A Generalization of Locally Finite Collections

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1. Introduction

The notions "locally finite" and "hereditarily closure preserving" are very important in general topology. Recently, in his paper [1], J.R. Boone first introduced the notion of cs-finiteness which is a generalization of locally-finiteness, and then he characterized paracompact spaces in terms of "cs-finite"; and in [2] he established a metrization theorem of developable spaces.

In this paper, we introduce the notions "as-finite", "almost as-finite" and "almost cs-finite" which are generalizations of "locally finite." Especially, in countably compact spaces, our as-finiteness has the property which J.R. Boone's cs-finiteness does not have (Theorem 3.1).

The purpose of this paper is to study the properties of (almost) as-finite collections in quasi-k-spaces (to be discussed in § 4); then, as applications, to characterize paracompact spaces in terms of "(almost) as-finite" and to establish a metrization theorem of semi-stratifiable spaces (to be treated in § 6).

We set the outline of this paper as follows: In § 2, we will develop the fundamental notions which are used in this paper; we will illustrate by Diagram 1 the basic implications which exist among, specifically, the properties of collections. In § 3, we will investigate the properties of as-finite (resp. cs-finite) collections in countably (resp. sequentially) compact spaces. In § 4, we will discuss in what classes of quasi-k-spaces the inverse implications in Diagram 1 are valid. In § 5, we will deal with the relations between mappings and (almost) as-finite collections. Lastly, in § 6, as applications of these notions, we will characterize paracompact spaces and collectionwise normal spaces in terms of (almost) as-finite collections, and we will establish a metrization theorem of semi-stratifiable spaces.

I wish to express my hearty thanks to Professor A. Okuyama of Osaka University of Education who has given me much kind advice.

2. Definitions and relations

In this section, we give the definitions of the terms which are used in this paper,

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and then we illustrate their relations.

First, let us recall the definitions of well-known terms. Let be X a topological space and let $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ be a collection of subsets of X. The collection $\mathfrak{F}$ is said to be *locally finite* if every point of X has a neighborhood which intersects only finitely many elements of $\mathfrak{F}$. The collection $\mathfrak{F}$ is called *closure preserving* if, for every subcollection $\mathfrak{G} \subset \mathfrak{F}$, the union of closures is the closure of the union (i.e. $\cup\{\bar{G} \mid G \in \mathfrak{G}\} = \overline{\cup\{G \mid G \in \mathfrak{G}\}}$). The collection $\mathfrak{F}$ is said to be *hereditarily closure preserving* if each collection $\{G_\alpha \mid \alpha \in A\}$ with $G_\alpha \subset F_\alpha$ is closure preserving. The space X is a *quasi-k-space* if and only if a subset F of X is closed whenever $F \cap C$ is closed in C for every countably compact subset C of X. A subset F of X is said to be *sequentially closed* if and only if no sequence in F converges to a point not in F. The space X is said to be a *sequential space* if each sequentially closed set of X is closed. The space X is a *singly bi-quasi-k-space* if and only if, whenever $x \in \bar{F}(F \subset X)$, there exists a q-sequence* $\{A_n\}$ in X such that $x \in \overline{F \cap A_n}$ for each n. The space X is *Fréchet space* if and only if, whenever $x \in \bar{F}$ in X, there is a sequence $\{x_n\}$ in F such that $\{x_n\}$ converges to x. The space X is said to be a *q-space* if every $x \in X$ has a q-sequence of neighborhoods. And, $\mathbf{N}$ denotes the positive integers set. As for other terms and symbols in general topology, see [9].

Next, we define the terms which are used uniquely in this paper.

Definition 2.1. A sequence $\{x_n\}$ of points of X is said to be an *ac-sequence* if each subsequence of $\{x_n\}$ has a cluster point in X.

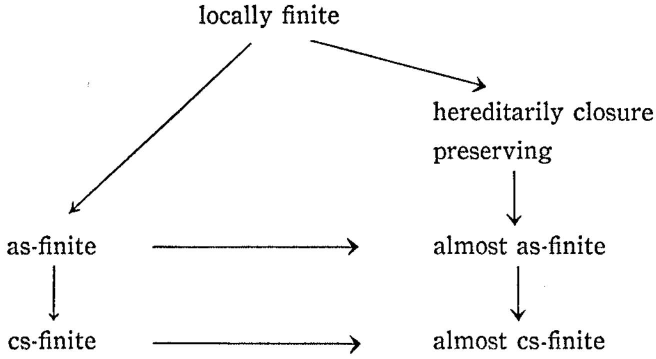
Definition 2.2. A collection $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ is *almost as-finite* if and only if, for every ac-sequence $\{x_n\}$, there exists a finite subset S_0 of $S=\{x_n \mid n \in \mathbf{N}\}$ such that $\{\alpha \in A \mid F_\alpha \cap (S-S_0) \neq \phi\}$ is finite. Especially, if $\{\alpha \in A \mid F_\alpha \cap S \neq \phi\}$ is finite for every ac-sequence $\{x_n\}$, we say that $\mathfrak{F}$ is *as-finite*.

Definition 2.3. A collection $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ is *almost cs-finite* if and only if, for every convergent sequence $\{x_n\}$, there exists a finite subset S_0 of $S=\{X_n \mid n \in \mathbf{N}\}$ such that $\{\alpha \in A \mid F_\alpha \cap (S-S_o) \neq \phi\}$ is finite.

According to J.R. Boone [1], a collection $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ is said to be *cs-finite* if $\{\alpha \in A \mid F_\alpha \cap S \neq \phi\}$ is finite for every convergent sequence $\{x_n\}$. It follows immediately from these definitions that $\mathfrak{F}$ is asfinite (resp. cs-finite) if and only if it is almost as-finite (resp. almost cs-finite) and point-finite.

