-called delta-sequence, i. e., a sequence of smooth functions within $-\infty<x<\infty$, satisfying

1° $\delta_k(x)=0$ for $|x|>\alpha_k$, where $\alpha_k>0$, $\alpha_k\to 0$;

2° $\int_{-\infty}^{\infty}\delta_k(x)dx=1$;

3° $|x^{l+1}\delta_k^{(l)}(x)|<M^l$ (M^l independent of k).

Using the equality (1) B. Fisher [1] gave many important results.

We define the quotient $\frac{\delta}{x}$ of the Dirac delta-distribution δ by the function x as the distributional limit of $\frac{\delta_{k,1}^1(x)}{x}$, where $\delta_{k,1}^1(x)$ is the sequence of functions within $-\infty<x<\infty$, satisfying

$$\begin{aligned}\delta_{k,1}^1(x)&=0 \quad \left(|x|>\frac{1}{2k}\right)\\ &=k \quad \left(|x|\leq\frac{1}{2k}\right).\end{aligned}$$

With this definition we have

$$\frac{\delta}{x}=-\delta'. \tag{2}$$

Using the equality (2) we have a few results.

The difference between (1) and (2) will be one of the characters which are called

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ambiguity by G. Takeuti [3].

Equality

$$\delta^2 - \frac{1}{\pi^2}\left(\frac{1}{x^2}\right) = -\frac{1}{\pi^2}\cdot\frac{1}{x^2} \tag{3}$$

is used in quantum mechanics, provided the difference on the left side is considered as a single entity, no meaning being related its particular members δ^2 and $\left(\frac{1}{x}\right)^2$.

J. Mikusiński [2] has justified this equality (3) using the Fourier Transform.

In this paper we intend to justify the following equality:

$$\delta^2 - \frac{1}{\alpha}\left(\frac{1}{x}\right)^2 = -\frac{1}{\alpha}\cdot\frac{1}{x^2} \qquad (\alpha > 0). \tag{4}$$

2. On the equality $\frac{\delta}{x} = -\delta'$

Let $a>0$, and for each integer $k \geq 1$, define

$$\delta^1_{k,a}(x) = 0 \qquad \left(|x| > \frac{a}{2k}\right)$$
$$= \frac{k}{a} \qquad \left(|x| \leq \frac{a}{2k}\right).$$

PROPOSITION 1. *Let f be a locally integrable function defined on R^1. Then we have*

$$st\left(\int_{-\infty}^{\infty}\delta^1_{1,a}(y-x)f(y)dy, \int_{-\infty}^{\infty}\delta^1_{2,a}(y-x)f(y)dy, \ldots, \int_{-\infty}^{\infty}{}^1_{k,a}\delta(y-x)f(y)dy, \ldots\right)$$
$$= f(x)$$

for almost every $x \in R^1$.

PROOF. This follows immediately from Lebesgue's theorem.

THEOREM 1. *The following equalities*

$$\frac{\delta}{x} = \frac{1}{x}\cdot\delta = \delta\frac{1}{x} = -\delta'$$

hold. Hence, for each $\varphi \in (\mathscr{D})$ we have

$$st\left(\frac{\delta^1_1}{x}, \varphi\right) = \varphi'(0) = -\delta'(\varphi),$$

PROOF. Let $\varphi \in (\mathscr{D})$, and let

where

$$\frac{\delta_1^1}{x}=\left(\frac{\delta_{1,1}^1}{x}, \frac{\delta_{2,1}^1}{x}, \dots, \frac{\delta_{k,1}^1}{x}, \dots\right).$$

$$m_k=\inf_{-\frac{1}{2k}\leq x\leq \frac{1}{2k}} \varphi'(x),\quad M_k=\sup_{-\frac{1}{2k}\leq x\leq \frac{1}{2k}} \varphi(x).$$

Since

$$\int_{-\frac{1}{2k}}^{-\varepsilon} \frac{k}{x}\varphi(x)dx+\int_{\varepsilon}^{\frac{1}{2k}} \frac{x}{k}\varphi(x)\,dx=2k\int_{\varepsilon}^{\frac{1}{2k}} \frac{\varphi(x)-\varphi(-x)}{2x}dx,$$

$$\frac{\varphi(x)-\varphi(-x)}{2x}=\varphi'(x(-1+2\theta)) \qquad (0<\theta<1)$$

we have

$$2km_k\left(\frac{1}{2k}-\varepsilon\right)\leq 2k\int_{\varepsilon}^{\frac{1}{2k}} \frac{\varphi(x)-\varphi(-x)}{2x}dx\leq 2kM_k\left(\frac{1}{2k}-\varepsilon\right),$$

where ε is a real number with $\frac{1}{2k}>\varepsilon>0$. Hence, we have

$$\begin{aligned}\lim_{k\to\infty} \text{v.p.}\int_{-\infty}^{\infty} \frac{\delta_{k,1}^1}{x}(x)\varphi(x)dx &= \lim_{k\to\infty} \text{v.p.}\int_{-\frac{1}{2k}}^{\frac{1}{2k}} \frac{k}{x}\varphi(x)dx \\ &= \lim_{k\to\infty}\left(\lim_{\varepsilon\to 0} 2k\int_{\varepsilon}^{\frac{1}{2k}} \frac{\varphi(x)-\varphi(-x)}{2x}dx\right) \\ &=\varphi'(0) \\ &= -\delta'(\varphi).\end{aligned}$$

