On the Hausdorff-Young inequality for Riemannian symmetric spaces

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1 Introduction

Hausdorff-Young theorem is an extension of the Plancherel formula to L^p space. This theorem has been extended to various cases. For example, it holds for separable locally compact type I group([L3] Theorem 2.2) on unitary dual. On the other hand, the Fourier transform of L^p function on a semisimple Lie group has a holomorphic extension to some domain T_p which depends on p and a Hausdorff-Young theorem holds on T_p. R.A.Kunze and E.M.Stein [K-S] obtained the first results for $SL(2, \boldsymbol{R})$ and R.L.Lipsman proved it for $SL(\mathrm{n}, \boldsymbol{C})$ ([L1],[L2]). In [E-K] the theorem was proved for the spherical Fourier transform on Riemannian symmetric spaces. In [E-K-T] it was proved for the Fourier transforms of K-finite functions on Riemannian symmetric spaces.

The purpose of the present paper is to prove an analogue of the Housdorff-Young inequality on G/K: If $1 \leq p \leq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left(\int_{\mathfrak{a}^*}\left(\int_{K/M}|\tilde{f}(\lambda, kM)|^2 dk_M\right)^{q/2}|c(\lambda)|^{-2} d\lambda\right)^{1/q} \leq C_p\|f\|_p$$

(Theorem 1). This inequality is the same as Theorem 1 of [E-K-T], however, in which it was assumed that functions f are K-finite. And if $1<p<\frac{4}{3}$ and $\xi+\sqrt{-1}\eta \in T_p$, then

$$\left\{\int_{\mathfrak{a}^*}\left|\int_{K/M}\overline{\tilde{f}(\xi-\sqrt{-1}\eta, kM)}\,\tilde{f}(\xi+\sqrt{-1}\eta, kM)\, dk_M\right|^{q/2}|c(\xi)|^{-2} d\xi\right\}^{1/q} \leq C_{p,\eta}\|f\|_p$$

(Theorem 2).

2 Notations and Preliminaries

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ an Iwasawa decomposition of $\mathfrak{g}$ such that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. We denote by $G=KAN$ the corresponding decomposition of G. For $x\in G$, $\kappa(x)\in K$ and $H(x)\in\mathfrak{a}$ denote the elements uniquely determined by $x\in\kappa(x)\exp(H(x))N$. For $a\in A$, we write $\log a$ for $H(a)$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and $\mathfrak{a}_C^*$ its complexification. Let M' and M be the normalizer and the centralizer of $\mathfrak{a}$ in K, respectively, and we denote by $W=M'/M$ be the Weyl group of G/K and $[W]$ its order. Let $\bar{\mathfrak{n}}=\theta(\mathfrak{n})$ and $\bar{N}$ denote the corresponding analytic subgroup of G. We fix an ordering on $\mathfrak{a}^*$ which is compatible with the above Iwasawa decomposition. Let ρ be the half of the sum of all positive roots of $(\mathfrak{g},\mathfrak{a})$. Let $\mathfrak{a}_+^*$ be the positive Weyl chamber of $\mathfrak{a}^*$ and put

$$\mathfrak{a}_+=\{H\in\mathfrak{a};\ \alpha(H)>0 \quad \text{for all} \quad \alpha\in\mathfrak{a}_+^*\}.$$

For any $\varepsilon>0$ we put

$$C_{\varepsilon\rho}=\{\lambda\in\mathfrak{a}^*;\ |(s\lambda)(H)|\le\varepsilon\rho(H) \ \text{ for all } H\in\mathfrak{a}_+ \text{ and } s\in W\}.$$

We define the tube domain T_p by $T_p=\mathfrak{a}^*+\sqrt{-1}C_{(2/p-1)\rho}$.