Now, we illustrate the basic implications which exist among these properties of collections (Diagram 1).

* A sequence $\{A_n\}$ of subsets is said to be a *q-sequence* if every sequence $\{x_n\}$ with $x_n \in A_n$ (for each n) has a cluster point in X.



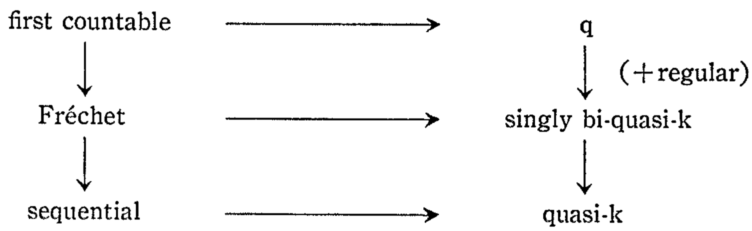
(Diagram 1.)

Most of these implications are either known or derived immediately from the definitions. So, we give the proof of only the following.

Proposition 2.1. *Let $\mathfrak{F}=\{F_\alpha \mid \alpha\in A\}$ be a collection of subsets of a topological space X. If $\mathfrak{F}$ is hereditarily closure preserving, then $\mathfrak{F}$ is almost as-finite.*

Proof. Suppose that $\mathfrak{F}$ is not almost as-finite. Then, there exists a distinct ac-sequence $\{x_n\}$ such that $\{\alpha\in A \mid F_\alpha\cap S_n\neq\phi\}$ is infinite for each n, where $S_n=\{x_i \mid i\geqq n\}$. Therefore, we can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a distinct sequence $\{\alpha_k\}$ in A such that $x_{n_k}\in F_{\alpha_k}$ for each $k\in\mathbf{N}$. Since $\mathfrak{F}$ is hereditarily closure preserving, $\{x_{n_k} \mid k\in\mathbf{N}\}$ is a discrete subset of X. This contradicts the fact that $\{x_n\}$ is an ac-sequence.

In §4, we will discuss in what classes of quasi-k-spaces the inverse implications in Diagram 1 are valid. Here, we illustrate the implications which exist among these notions of the spaces to be treated in §4 (Diagram 2.). (Cf. [8])



(Diagram 2.)

Throughout this paper, topological spaces are assumed to be T_1-spaces.

3. As (cs)-finitenes in countably (sequentially) compact spaces

THEOREM 3.1. *Let X be a countably compact space. If $\mathfrak{F}=\{F_\alpha | \alpha \in A\}$ is an almost as-finite collection in X, then the set*

$$X_0=\{x \in X | \{\alpha \in A | x \in F_\alpha\} \text{ is infinite}\}$$

is finite, and the set

$$A'=\{\alpha \in A | F_\alpha \cap (X-X_0) \neq \phi\}$$

is finite.

PROOF. On the contrary, suppose that X is infinite. Then, we can choose a distinct sequence $\{x_n\}$ in X and a distinct sequence $\{\alpha_n\}$ in A such that $x_n \in F_{\alpha_n}$ for each $n \in \mathbf{N}$. Since $\mathfrak{F}$ is almost as-finite, $\{x_n\}$ is not an ac-sequence. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ does not cluster in X. Since X is a T_1-space, the set $\{x_{n_k} | k \in \mathbf{N}\}$ is an infinite, discrete set. This contradicts the fact that X is countably compact, and hence the first half of Theorem 3.1 is proved.

To complete the proof of Theorem 3.1, assume that A' is infinite. Since $\mathfrak{F}$ is point-finite in $X-X_0$, we can choose a distinct sequence $\{x_n\}$ in $X-X_0$ and a distinct sequence $\{\alpha_n\}$ in A' such that $x_n \in F_{\alpha_n}$ for each $n \in \mathbf{N}$. Similarly, a contradiction follows from this. The proof is complete.

REMARK. Theorem 3.1 does not necessarily hold for a cs-finite (almost cs-finite) collection even if a space X is compact. This is shown by the following example.

EXAMPLE 3.1. Let $X=\beta\mathbf{N}$ be the Stone-Čech compactification of the integers $\mathbf{N}$, and put $\mathfrak{F}=\{\{n\} | n \in \mathbf{N}\}$. Then, X is a compact T_2-space and any convergent sequence $\{x_n\}$ in X does not contain infinitely many points of $\mathbf{N}$. Therefore, $\mathfrak{F}$ is a cs-finite collection in X. Nevertheless, $X_0=\phi$ and $A'=\mathbf{N}$ is infinite.

COROLLARY 3.2. *Let be a countably compact space. If $\mathfrak{F}$ is an as-finite collection in X, then $\mathfrak{F}$ is finite.*

COROLLARY 3.3. *Let X be a space and let $\mathfrak{F}=\{F_\alpha | \alpha \in A\}$ be an almost as-finite collection in X. If C is a countably compact subset of X, then there exists a finite subset C_0 of C such that $\{\alpha \in A | F_\alpha \cap (C-C_0) \neq \phi\}$ is finite.*

COROLLARY 3.4 (A. Okuyama [11, Theorem 2.1]). *Let $\mathfrak{F}=\{F_\alpha | \alpha \in A\}$ be a hereditarily closure preserving closed cover of a space X and C a countably compact set of X. Then, there exist $x_1, x_2, \cdots, x_n$ in C such that $\mathfrak{F}$ is locally finite at any $x \in C-\{x_1, \cdots x_n\}$, and only finitely many members of $\mathfrak{F}$ meet $C-\{x_1, \cdots, x_n\}$.*

THEOREM 3.5. *Let X be a sequentially compact space. If $\mathfrak{F}=\{F_\alpha \mid \alpha\in A\}$ is an almost cs-finite collection in X, then the set $X_0=\{x\in X \mid \{\alpha\in A \mid x\in F_\alpha\}$ is infinite$\}$ is finite, and the set $A'=\{\alpha\in A \mid F_\alpha\cap(X-X_0)\neq\phi\}$ is finite.*

PROOF. The proof is similar to that of Theorem 3.1.