This shows that

$$\frac{\delta}{x}=-\delta'.$$

It is clear that

$$\frac{\delta}{x}=\frac{1}{x}\cdot\delta=\delta\cdot\frac{1}{x}. \qquad \text{Q.E.D.}$$

Corollary 1. *Let $\varphi\in(\mathscr{D})$, then*

$$\lim_{k\to\infty} \text{v.p.}\int_{-\infty}^{\infty} \frac{\delta_{k,a}^1(x)}{x}\varphi(x)dx=\varphi'(0)=-\delta'(\varphi).$$

Hence, we have

$$st\left(\frac{\delta_a^1}{x}, \varphi\right)=\varphi'(0)=-\delta'(\varphi)$$

for each $\varphi \in (\mathscr{D})$, *where*

$$\frac{\delta_a^1}{x} = \left(\frac{\delta_{1,a}^1}{x}, \frac{\delta_{2,a}^1}{x}, \dots, \frac{\delta_{k,a}^1}{x}, \dots \right).$$

PROOF. This follows immediately from Theorem 1.

For each integer $k \geq 1$, define

$$\begin{aligned} \delta_k(x) &= 0 && \left(x \leq -\frac{1}{k} \right) \\ &= k + k^2 x && \left(-\frac{1}{k} \leq x \leq 0 \right) \\ &= k - k^2 x && \left(0 \leq x \leq \frac{1}{k} \right) \\ &= 0 && \left(\frac{1}{k} \leq x \right). \end{aligned}$$

PROPISITION 2. *Let* $\varphi \in (\mathscr{D})$, *then*

$$\lim_{k \to \infty} \text{v. p.} \int_{-\infty}^{\infty} \frac{\delta_k(x)}{x} \varphi(x) dx = \varphi'(0) = -\delta'(\varphi).$$

Hence, we have

$$st\left(\frac{\delta}{x}, \varphi \right) = \varphi'(0) = -\delta'(\varphi).$$

for each $\varphi \in (\mathscr{D})$, *where*

$$\frac{\delta}{x} = \left(\frac{\delta_1}{x}, \frac{\delta_2}{x}, \dots, \frac{\delta_k}{x}, \dots \right).$$

PROOF. Let $\delta_k'(x)$ be the derivative in the sense of distribution of the function $\delta_k(x)$, then

$$\frac{\delta_k(x)}{x} = \delta_k'(x) + 2 \cdot \frac{\delta_{2,2}^1(x)}{x}$$

for almost every $x \in R^1$.

Let $\varphi \in (\mathscr{D})$, and let

$$m_k = \inf_{-\frac{1}{k} \leq x \leq \frac{1}{k}} \varphi(x), \ M_k = \sup_{-\frac{1}{k} \leq x \leq \frac{1}{k}} \varphi'(x).$$

Since

$$\int_{-\infty}^{\infty} \delta_k'(x) \varphi(x) dx = -k^2 \int_0^{\frac{1}{k}} \frac{\varphi(x) - \varphi(-x)}{2x} 2x dx$$

$$= -k^2 \int_0^{\frac{1}{k}} \varphi'(x(-1+2\theta))2x dx \qquad (0<\theta<1),$$

we have

$$M_k = k^2 \int_0^{\frac{1}{k}} M_k 2x dx \geq -\int_{-\infty}^{\infty} \delta_k'(x)\varphi(x)dx$$

$$\geq k^2 \int_0^{\frac{1}{k}} m_k 2x dx = m_k.$$

Hence, we have

(i) $$\lim_{k\to\infty}\int_{-\infty}^{\infty} \delta_k'(x)\varphi(x)dx = -\varphi'(0) = \delta'(\varphi)$$

for every $\varphi \in (\mathscr{D})$.
Let $\varphi \in (\mathscr{D})$, then

(ii) $$\lim_{k\to\infty} \text{v.p.} \int_{-\infty}^{\infty} 2 \cdot \frac{\delta_{k,2}^1(x)}{x} \varphi(x)dx = 2\varphi'(0) = -2\delta'(\varphi).$$

The relation between (i) and (ii) implys that

$$\lim_{k\to\infty} \text{v.p.} \int_{-\infty}^{\infty} \frac{\delta_k(x)}{x} \varphi(x)dx$$

$$= \lim_{k\to\infty}\int_{-\infty}^{\infty} \delta_k'(x)\varphi(x)dx + \lim_{k\to\infty} \text{v.p.} \int_{-\infty}^{\infty} 2 \cdot \frac{\delta_{k,2}^1(x)}{x} \varphi(x)dx$$

$$= \delta'(\varphi) - 2'(\varphi)$$

$$= -\delta'(\varphi)$$

for every $\varphi \in (\mathscr{D})$.