We denote by $C_c^\infty(G)$ the space of all compactly supported C^∞-functions on G and $C_c^\infty(G/K)$ and $C_c^\infty(K/G/K)$ be the subspaces of $C_c^\infty(G)$ of right K-invariant and K-biinvariant functions, respectively. The Killing form of $\mathfrak{g}$ induces euclidean measures on A and $\mathfrak{a}^*$. We normalize them by multiplying the factor $(2\pi)^{-\dim\mathfrak{a}^*/2}$ and denote them by da and $d\nu$, respectively. Let dk be the normalized Haar measure on K so that the total measure is one. The Haar measures on N and $\bar{N}$ are normalized so that

$$\theta(dn)=d\bar{n}, \qquad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))}d\bar{n}=1.$$

And we normalize the Haar measure dx on G so that

$$\int_G f(x)\,dx=\int_{KAN} f(kan)\,e^{2\rho(\log a)}\,dkdadn, \qquad f\in C_c^\infty(G).$$

Let dk_M be the K invariant measure on K/M of the total measure one.

Let

$$\varphi_\lambda(x)=\int_K e^{(\sqrt{-1}\lambda-\rho)(H(xk))}dk, \qquad x\in G,$$

be the elementary spherical function. We denote the Harish-Chandra c-function by

$$c(\lambda)=\int_{\bar{N}} e^{(-\sqrt{-1}\lambda+\rho)(H(\bar{n}))}d\bar{n}.$$

3 Fourier transforms

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the Banach space of all bounded linear operators on $\mathcal{H}$. The operator norm of $\mathcal{B}(\mathcal{H})$ is denoted by $\|B\|_\infty$ and the p-norm $\|B\|_p$ of B is defined by $\|B\|_p=(\mathrm{tr}((B^*B)^{p/2}))^{1/p}$ for $1\leq p<\infty$, where tr is the trace.

Let $L^2(K/M)$ be the subspace of right M-invariant functions in $L^2(K)$ and denote by $(\cdot,\cdot)$ the inner product in $L^2(K/M)$. Let $L^p(\mathfrak{a}^*, \mathcal{B}(L^2(G/K)))^W$ be the Banach space of all $\mathcal{B}(L^2(G/K))$-valued W-invariant functions F on $\mathfrak{a}^*$ such that

(1) if $1\leq p<\infty$,

$$\|F\|_p=\left(\int_{\mathfrak{a}^*}\|F(\lambda)\|_p^p|c(\lambda)|^{-2}d\lambda\right)^{1/p}<\infty,$$

(2) if $p=\infty$,

$$\|F\|_\infty=\mathrm{ess\ sup}_{\lambda\in\mathfrak{a}^*}\|F(\lambda)\|_\infty<\infty,$$

where we identify two functions if they differ only on null sets.

For each $\lambda\in\mathfrak{a}_C^*$ the induced representation π_λ of G on $L^2(K/M)$ is defined by

$$(\pi_\lambda(x)\Phi)(k)=e^{(\sqrt{-1}\lambda-\rho)(H(x^{-1}k))}\Phi(\kappa(x^{-1}k)),\ (\Phi\in L^2(K/M),\ x\in G,\ k\in K).$$

Then if $\lambda\in\mathfrak{a}^*$, then π_λ is a unitary representation. For $f\in C_c^\infty(G/K)$ and $\lambda\in\mathfrak{a}_C^*$ we put

$$\pi_\lambda(f)=\int_G f(x)\pi_\lambda(x)dx.$$

Then

$$(\pi_\lambda(f)\Phi)(k)=\int_K K_f(\lambda; k, k_1)\Phi(k_1)dk_1,$$

where

$$K_f(\lambda; k, k_1)=\int_{AN} f(kank_1)e^{(-\sqrt{-1}\lambda+\rho)(\log a)}dadn$$
$$=\int_{AN} f(kan_1)e^{(-\sqrt{-1}\lambda+\rho)(\log a)}dadn$$

We define the Fourier transform $\tilde{f}$ of $f\in C_c^\infty(G/K)$ by

$$\tilde{f}(\lambda, kM)=\int_{AN} f(kan)e^{(-\sqrt{-1}\lambda+\rho)(\log a)}dadn \quad (\lambda\in\mathfrak{a}^*, k\in K).$$