COROLLARY 3.6. *Let X be a sequentially compact space. If $\mathfrak{F}$ is a cs-finite collection in X, then $\mathfrak{F}$ is finite.*

COROLLARY 3.7. *Let X be a space and let $\mathfrak{F}=\{F_\alpha \mid \alpha\in A\}$ be an almost cs-finite collection in X. If C is sequentially compact subset of X, then there exists a finite subset C_0 of C such that $\{\alpha\in A \mid F_\alpha\cap(C-C_0)\neq\phi\}$ is finite.*

Now, we introduce a certain property which all subparacompact* spaces have.

DEFINITION 3.1. A space X has *the property* (C) (resp. *the property* (C')) if and only if, for every open covering $\mathfrak{G}$ of X, $\omega(\mathfrak{G})$ has a σ-almost as-finite (resp. σ-almost cs-finite) refinement, where $\omega(\mathfrak{G})$ denotes the collection of all countable unions of members of $\mathfrak{G}$.

THEOREM 3.8. *If X is a countably compact space with the property (C), then X is compact.*

PROOF. Let $\mathfrak{G}$ be an open covering of X. Since X has the property (C), there exists a σ-almost as-finite refinement $\mathfrak{F}=\bigcup_{n=1}^{\infty}\mathfrak{F}_n$ of $\omega(\mathfrak{G})$, where $\mathfrak{F}_n$ is almost as-finite for each $n\in\mathbf{N}$. By Theorem 3.1, for each $n\in\mathbf{N}$, there exists a finite subset X_n of X such that $\mathfrak{F}_n'=\{F\in\mathfrak{F}_n \mid F\cap(X-X_n)\neq\phi\}$ is finite. Put $Y_n=\cup\{F \mid F\in\mathfrak{F}_n\}$ and $Y_n'=\cup\{F \mid F\in\mathfrak{F}_n'\}$, then we obtain the following:

$$Y_n'\supset Y_n-X_n, \qquad \bigcup_{n=1}^{\infty}Y_n=X.$$

Therefore, $\bigcup_{n=1}^{\infty}Y_n'\supset X-\bigcup_{n=1}^{\infty}X_n$. Since $\bigcup_{n=1}^{\infty}X_n$ is a countable subset, there exists a countable subcollection $\mathfrak{F}_0'$ of $\mathfrak{F}$ such that $\bigcup_{n=1}^{\infty}X_n\subset\cup\{F \mid F\in\mathfrak{F}_0'\}$. Then, $\mathfrak{F}'=\bigcup_{i=0}^{\infty}\mathfrak{F}_i'$ is a countable covering of X and a refinement of $\omega(\mathfrak{G})$. Therefore, $\omega(\mathfrak{G})$ has a countable subcovering $\{W_n \mid n\in\mathbf{N}\}$. Since W_n is a countable union of elements of $\mathfrak{G}$ for each $n\in\mathbf{N}$, we can denote $W_i=\bigcup_{j=1}^{\infty}G_{ij}$ for each $i\in N$, where $G_{ij}\in\mathfrak{G}$. Since X is countably compact, the countable covering $\{G_{ij} \mid i, j\in\mathbf{N}\}$ has a finite subcovering. Hence, X is compact. The proof is complete.

* A space X is said to be a *subparacompact space* if every open covering of X has A σ-discrete, closed refinement.

COROLLARY 3.9 (D.K. Burke [3]). *A countably compact, subparacompact space is compact.*

THEOREM 3.10. *A sequentially compact space with the property (C') is compact.*

PROOF. The proof is similar to that of Theorem 3.7.

4. As (cs)-finiteness in quasi-k-spaces

In this section, we discuss in what classes of quasi-k-spaces the inverse implications in Diagram 1 are valid.

THEOREM 4.1. *In a singly bi-quasi-k-space X, a collection $\mathfrak{F}=\{F_\alpha \mid \alpha\in A\}$ of subsets of X is hereditarily closure preserving if and only if $\mathfrak{F}$ is almost as-finite.*

PROOF. The necessity is clear. To prove the sufficiency, assume that $\mathfrak{F}$ is not hereditarily closure preserving. Then, there exists a collection $\{H_\alpha \mid \alpha\in A\}$ such that

$$H_\alpha \subset F_\alpha \qquad (\alpha\in A),$$

$$\overline{\cup\{H_\alpha \mid \alpha\in A\}} - \cup\{\bar{H}_\alpha \mid \alpha\in A\} \neq \phi.$$

Let p be a point of $\overline{\cup\{H_\alpha \mid \alpha\in A\}} - \cup\{\bar{H}_\alpha \mid \alpha\in A\}$. Since X is a singly bi-quasi-k-space, there exists a q-sequence $\{B_n\}$ such that

(1) $$p\in \overline{[\cup\{H_\alpha \mid \alpha\in A\}]\cap B_n} \qquad (\text{for each } n\in \mathbf{N}).$$