B. Fisher [1] proved the following proposition.

PROPOSITION 3. *The following equalities hold*:

(5) $$x \cdot \left(\frac{1}{x}\right) \cdot \delta = -\frac{1}{2} x \cdot \delta' = \frac{1}{2}\delta,$$

(6) $$\left(x \cdot \frac{1}{x}\right) \cdot \delta = \delta,$$

(7) $$(x \cdot \delta) \cdot \frac{1}{x} = 0.$$

By Theorem 1 we immediately have the following proposition.

PROPOSITION 4. *The following equalities hold*:

(5)′ $$x\cdot\left(\frac{1}{x}\right)\cdot\delta=-x\cdot\delta'=\delta,$$

(6)′ $$\left(x\cdot\frac{1}{x}\right)\cdot\delta=\delta,$$

(7)′ $$(x\cdot\delta)\cdot\frac{1}{x}=\delta.$$

3. On the equality $\delta^2-\frac{1}{\alpha}\left(\frac{1}{x}\right)^2=-\frac{1}{\alpha}\cdot\frac{1}{x^2}$

For each integer $k\geq 1$, define

$$X_k(x)=\frac{1}{x^2}\qquad\left(|x|\geq\frac{1}{k}\right)$$
$$=0\qquad\left(|x|<\frac{1}{k}\right).$$

The following theorem justifies the equality (4).

THEOREM 2. *For each $\alpha>0$, there exists an $a>0$ and for each $\varphi\in(\mathscr{D})$, the equality*

$$st(((\delta_a^1)^2,\varphi)-\frac{1}{\alpha}(X,\varphi))=\left(\text{Pf.}\,\frac{1}{\alpha}\cdot\frac{1}{x^2},\varphi\right)$$

holds, where

$$X=(X_1, X_2,\dots, X_k,\dots),$$
$$(\delta_a^1)^2=((\delta_{1,a}^1)^2, (\delta_{2,a}^1)^2,\dots, (\delta_{k,a}^1)^2,\dots).$$

PROOF. Let $\varphi\in(\mathscr{D})$.
Since

$$2a\int_{-\infty}^{\infty}(\delta_{k,a}^1)^2(x)\varphi(x)dx=2a\int_{-\frac{a}{2k}}^{\frac{a}{2k}}\frac{k^2}{a^2}\varphi(x)dx$$
$$=2k\varphi(0)+O\left(\frac{1}{k}\right),$$

$$\int_{-\infty}^{\infty}X_k(x)\varphi(x)dx=\int_{-\infty}^{-\frac{1}{k}}\frac{1}{x^2}\varphi(x)+\int_{\frac{1}{k}}^{\infty}\frac{1}{x^2}\varphi(x)dx$$
$$=2k\varphi(0)+O\left(\frac{1}{k}\right)+\int_{-\infty}^{-\frac{1}{k}}\frac{1}{x}\varphi'(x)dx+\int_{\frac{1}{k}}^{\infty}\frac{1}{x}\varphi'(x)dx,$$

we have

$$\int_{-\infty}^{\infty}(\delta_{k,a}^{1})^{2}(x)\varphi(x)dx-\frac{1}{2a}\int_{-\infty}^{\infty}X_{k}(x)\varphi(x)dx$$

$$=O\left(\frac{1}{k}\right)-\frac{1}{2a}\left(\int_{-\infty}^{-\frac{1}{k}}\frac{1}{x}\varphi'(x)dx+\int_{\frac{1}{k}}^{\infty}\frac{1}{x}\varphi'(x)dx\right).$$

We immediately have

$$\lim_{k\to\infty}\frac{1}{2a}\left(\int_{-\infty}^{-\frac{1}{k}}\frac{1}{x}\varphi'(x)dx+\int_{\frac{1}{k}}^{\infty}\frac{1}{x}\varphi'(x)dx\right)$$

$$=-\frac{d}{dx}\left(\mathrm{Pf.}\frac{1}{2a}\cdot\frac{1}{x}\right)(\varphi)$$

$$=\mathrm{Pf.}\frac{1}{2a}\cdot\frac{1}{x^{2}}(\varphi).$$

Hence

$$St(((\delta_{a}^{1})^{2},\varphi)-\frac{1}{2a}(X,\varphi))=\left(\mathrm{Pf.}\frac{1}{2a}\cdot\frac{1}{x^{2}},\varphi\right).$$

If we put $a=\frac{\alpha}{2}$ then we obtain

$$St(((\delta_{\frac{\alpha}{2}}^{1})^{2},\varphi)-\frac{1}{\alpha}(X,\varphi))=\left(\mathrm{Pf.}\frac{1}{\alpha}\cdot\frac{1}{x^{2}},\varphi\right). \qquad \text{Q.E.D.}$$

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References

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