Then it can be extended to a holomorphic function on $\mathfrak{a}_C^*$. Let Φ_0 be the identity function on K. Then we have

$$(\pi_\lambda(f)\Phi)(k)=\tilde{f}(\lambda, kM)\int_K\Phi(k_1)dk_1=\tilde{f}(\lambda, kM)(\Phi, \Phi_0).$$

On the other hand,

$$(\pi_\lambda(f)^*\pi_\lambda(f)\Phi)(k)=\int_K\left(\int_K\overline{K_f(\lambda; k_1, k)}K_f(\lambda; k_1, k_2)dk_1\right)\Phi(k_2)dk_2$$
$$=\int_{K/M}|\tilde{f}(\lambda, k_1M)|^2dk_{1M}\int_K\Phi(x_2)dk_2$$
$$=\int_{K/M}|\tilde{f}(\lambda, k_1M)|^2dk_{1M}(\Phi, \Phi_0).$$

Hence

$$\|\pi_\lambda(f)\|_p=\left(\int_K|\tilde{f}(\lambda, kM)|^2dk_M\right)^{1/2}, \qquad (1\le p\le\infty).$$

We put $\mathcal{F}(f)(\lambda)=\pi_\lambda(f)$. If $\lambda\in\mathfrak{a}^*$, then $\|\pi_\lambda(x)\|_\infty=1$ and

$$\|\mathcal{F}(f)\|_\infty \leq \|f\|_1. \tag{3.1}$$

Therefore, the Fourier transform can be extended to $L^1(G/K)$ satisfying (3.1).

The following is the Plancherel formula([H] Theorem 2.6): If $f \in C_c^\infty(G/K)$, then we have

$$\begin{aligned}\int_G |f(x)|^2 dx &= [W]^{-1} \int_{\mathfrak{a}^* \times K/M} |\tilde{f}(\lambda, kM)|^2 |c(\lambda)|^{-2} d\lambda\, dk_M \\ &= [W]^{-1} \int_{\mathfrak{a}^*} \|\pi_\lambda(f)\|_2^2 |c(\lambda)|^{-2} d\lambda.\end{aligned}$$

Hence

$$[W]^{-1/2} \|\mathcal{F}(f)\|_2 = \|f\|_2 \tag{3.2}$$

and it is known that the mapping $\mathcal{F}: f \to \mathcal{F}(f)$ can be extended to an isometry of $L^2(G/K)$ onto $L^2(\mathfrak{a}^*, \mathcal{B}(L^2(G/K)))^W$.

4 Hausdorff-Young inequality in the real case

To prove the Hausdorff-Young inequality on $\mathfrak{a}^*$, we use the Riesz-Thorin interpolation theorem for operator valued functions which can easily be led from the Kunze-Stein interpolation theorem in the general case([K-S] Theorem 3). Let (X, μ) and (X', μ') be two σ-finite measure spaces. A complex valued function on X is called simple if it can be expressed as a finite linear combination of characteristic functions of measurable sets of finite measure. Let $\mathcal{B}(\mathcal{H})$ be, as above, the Banach space of all bounded operators on a Hilbert space $\mathcal{H}$.

LEMMA 1. *Let T be a linear map of the space of all compactly supported simple functions on X to the space of all measurable $\mathcal{B}(\mathcal{H})$-valued functions on X'. Let p_i, q_i $(i=1,2)$ be $1 \leq p_i, q_i \leq \infty$ and $q_0 \neq \infty$ or $q_1 \neq \infty$. Suppose that there exists $k_i > 0$ such that*

$$\|T(f)\|_{q_i} \leq k_i \|f\|_{p_i}$$

for any compactly supported simple functions on X. For each $0<t<1$ we define p_t and q_t *by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Then there exits a constant k_t *such that* $k_t \leq k_0^{1-t}k_1^t$ *and*

$$\| T(f) \|_{q_t} \leq k_t \| f \|_{p_t},$$

for any compactly supported simple function f on X. And if $p_t < \infty$, *then T can be extended to an operator of* $L^{p_t}(X)$.