Then, for each $n\in\mathbf{N}$ and for every neighborhood V of p

(2) $$\{\alpha\in A \mid H_\alpha\cap B_n\cap V\neq\phi\} \text{ is infinite.}$$

In fact: Assume that $\{\alpha\in A \mid H_\alpha\cap B_n\cap V\neq\phi\}$ is finite for some $n\in\mathbf{N}$ and some open neighborhood V of p. Then, we can put $\{\alpha\in A \mid H_\alpha\cap B_n\cap V\neq\phi\}=\{\alpha_1, \alpha_2,\cdots,\alpha_k\}$. From (1),

$$\begin{aligned} p &\in \overline{[\cup\{H_\alpha \mid \alpha\in A\}]\cap B_n}\cap V \\ &\subset \overline{[\cup\{H_\alpha \mid \alpha\in A\}]\cap B_n\cap V} \\ &\subset \overline{\bigcup_{i=1}^{k} H_{\alpha i}} \subset \cup\{\bar{H}_\alpha \mid \in A\}. \end{aligned}$$

This contradicts the fact that $p\notin \cup\bar{H}_\alpha \mid \alpha\in A\}$. Hence, (2) is valid.

Since $p\in \overline{[\cup\{H_\alpha \mid \alpha\in A\}]\cap B_1} - \cup\{H_\alpha \mid \alpha\in A\}$, there exist an element $\alpha_1\in A$

and a point x_1 in X such that

$$x_1 \in H_{\alpha_1} \cap B_1, \quad x_1 \neq p.$$

The space X is T_1, so there exists an open neighborhood $V_1(p)$ of p such that $x_1 \notin V_1(p)$. Now, from (1),

$$\begin{aligned} p \in & \overline{[\cup\{H_\alpha \mid a \in A\}] \cap B_2} \cap V_1(p) \\ & \subset \overline{[\cup\{H_\alpha \mid \alpha \in A\}] \cap B_2 \cap V_1(p)}. \end{aligned}$$

Then, by (2), there exist an element $\alpha_2 \in A$ and a point x_2 in X such that

$$x_2 \in H_{\alpha_2} \cap B_2 \cap V_1(p), \quad \alpha_2 \neq \alpha_1, \quad x_2 \neq p.$$

Since X is T_1, there exists an open neighborhood $V_2(p)$ of p such that

$$x_2 \notin V_2(p), \quad V_2(p) \subset V_1(p).$$

By induction, we obtain a distinct point sequence $\{x_n\}$ in X and a distinct sequence $\{\alpha_n\}$ in A such that

$$x_n \in H_{\alpha_n} \cap B_n \subset F_{\alpha_n}, \quad x_n \neq p$$

for each $n \in \mathbf{N}$. Since $\{B_n\}$ is a q-sequence, $\{x_n\}$ is an ac-sequence in X. This contradicts the fact that $\mathfrak{F}$ is almost as-finite in X, because both $\{x_n\}$ and $\{\alpha_n\}$ are distinct sequences. Therefore, $\mathfrak{F}$ is hereditarily closure preserving. The proof is complete.

REMARK 1. In a sequential space X, an almost as-finite collection $\mathfrak{F}$ in X need not be hereditarily closure preserving. This is shown by Example 4.1.

REMARK 2. In Theorem 4.1, we cannot replace the term "almost as-finite" by "as-finite." We can see this from Example 4.2.

EXAMPLE 4.1 Let X be the space defined by S.P. Franklin [4, Example 1.8]. Franklin established that X is a sequential space which is not Fréchet. Put $F_n = \left(\frac{1}{n+1}, \frac{1}{n}\right)$ and $\mathfrak{F} = \{F_n \mid n \in \mathbf{N}\}$. Then, $\mathfrak{F}$ is an almost as-finite collection which is not hereditarily closure preserving.

PROOF. To show that $\mathfrak{F}$ is almost as-finite, assume that $\mathfrak{F}$ is not almost as-finite. Then, there exists a distinct ac-sequence $\{x_n\}$ in X such that $\{n \in \mathbf{N} \mid F_n \cap S_m \neq \phi\}$ is infinite for each $m \in \mathbf{N}$, where $S_m = \{x_i \mid i \geqq m\}$. So, we can extract two increasing sequences $\{n_i\}$, $\{m_i\}$ in $\mathbf{N}$ such that $x_{n_i} \in F_{m_i}$ for each $i \in \mathbf{N}$. Put

$$V(0)=\{0\} \cup [\bigcup_{i=1}^{\infty} (x_{n_{i+1}}, x_{n_i})],$$

then $V(0)$ is an open neighborhood of in X and $x_{n_i} \notin V(0)$ for each $i \in \mathbf{N}$. Therefore, $\{x_{n_i}\}$ does not cluster at any point of X. This contradicts the fact that $\{x_n\}$ is an ac-sequence. Hence, $\mathfrak{F}$ is almost as-finite. On the other hand, $\mathfrak{F}$ is not hereditarily closure preserving, since

$$\cup\{\overline{F_n} \mid n \in \mathbf{N}\}=(0, 1] \neq [0, 1]=\overline{\cup\{F_n \mid n \in \mathbf{N}\}}.$$

EXAMPLE 4.2. Let Y be the disjoint union of a sequence $\{I_n\}$ of copies of the interval I, let $A=\{0_n \in I_n \mid n \in \mathbf{N}\}$, and let $X=Y/A$ be the quotient space obtained from Y by identifying A to a point x_0. Let $f: Y \to X$ be the quotient map, and put $\mathfrak{F}=\{f(I_n) \mid n \in \mathbf{N}\}$. Then, X is a Fréchet Space, and $\mathfrak{F}$ is a hereditarily closure preserving collection which is not as-finite.

