On the non-differentiability of special exponential sums with Möbius weight; $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$

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Abstract

Fröberg [2] said that he believes $f(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$ being non-differentiable everywhere by computer computation, where $i=\sqrt{-1}$ and $\mu(n)$ is the Möbius function.

In this paper, we show in Theorem 1.1 that for any interval in $[0,2\pi]$ there exists a positive Lebesgue measurable (L^1-measurable) set such that $f'(x)$ is not L^2-measurable on its interval.

1 Theorems

Let $\mu(n)$ be the Möbius function. We define $f(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$. Then $f(x)$ has a period 2π. Bateman and Chowla [1] show that $f(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$ is continuous. By numerical computations, Fröberg [2, p.210] said that it is perfect clear that the function is not differentiable. But he does not give the proof.

Lemma 1.1 ([3, Theorem 68]). *If both $g(x)$ and $g'(x)$ belong to $L^2(-\infty,\infty)$, then both $G(x)$ and $xG(x)$ belong to L^2 ; and vice versa, where*

$$G(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{ixt}dt.$$

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Lemma 1.2 ([1, Lemma 2])**.** *The series* $\displaystyle\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$ *converges uniformly in* $x\in\mathbf{R}$.

Theorem 1.1. *Let* $f(x)=\displaystyle\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$. *Then for any* $[\alpha,\beta]\subset[0,2\pi]$, $f'(x)$ *does not belong to* $L^2[\alpha,\beta]$ *,i.e.,* $f'(x)\notin L^2[\alpha,\ \beta]$.

Proof. It was proved by Bateman and Chowla [BC] that $f(x)$ converges uniformly in x for real x. This implies that $f(x)$ is continuous. We define

$$g(x)=\begin{cases} f(x) & \text{if} \quad \alpha\le x\le\beta, \\ \\ 0, & \text{otherwise.}\end{cases}$$

By $f(x)$ being continuous, $g(x)$ is $L^2(-\infty,\infty)$-measurable. Since $f(x)$ converges uniformly in x, the Fourier transform of $g(x)$ is

$$G(y)=\hat{g}(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{ixy}dx=\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\int_{\alpha}^{\beta}e^{inx}e^{ixy}dx.$$

Thus

$$G(y)=\begin{cases}\displaystyle\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{1}{i(n+y)}(e^{i(n+y)\beta}-e^{i(n+y)\alpha}) & \text{if } y\neq -n, \\ \displaystyle\qquad\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}(\beta-\alpha)=0 & \text{if } y=-n.\end{cases}$$

Suppose $g'(x)\in L^2[\alpha,\beta]$ on some $[\alpha,\beta]$, that is, $g'(x)\in L^2(-\infty,\infty)$. Both g and g' belong to $L^2(-\infty,\ \infty)$, then $\hat{g'}(y)=(-i)yG(y)\in L^2(-\infty,\infty)$ by Lemma 1.1. Since, for $y\neq -n$,

$$\hat{g'}=(-i)yG(y)=\frac{-i}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{y}{i(n+y)}\{e^{i(n+y)\beta}-e^{i(n+y)\alpha}\},$$

we have

$$\begin{aligned}\infty \quad > \quad & \int_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{y}{(n+y)}\left\{e^{i(n+y)\beta}-e^{i(n+y)\alpha}\right\}\right|^2dy \\ = \quad & \int_{-\infty}^{\infty}\left|\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{t-n}{t}(e^{it\beta}-e^{it\alpha})\right|^2dt=\int_{-\infty}^{\infty}\frac{|e^{it\beta}-e^{it\alpha}|^2}{t^2}\left|\sum_{n=1}^{\infty}\frac{t-n}{n}\mu(n)\right|^2dt \\ = \quad & \int_{-\infty}^{\infty}\frac{4\sin^2\frac{(\beta-\alpha)}{2}t}{t^2}\left|\sum_{n=1}^{\infty}\mu(n)\right|^2dt=\left|\sum_{n=1}^{\infty}\mu(n)\right|^2\int_{-\infty}^{\infty}\frac{4\sin^2\frac{(\beta-\alpha)}{2}t}{t^2}dt,\end{aligned}$$

by $\displaystyle\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$.

Thus $\left|\sum_{n=1}^{\infty}\mu(n)\right|^2 < \infty$, which contradicts to [4, Theorem 14.26(B)], that is,

$$\limsup_{x\to\infty}\frac{1}{\sqrt{x}}\left|\sum_{n\le x}\mu(n)\right| > 0.$$

Thus we obtain $g'(x)\notin L^2[\alpha,\beta]$, *i.e.,* $f'(x)\notin L^2[\alpha,\beta]$, which completes the proof.

Theorem 1.2. *Let* $f(x)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}$. *If* $f'(x)$ *exists for some* x, *then* $f'(x)=0$.

Moreover, if $f'_+(x)$ *or* $f'_-(x)$ *exists for some* x, *then* $f'_+(x)=0$ *or* $f'_-(x)=0$, *respectively.*

Proof. Since

$$\sum_{n=1}^{\infty}\left|\frac{\mu(n)}{n^2}e^{inx}(1-e^{inh})\right| \le \sum_{n=1}^{\infty}\frac{2}{n^2} < \infty,$$

the function

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{inx}(1-e^{inh})$$

absolutely converges.