If we choose that $X = G/K$, $X' = \mathfrak{a}^*$, $T = \mathcal{F}$, $p_0 = 1$, $q_0 = \infty$ *and* $p_1 = q_1 = 2$, then we have the following theorem since the inequalities (3.1) and (3.2) hold for compactly supported simple function f on G/K.

THEOREM 1. *If* $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then there exists a positive constant* C_p, $0 < C_p < [W]^{1/q}$ *such that*

$$\|\mathcal{F}(f)\|_q = \left\{ \int_{\mathfrak{a}^*} \left(\int_{K/M} | \tilde{f}(\lambda, kM) |^2 dk_M \right)^{q/2} | c(\lambda) |^{-2} d\lambda \right\}^{1/q} \leq C_p \| f \|_p$$

for any $f \in L^p(G/K)$.

In [E-K-T] this theorem was proved only for K-finite functions on G/K.

If f is K-biinvariant, then

$$\tilde{f}(\lambda, kM) = \int_{AN} f(an)\, e^{(-\sqrt{-1}\lambda + \rho)(log a)} da dn$$
$$= \int_G f(x) \varphi_\lambda (x^{-1})\, dx.$$

Hence $\tilde{f}(\lambda, kM)$ is the spherical transform of $f \in C_c^\infty(K \backslash G/K)$. We denote it by $\tilde{f}(\lambda)$. And the Plancherel formula and the inversion formula are as follows: If $f \in C_c^\infty(K \backslash G/K)$, then we have

$$\int_G | f(x) |^2 dx = [W]^{-1} \int_{\mathfrak{a}^*} | \tilde{f}(\lambda) |^2 | c(\lambda) |^{-2} d\lambda$$

and

$$f(x) = [W]^{-1} \int_{\mathfrak{a}^*} \tilde{f}(\lambda) \varphi_\lambda(x) | c(\lambda) |^{-2} d\lambda .$$

Restricting ourseves to consider K-biinvariant functions in Theorem 1, we get the following corollary ([E-K] Lemma 8).

COROLLARY. *If* $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then there exists a positive constant* C_p, $0 < C_p < [W]^{1/q}$ *such* *that*

$$\left(\int_{\mathfrak{a}^*} |\tilde{f}(\lambda)|^q |c(\lambda)|^{-2} d\lambda\right)^{1/q} \leq C_p \|f\|_p$$

for any $f \in L^p(K\backslash G/K)$.

For the Hausdorff-Young inequality in $T_p = \mathfrak{a}^* + \sqrt{-1}C_{\varepsilon_p}$ it is necessary to have an inequality

$$\sup_{\lambda \in T_1} \| (\mathcal{F}f)(\lambda) \|_\infty \leq \|f\|_1 \tag{4.1}$$

instead of (3.1). Kunze-Stein ([K-S]) and Lipsman ([L1], [L2]) constructed uniformly bounded representations to have an inequality of this type. In [E-K] the boundedness of φ_λ on T_1 due to Helgason-Johnson [H-J] was used. In [E-K-T] the authors proved that if $f \in C_c^\infty(G/K)$ is K-finite of type F, then there exists a constant $C_F > 0$ such that

$$\sup_{\lambda \in T_1} \| (\mathcal{F}f)(\lambda) \|_\infty \leq C_F \|f\|_1.$$

5 An inequality in T_p

Next is the Young inequality due to A. Weil ([W], p.55)

LEMMA 2. *Let G be a locally compact unimodular group. We denote by* $f * g$ *the convolution of functions f and g on G. Let p and q be real numbers such that* $1 < p < \infty$, $1 < q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} > 1$. *Let* $r = \frac{pq}{p+q-pq}$. *If* $f \in L^p(G)$ *and* $g \in L^q(G)$, *then* $f * g$ *is in* $L^r(G)$ *and the inequality*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q$$

holds.

As a direct consequence of Lemma 1 we have the following lemma.