PROOF. Since f is a closed map, X is a Fréchet space and $\mathfrak{F}$ is hereditarily closure preserving. Nevertheless, $\mathfrak{F}$ is not point-finite at x_0; therefore, $\mathfrak{F}$ is not as-finite.

THEOREM 4.2. *In sequential space X, a collection $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ of subsets of X is almost as-finite if and only if $\mathfrak{F}$ is almost cs-finite.*

PROOF. The necessity is obvious. To prove the sufficiency, suppose that $\mathfrak{F}$ is not almost as-finite. Then, there exists a distinct ac-sequence $\{x_n\}$ such that $\{\alpha \in A \mid F_\alpha \cap S_n \neq \phi\}$ is infinite for each $n \in \mathbf{N}$. So, we can choose a subsequence $\{x'_n\}$ of $\{x_n\}$ and a distinct sequence $\{\alpha_n\}$ in A such that $x'_n \in F_{\alpha_n}$. Since $\{x_n\}$ is an ac-sequence, $\{x'_n\}$ has a cluster point $y \in X$. Here, we can assume without loss of generality that $y \in \overline{S'}-S'$, where $S'=\{x'_n \mid n \in \mathbf{N}\}$. Since X is T_1 and sequential, there exists a distinct sequence $\{y_n\}$ in S' which converges to a point $y' \notin S'$. We put $y_k=x'_{n_k}$ $(k=1,2\cdots)$. Here, we will construct subesquence $\{x'_{n(l)}\}$ of $\{x'_n\}$ which converges to y'. Put

$$x'_{n(1)}=y_1=x'_{n_1}, \quad U_1=X-\{x'_i \mid i \leqq n_1\}.$$

Since U_1 is an open neighborhood of y', there exists an integer $k_1>1$ such that $y_{k_1}=x'_{n_{k_1}} \in U_1$. So, put

$$x'_{n(2)}=y_{k_1}=x'_{n_{k_1}}, \quad U_2=X-\{x'_i \mid i \leqq n_{k_1}\}.$$

In the same way, we can choose an integer $k_2>k_1$ such that $y_{k_2}=x'_{n_{k_2}} \in U_2$, and so put

$$x'_{n(3)}=y_{k_2}=x'_{n_{k_2}}, \quad U_3=X-\{x'_i \mid i\leqq n_{k_2}\}.$$

By induction, we obtain a sequence $\{x'_{n(l)}\}$ which is a subsequence of both $\{x'_n\}$ and $\{y_n\}$. Hence, $\{x'_{n(l)}\}$ is a distinct subsequence of $\{x'_n\}$ and converges to y'. Now, $\{\alpha\in A \mid F_\alpha\cap(T-T_0)\neq\phi\}$ is infinite for each finite subset of T_0 of $T=\{x'_{n(l)} \mid l\in\mathbf{N}\}$. This contradicts the fact that $\mathfrak{F}$ is almost cs-finite. The proof is complete.

COROLLARY 4.3 *In a sequential space X, a collection $\mathfrak{F}$ of subsets of X is as-finite if and only if it is cs-finite.*

COROLLARY 4.4 *In a sequential, singly bi-quasi-k-space X, a collection $\mathfrak{F}$ of subsets of X is hereditarily closure preserving if and only if it is almost cs-finite.*

PROOF. This follows immediately from Theorems 4.1 and 4.2.

Franklin's example ([5], Example 7.1) shows that a sequential, singly bi-quasi-k-space need not be a Fréchet space. Therefore, the following is at least formally stronger than J.R. Boone's result [1, Lemma 3.4].

COROLLARY 4.5. *Let X be a sequential and singly bi-quasi-k-space. If $\mathfrak{F}$ is a cs-finite collection of subsets of X, then $\mathfrak{F}$ is hereditarily closure preserving.*

THEOREM 4.6 *In a q-space X, a collection $\mathfrak{F}=\{F_\alpha \mid \alpha\in A\}$ of subsets of X is locally finite if and only if it is as-finite.*

PROOF. The necessity is obvious. To prove the sufficiency, assume that $\mathfrak{F}$ is not locally finite. Then, there exists a point x of X such that every neighborhood of x meets infinitely many elements of $\mathfrak{F}$. Since X is a q-space, a decreasing q-sequence $\{U_n\}$ of neighborhoods of x exists. And, since $\mathfrak{F}$ is point-finite and X is a T_1-space, there exist a distinct sequence $\{x_n\}$ of points in X and a distinct sequence $\{\alpha_n\}$ in A such that $x_n\neq x$ and $x_n\in U_n\cap F_{\alpha_n}$ for each $n\in\mathbf{N}$. Furthermore, there exists a finite subset S_0 of $S=\{x_n \mid n\in\mathbf{N}\}$ such that $\{\alpha\in A \mid F_\alpha\cap(S-S_0)\neq\phi\}$ is finite, since $\{x_n\}$ is an ac-sequence and $\mathfrak{F}$ is almost as-finite. This contradicts the choices of sequences $\{x_n\}$ and $\{\alpha_n\}$. Therefore, $\mathfrak{F}$ is locally finite. The proof is complete.

REMARK. In Theorem 4.6, we cannot replace the term "q-space" by "Fréchet space". This is shown by the following example.