For $t>0$, we set the function

$$g(x,t)=\frac{-2}{t}f(x)+\frac{2i}{t^2}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{inx}(1-e^{int}). \tag{1}$$

By Lemma 1.2, the function $f(x)$ converges uniformly in x.

Thus

$$f(x+h)-f(x) \quad = \quad \sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}(e^{inh}-1)$$

converges uniformly in h. Therefore

$$\begin{aligned}\int_0^t\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}\left(e^{inh}-1\right)dh &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\int_0^t e^{inx}\left(e^{inh}-1\right)dh\\
&= -t\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}+i\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{inx}(1-e^{int}).\\
&= -tf(x)+i\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{inx}(1-e^{int}). \qquad (2)\end{aligned}$$

Thus

$$
\begin{aligned}
g(x,t) &= \frac{1}{\frac{1}{2}t^2}\left\{-tf(x)+i\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{inx}\left(1-e^{int}\right)\right\}\\
&= \frac{\displaystyle\int_0^t\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}\left(e^{inh}-1\right)dh}{\displaystyle\int_0^t h\,dh}.
\end{aligned}
$$

Applying Cauchy's mean value theorem, we have, for some h with $0<h<t$,

$$
g(x,t) = \frac{\displaystyle\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{inx}\left(e^{inh}-1\right)}{h} = \frac{1}{h}(f(x+h)-f(x)). \tag{3}
$$

Since $\dfrac{1-e^{int}}{t}=-ine^{int}+o(1)$ as $t\to 0$, where o is the Landau's small o,

we have for fixed N,

$$
\lim_{t\to 0}\sum_{n=1}^{N}\frac{\mu(n)}{n^2}e^{inx}\frac{1-e^{int}}{t}=\lim_{t\to 0}\left(-i\sum_{n=1}^{N}\frac{\mu(n)}{n}e^{in(x+t)}\right), \tag{4}
$$

and for fixed $t\neq 0$,

$$
\begin{aligned}
&\left|\sum_{n=N}^{\infty}\frac{\mu(n)}{n^2}e^{inx}\frac{1-e^{int}}{t}-\left(-i\sum_{n=N}^{\infty}\frac{\mu(n)}{n}e^{in(x+t)}\right)\right|\\
\leq\ &\left|\sum_{n=N}^{\infty}\frac{\mu(n)}{n}e^{in(x+t)}\right|+\frac{2}{|t|}\sum_{n=N}^{\infty}\frac{1}{n^2}<\epsilon,
\end{aligned}
$$

for any positive $\epsilon>0$ as $N\to\infty$, because $f(x+t)=\displaystyle\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{in(x+t)}$ exists.

Therefore from (4), we have

$$
\lim_{t\to 0}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{inx}\frac{1-e^{int}}{t}=-if(x+t).
$$

Thus from (1), we have

$$
\lim_{t\to 0}g(x,t)=\lim_{t\to 0}\frac{2(f(x+t)-f(x))}{t} \tag{5}
$$

if $f'(x)$ exists for some x. Therefore, by (3) and (5), we have $2f'(x)=f'(x)$, that is, $f'(x)=0$.

If $f'_+(x)$ or $f'_-(x)$ exists for some x, then we replace $t\to 0$ with $t\to +0$ or $t\to -0$ in the above proof . Thus we have $f'_+(x)=0$ or $f'_-(x)=0$, respectively.

By Theorem 1.1 and Theorem 1.2, we have

Corollary 1.1. $f(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} e^{inx}$ *is non-differential except for* $\{x|\ f_{\pm}(x) = 0\}$.

Theorem 1.3. *We have for* $T = \{x|\ f'_{+}(x)$ *or* $f'_{-}(x)$ *does not exist* $\}$

$$\left| \sum_{n=1}^{N} \mu(n) e^{inx} \right| \to \infty \quad \textit{as} \quad N \to \infty \textit{ in } x \in T.$$

Proof. By the fact $\lim_{h \to 0} \frac{1}{h} \int_0^h e^{int} dt = 1$, we have

$$\lim_{h \to 0} \frac{1}{h}(f(x+h) - f(x)) = \lim_{h \to 0} \frac{1}{h} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (e^{in(x+h)} - e^{inx})$$

$$= \lim_{h \to 0} \sum_{n=1}^{\infty} i\mu(n) e^{inx} \frac{1}{h} \int_0^h e^{int} dt = i \sum_{n=1}^{\infty} \mu(n) e^{inx} (1 + o(1)) \quad \text{as} \quad h \to 0.$$

In the above equations, if we replace $h \to 0$ with $h \to +0$ or $h \to -0$, we have the same equations. By Theorem 1.1 and 1.2, if $f'_{\pm}(x)$ is exists, then $f'_{\pm}(x) = 0$.

Thus we have

$$\left| \sum_{n=1}^{N} \mu(n) e^{inx} \right| \to \infty \quad \text{as} \quad N \to \infty \text{ in } x \in T,$$

which completes the proof.

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