LEMMA 3. *Let K be a compact subgroup of a locally compact unimodular group G. For a function f on G we put* $f^*(x) = \overline{f(x^{-1})}$. *Let* $1 < p < 2$. *If f is in* $L^p(G/K)$, *then* $f^* * f$ *is in* $L^{p/2-1}(K\backslash G/K)$.

Now we return to our Riemannian symmetric space case. Next lemma is the Hausdorff-Young theorem for $L^p(K\backslash G/K)$ ([E-K] Theorem 1). For a subset S of $\mathfrak{a}_C^*$ we denote by Int$S$ the interior of S.

LEMMA 4. *Let* $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\varepsilon = \frac{2}{p} - 1$. *If* $f \in L^p(K\backslash G/K)$, *then the Fourier transform* $\tilde{f}$

can be holomorphically extended to the tube domain $IntT_p$ *and for any* $\eta \in IntC_{\varepsilon\rho}$, *there exists a positive constant* $C_{p,\eta}$ *such that*

$$\left(\int_{\mathfrak{a}^*} |\tilde{f}(\xi+\sqrt{-1}\eta)|\right)^{1/q} \leq C_{p,\eta} \|f\|_p \quad (f \in L_p(K\backslash G/K)).$$

Let $f \in C_c^\infty(G/K)$ and $\lambda \in \mathfrak{a}_C^*$.

$$(\widetilde{f^* * f})(\lambda) = \int_{AN} (f^* * f)(an)\, e^{(-\sqrt{-1}\lambda+\rho)(\log a)}\, dadn$$

$$= \int_{AN}\left(\int_G f^*(x^{-1}) f(xan)\, dx\right) e^{(-\sqrt{-1}\lambda+\rho)(\log a)}\, dadn$$

$$= \int_{AN}\left(\int_{KAN} \overline{f(ka'n')} f(ka'a(a^{-1}n'a)n)\, e^{2\rho(\log a')}\, dkda'\, dn'\right) e^{(-\sqrt{-1}\lambda+\rho)(\log a)}\, dadn$$

$$= \int_{KAN} \overline{f(ka'\, n')} \left(\int_{AN} f(kan)\, e^{(-\sqrt{-1}\lambda+\rho)(\log a - \log a')}\, dadn\right) e^{2\rho(\log a')}\, dkda'\, dn'$$

$$= \int_K \left(\int_{AN} \overline{f(ka'\, n')}\, e^{(\sqrt{-1}\bar{\lambda}+\rho)(\log a')}\, da'\, an'\right)\left(\int_{AN} f(kan)\, e^{(-\sqrt{-1}\lambda+\rho)(\log a)}\, dadn\right) dk$$

$$= \int_{K/M} \overline{\tilde{f}(\bar{\lambda},\, kM)}\, \tilde{f}(\lambda,\, kM)\, dk_M.$$

Thus we have the following lemma.

LEMMA 5. *Let* $f \in C_c^\infty(G/K)$. *Then*

$$(\widetilde{f^* * f})(\lambda) = \int_{K/M} \overline{\tilde{f}(\bar{\lambda},\, kM)}\, \tilde{f}(\lambda,\, kM)\, dk_M, \qquad (\lambda \in \mathfrak{a}_C^*).$$

From Lemmas 2, 3 ,4 and 5 we have the following theorem.

THEOREM 2. *Let* $1 < p < \frac{4}{3}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *For any* $\eta \in IntC_{(4/p-3)\rho}$, *there exists a positive constant* $C_{p,\eta}$ *such that*

$$\left(\int_{\mathfrak{a}^*} \left| \int_{K/M} \overline{\tilde{f}(kM, \xi-\sqrt{-1}\eta)}\, \tilde{f}(kM, \xi+\sqrt{-1}\eta)\, dk_M \right|^{q/2} |c(\xi)|^{-2} d\xi \right)^{1/q} \leq C_{p,\eta} \|f\|_p$$

for any $f \in C_c^\infty(G/K)$.

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