EXAMPLE 4.3. Let $\mathbf{R}$ be the real line, let $\mathbf{Z}$ be the integers, and let $X=\mathbf{R}/\mathbf{Z}$ be the quotient space obtained from $\mathbf{R}$ by identifying $\mathbf{Z}$ to a point x_0. Let $f:\mathbf{R}\rightarrow X$ be the quotient map, and put $\mathfrak{F}=\{F_n=f[(n, n+1)] \mid n\in\mathbf{Z}\}$. Then, X is a Fréchet space, and $\mathfrak{F}$ is an as-finite collection which is not locally finite.

PROOF. Since f is a pseudo-open map and $\mathbf{R}$ is a Fréchet space, X is a Fréchet space. Let $\{x_n\}$ be an ac-sequence in X, then $S=\{x_n \mid n\in\mathbf{N}\}$ meets at most finitely many elements of $\mathfrak{F}$. In fact, assume that $\{m \in \mathbf{N} \mid S \cap F_m \neq \phi\}$ is infinite. Then, we can extract a distinct subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a distinct sequence $\{m_k\}$ in $\mathbf{N}$ such that $x_{n_k} \in F_{m_k}$ for each $k \in \mathbf{N}$. Yet, the sequence $\{x_{n_k}\}$ does not cluster in the space X, because $x_{n_k} \in f[(m_k, m_k+1)]$ for each $k \in \mathbf{N}$. This contradicts the fact that $\{x_n\}$ is an ac-sequence. On the other hand, $\mathfrak{F}$ is not locally finite at x_0.

Franklin's example [5, Example 7.1] also shows that a sequential, q-space need not be a first countable space. Therefore, the following is at least formally stronger than Boone's result [1, Lemma 3.9].

Corollary 4.7. *In a sequential, q-space, $\mathfrak{F}$ a collection of subsets of X is locally finite if and only if it is cs-finite.*

Proof. This follows immediately from Theorem 4.6 and Corollary 4.3.

Theorem 4.8. *Let X be a sequential space and let $\mathfrak{F} = \{F_\alpha \mid \alpha \in A\}$ be a collection of closed subsets of X. Then, $\mathfrak{F}$ is hereditarily closure preserving if and only if it is almost cs-finite.*

Proof. The necessity is obvious. To prove the sufficiency, suppose that $\mathfrak{F}$ is not hereditarily closure preserving. Then, there exists a collection $\{H_\alpha \mid \alpha \in A\}$ such that

$$H_\alpha \subset F_\alpha \qquad \text{for each } \alpha \in A,$$

$$\overline{\cup\{H_\alpha \mid \alpha \in A\}} \neq \cup\{\overline{H_\alpha} \mid \alpha \in A\}.$$

Therefore, $\cup\{\overline{H_\alpha} \mid \alpha \in A\}$ is not closed in X. Since X is sequential, there exists a sequence $\{x_n\}$ in $\cup\{\overline{H_\alpha} \mid \alpha \in A\}$ which converges to a point x not in $\cup\{\overline{H_\alpha} \mid \alpha \in A\}$. Here, $\{\overline{H_\alpha} \mid \alpha \in A\}$ is almost cs-finite, because $\mathfrak{F}$ is almost cs-finite and $\overline{H_\alpha} \subset F_\alpha$ for each $\alpha \in A$. Then, there exist an integer m and a finite subset $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of A such that

$$\{x_n \mid n \geqq m\} \subset \bigcup_{i=1}^{k} H_{\alpha_i}.$$

Consequently, $x \in \overline{\{x_n \mid n \geqq m\}} \subset \bigcup_{i=1}^{k} \overline{H_{\alpha_i}} \subset \cup\{\overline{H_\alpha} \mid \alpha \in A\}$. This contradicts the fact that $x \notin \cup\{\overline{H_\alpha} \mid \alpha \in A\}$. Therefore, $\mathfrak{F}$ is hereditarily closure preserving. The proof is complete.

Corollary 4.9 (J. R. Boone [1]). *Let X be a sequetnial space and let $\mathfrak{F}$ be a collection of closed subsets of X. Then, $\mathfrak{F}$ is locally finite if and only if it is cs-finite.*

Theorem 4.10. *Let X be a quasi-k-space and let $\mathfrak{F} = \{F_\alpha \mid \alpha \in A\}$ be a collection of closed subsets of X. Then, $\mathfrak{F}$ is hereditarily closure preserving if and only if it is* *almost as-finite.*

PROOF. The necessity is clear. To prove the sufficiency, assume that $\mathfrak{F}$ is not hereditarily closure preserving. Then, there exists a collection $\{H_\alpha \mid \alpha \in A\}$ such that

$$H_\alpha \subset F_\alpha \qquad \text{for each } \alpha \in A,$$

$$\overline{\cup\{H_\alpha \mid \alpha \in A\}} - \cup\{\overline{H}_\alpha \mid \alpha \in A\} \neq \phi.$$

Therefore, $\cup\{\overline{H}_\alpha \mid \alpha \in A\}$ is not closed. Since X is a quasi-k-space, there exist a countably compact set K and a point p such that

$$p \in \overline{\cup\{\overline{H}_\alpha \mid \alpha \in A\} \cap K} - \cup\{\overline{H}_\alpha \mid \alpha \in A\}. \tag{3}$$

Here, $\{\overline{H}_\alpha \mid \alpha \in A\}$ is almost as-finite, because $\mathfrak{F}$ is almost as-finite and $\overline{H}_\alpha \subset F_\alpha$ for each $\alpha \in A$. By Corollary 3.3, there exists a finite subset K_0 of K such that $\{\alpha \in A \mid \overline{H}_\alpha \cap (K - K_0) \neq \phi\}$ is finite; then we denote $\{\alpha \in A \mid \overline{H}_\alpha \cap (K - K_0) \neq \phi\} = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$. From (3), without loss of generality, we can assume that p does not belong to K_0. Since X is T_1 and K_0 is a finite set, we can choose an open neighborhood $V(p)$ of p with the property $V(p) \cap K_0 = \phi$. Hence, from (3)

$$\begin{aligned} p \in\ & \overline{V(p) \cap [\cup\{\overline{H}_\alpha \mid \alpha \in A\}] \cap K} \\ \subset\ & \overline{V(p) \cap [\cup\{H_{\alpha_i} \mid i = 1, 2, \cdots, k\}]} \subset \bigcup_{i=1}^{k} \overline{H}_{\alpha_i} \\ \subset\ & \cup\{\overline{H}_\alpha \mid \alpha \in A\}. \end{aligned}$$

This contradicts the fact that $p \notin \cup\{\overline{H}_\alpha \mid \alpha \in A\}$. Consequently, $\mathfrak{F}$ is hereditarily closure preserving. The proof is complete.

REMARK. If we drop the condition that $\mathfrak{F}$ is a collection of closed subsets of X, Theorem 4.10 does not hold. We can see this from Example 4.1.

COROLLARY 4.11. *Let X be a quasi-k-space and let $\mathfrak{F} = \{F_\alpha \mid \alpha \in A\}$ is a collection of closed subsets of X. Then, $\mathfrak{F}$ is locally finite if and only if it is as-finite.*

5. Mappings and as-finite collections

THEOREM 5.1 *Let $f: X \to Y$ be a continuous, closed map. If $\mathfrak{F} = \{F_\alpha \mid \alpha \in A\}$ is an almost as-finite collection of subsets of X, then $f(\mathfrak{F}) = \{f(F_\alpha) \mid \alpha \in A\}$ is almost as-finite in Y.*

PROOF. On the contrary, suppose that $f(\mathfrak{F})$ is not almost as-finite. Then, there exists a distinct ac-sequence $\{y_n\}$ in Y such that $\{\alpha\in A\,|\,f(F_\alpha)\cap S_n\neq\phi\}$ is infinite for each $n\in\mathbf{N}$, where $S_n=\{y_i\,|\,i\geqq n\}$. Hence, we can extract a distinct subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and a distinct sequence $\{\alpha_k\}$ in A such that $y_{n_k}\in f(F_{\alpha_k})$ for each $k\in\mathbf{N}$. Now, choose a point $x_k\in f^{-1}(y_{n_k})\cap F_{\alpha_k}$ for each $k\in\mathbf{N}$. Then, $\{x_k\}$ is not an ac-sequence, because $\{x_k\}$ is distinct and $\mathfrak{F}$ is almost as-finite. Therefore, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which does not cluster in X. Since X is a T_1-space, $\{x_{k_i}\,|\,i\in\mathbf{N}\}$ is a discrete set. By the closedness of f, $\{y_{n_{k_i}}\,|\,i\in\mathbf{N}\}$ is a discrete subset of Y. This contradicts the fact that $\{y_n\}$ is an ac-sequence. Consequently, $f(\mathfrak{F})$ is almost as-finite. This completes the proof.

Remark. In Theorem 5.1, we cannot replace the term "almost as-finite" by "as-finite." We can see this from Example 4.2.

Corollary 5.2. *Let $f: X\to Y$ be a quasi-perfect* map. If $\mathfrak{F}=\{F_\alpha\,|\,\alpha\in A\}$ is an as-finite collection of subsets of X, then $f(\mathfrak{F})$ is as-finite in Y.*

Proof. This follows immediately from Theorem 5.1 and Corollary 3.2.

Theorem 5.3. *Let $f: X\to Y$ be a continuous map. If $\mathfrak{F}=\{F_\alpha\,|\,\alpha\in A\}$ is an as-finite (resp. a cs-finite) collection of subsets of Y, then $f^{-1}(\mathfrak{F})=\{f^{-1}(F_\alpha)\,|\,\alpha\in A\}$ is as-finite (resp. cs-finite) in X.*

Proof. We prove this theorem only for the "as-finite" case; the "cs-finite" case follows similarly. It is clear that $f^{-1}(\mathfrak{F})$ is point-finite. Assume that $f^{-1}(\mathfrak{F})$ is not almost as-finite, then there exists a distinct ac-sequence $\{x_n\}$ in X such that $\{\alpha\in A\,|\,f^{-1}(F_\alpha)\cap S_n\neq\phi\}$ is infinite for each $n\in\mathbf{N}$, where $S_n=\{x_m\,|\,m\geqq n\}$. So, we can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a distinct sequence $\{\alpha_k\}$ in A such that $x_{n_k}\in f^{-1}(F_{\alpha_k})$ for each $k\in\mathbf{N}$. Put $y_k=f(x_{n_k})$ for each $k\in\mathbf{N}$, then $\{y_k\}$ is an ac-sequence in Y and $y_k\in F_{\alpha_k}$ for each $k\in\mathbf{N}$. Since $\mathfrak{F}$ is point-finite, $\{y_k\,|\,k\in\mathbf{N}\}$ is infinite; this contradicts the fact that $\mathfrak{F}$ is as-finite. Consequently, $f^{-1}(\mathfrak{F})$ is as-finite. The proof is complete.

Remark. In Theorem 5.3, we cannot replace the term "as-finite" (resp. "cs-finite") by "almost as-finite" (resp. "almost cs-finite"). We can see this from the following example.

Example 5.1. Let X be the real line, let $A=\{\frac{1}{n}\,|\,n\in\mathbf{N}\}$, and let $Y=X/A$ be the quotient space obtained from X by identifying A to a point y_0. Let $f: X\to Y$ be the quotient map and put $\mathfrak{F}=\{\{y_0,n\}\,|\,n\in\mathbf{N}\}$. Then, $\mathfrak{F}$ is almost as-finite (resp. almost cs-finite) in Y but $f^{-1}(\mathfrak{F})$ is not almost as-finite (resp. not almost cs-finite) in X.

* A continuous map $f: X\to Y$ is said to be *quasi-perfect* iff it is a closed map and $f^{-1}(y)$ is countably compact for each $y\in Y$.

THEOREM 5.4. *Let $f : X \to Y$ be a finite to one, continuous map. If $\mathfrak{F}=\{F_\alpha \mid \alpha \in A\}$ is an almost as-finite (resp. almost cs-finite) collection of subsets of Y, then $f^{-1}(\mathfrak{F})$ is almost as-finite (resp. almost cs-finite) in X.*

PROOF. Let $\{x_n\}$ be a distinct ac-sequence in X, and put $y_n=f(x_n)$ for each $n \in \mathbf{N}$. Then, $\{y_n\}$ is an ac-sequence in Y. Since $\mathfrak{F}$ is almost as-finite, there exists a finite subset S_0 of $S=\{y_n \mid n \in \mathbf{N}\}$ such that $\{\alpha \in A \mid F_\alpha \cap (S-S_0) \neq \phi\}$ is finite. Therefore,

$$\{\alpha \in A \mid f^{-1}(F_\alpha) \cap [\{x_n \mid n \in \mathbf{N}\} - f^{-1}(S_0)] \neq \phi\}$$

is finite. And since $f^{-1}(S_0)$ is a finite set, $f^{-1}(\mathfrak{F})$ is almost as-finite in X. The "almost cs-finite" case follows similarly. The proof is complete.

6. Applications

In this section, as applications of the as-finiteness, we mention some characterizations of paracompact spaces and give a metrization theorem of semi-stratifiable spaces.

In [2], J. R. Boone defines property (ω) as follows: A space X is said to *have property* (ω) if for each discrete collection $\{F_\alpha \mid \alpha \in A\}$ of closed subsets of X, there exists a cs-finite collection $\{G_\alpha \mid \alpha \in A\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$, for each $\alpha \in A$ and $G_\alpha \cap F_\beta = \phi$, if $\alpha \neq \beta$. Now, modifying this definition, we introduce the notion of spaces with property (ω^*).

DEFINITION 6.1. A space X said to *have property* (ω^*) if and only if for each discrete collection of closed sets $\{F_\alpha \mid \alpha \in A\}$ in X, there exists an almost as-finite collection of open sets $\{G_\alpha \mid \alpha \in A\}$ such that $F_\alpha \subset G_\alpha$, for each $\alpha \in A$ and $G_\alpha \cap F_\beta = \phi$, if $\alpha \neq \beta$.

The following implications are derived immediately from the above definitions.

with property (ω)

collectionwise normal ↗ ↘

with property (ω^*)

And, by Theorem 4.2, a sequential space with property (ω) has property (ω^*). Therefore, the following proposition is at least formally stronger than Boone's result [2, Corollary 3.2].

PROPOSITION 6.1. *A quasi-k-space X is collectionwise normal if and only if it is a normal space with property (ω^*).*

PROOF. Cf. Theorem 4.2 and the proof of [2, Corollary 3.2].

Proposition 6.2. *The following properties of a regular, singly bi-quasi-k-space X are equivalent.*

(a) *X is paracompact.*

(b) *Every open covering of X has a σ-almost as-finite open refinement.*

(c) *X is a subparacompact space with property* (ω^*).

Proof. It follows immediately from Michael's theorem [7, Theorem 2] and Theorem 4.1 that (a) and (b) are equivalent. (a)$\Rightarrow$(c): This is obvious. (c)$\Rightarrow$(b): Let $\mathfrak{G}$ be an open covering of X. Since X is subparacompact, there exists a σ-discrete closed refinement $\bigcup_{n=1}^{\infty} \mathfrak{F}_n$ of $\mathfrak{G}$, where $\mathfrak{F}_n=\{F_\alpha \mid \alpha \in A_n\}$ is a discrete collection of closed sets in X. Since X has property (ω^*), there exists an almost as-finite collection $\mathfrak{H}_n =\{H_\alpha \mid \alpha \in A_n\}$ of open sets such that $F_\alpha \subset H_\alpha$ and $\mathfrak{H}_n < \mathfrak{G}$. Therefore, $\bigcup_{n=1}^{\infty} \mathfrak{H}_n$ is a σ-almost as-finite open refinement of $\mathfrak{G}$. The proof is complete.

Proposition 6.3. *A regular, quasi-k-space X is paracompact if and only if every open covering of X has an almost as-finite closed refinement.*

Proof. This follows immediately from Michael's theorem [7, Theorem 1] and Theorem 4.10.

The following is a generalization of [12, Theorem 9] as well as [6, Theorem 3.1].

Proposition 6.4. *A semi-stratifiable space X is metrizable if and only if X is a regular $w\Delta$-space with property* (ω^*).

Proof. The necessity is obvious. To prove the sufficiency: Since X is a regular, semi-stratifiable space, X is a subparacompact space with G_δ-diagonal. Then, by Proposition 6.2, X is paracompact, since a $w\Delta$-space is a singly bi-quasi-k-space. Therefore, X is a paracompact, T_2, M-space with G_δ-diagonal. By Okuyama's theorem [10, Theorem 1], X is metrizable. The proof is complete.

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