MINOR SUMMATION FORMULA AND SCHUR FUNCTION IDENTITIES

Masao ISHIKAWA[1]

(Accepted 21 June 1994)

Department of Mathematics
Faculty of General Education
University of Tottori

1. Introduction

In the paper [IW], we exploited a minor summation formula which include Pfaffians. In this paper we consider applications of this formula to the Schur function identities. First we give a new proof on the Schur function identities called Littlewood identities by the minor summation formula, and then we exploit some new identities on Schur functions, which is obtained by a deformation of the proof given above. The main result of this paper is Theorem 5.4. We are now studying this subject intensively and our survey is continuing. So the results described here might be a part of our results and it is probable that more fantastic results might be discovered later. The detailed proof will be omitted and described in another place.

2. Summation Formula of Pfaffians

In the paper [IW] we exploited a minor summation formula of Pfaffians. Now we briefly review this formula in this section. First of all we fix some basic notation here. In Section 2 of [IW] we treated the general quantum situation, but here we only need $q=1$ case so that we assume this condition from the beginning and only treat this case in this paper.

The auther would like to express his thanks to Professor H.Kimura, Professor S.Okada and Professor M.Wakayama for their valuable comments.

[1] Partially supported by Inoue Foundation for Science and partially supported by Grant-in Aid for Scientific Research No. 06740027, the Ministry of Education, Science and Culture of Japan.

Let r, m, n be positive integers such that $r \leq m, n$. Let T be an arbitrary m by n matrix. For two sequences $i=(i_1, \ldots, i_r)$ and $\boldsymbol{k}=(k_1, \ldots, k_r)$, let $T^{\boldsymbol{i}}_{\boldsymbol{k}}=T^{i_1 \ldots i_r}_{k_1 \ldots k_r}$ denote the sub-matrix of T obtained by picking up the rows and columns indexed by $\boldsymbol{i}$ and $\boldsymbol{k}$, respectively.

Assume $m \leq n$ and let B be an arbitrary n by n antisymmetric matrix, that is, $B=(b_{ij})$ satisfies $b_{ij}=-b_{ji}$. As long as B is a square antisymmetric matrix, we write $B_i=B_{i_1 \ldots i_r}$ for $B^{\boldsymbol{i}}_{\boldsymbol{i}}=B^{i_1 \ldots i_r}_{i_1 \ldots i_r}$ in abbreviation. One of the main result in [IW] is the following theorem. (See p.6, Theorem 1 of [IW].)

Theorem 2.1. *Let $m \leq n$ and $T=(t_{ik})$ be an arbitrary m by n matrix.*

(1) *Let m be even and $B=(b_{ik})$ be any n by n antisymmetric matrix with entries b_{ik}. Then*

$$\sum_{1 \leq k_1 < \cdots < k_m \leq n} \operatorname{pf}(B_{k_1 \ldots k_m}) \det(T^{1 \ldots m}_{k_1 \ldots k_m}) = \operatorname{pf}(Q), \tag{2.1}$$

where Q is the m by m antisymmetric matrix defined by $Q=TB^tT$, i.e.

$$Q_{ij}=\sum_{1 \leq k < l \leq n} b_{kl} \det(T^{ij}_{kl}), \qquad (1 \leq i, j \leq m). \tag{2.2}$$

(2) *Let m be odd and $B=(b_{ik})_{0 \leq i, k \leq n}$ be any $(n+1)$ by $(n+1)$ antisymmetric matrix. Then*

$$\sum_{1 \leq k_1 < \cdots < k_m \leq n} \operatorname{pf}(B_{0 k_1 \ldots k_m}) \det(T^{1 \ldots m}_{k_1 \ldots k_m}) = \operatorname{pf}(Q'), \tag{2.3}$$

where

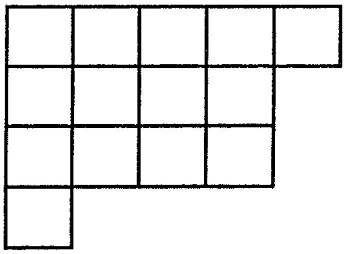
$$Q'_{ij} = \begin{cases} 0 & \text{if } i=j=0, \\ \sum_{k=1}^{n} b_{0k} t_{jk} & \text{if } i=0 \text{ and } 1 \leq j \leq m, \\ \sum_{k=1}^{n} b_{k0} t_{jk} & \text{if } i=0 \text{ and } 1 \leq i \leq m, \\ \sum_{1 \leq k < l \leq n} b_{kl} \det(T^{ij}_{kl}) & \text{if } 1 \leq i, j \leq m. \end{cases} \tag{2.4}$$

We regard the Pfaffian pf $(B_{k_1 \ldots k_m})$ as certain "weights" of the sub-determinants $\det(T^{1 \ldots m}_{k_1 \ldots k_m})$. By changing this antisymmetric matrix we obtain a considerably wide variation of the minor summation formula. The prototype of this formula appeared in [Ok] and [Ste].

3. Littlewood's Formulas

In this section we prove several identities on the Schur functions, known as Littlewood's formula. (See Section 5, Ex. 7,8,9 of [Mc].)

We define the Schur function here. A *partition* is a weakly decreasing sequence of nonnegative integers $\lambda := (\lambda_1, \cdots \lambda_m)$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$. The number of nonzero parts of λ is denoted by $\ell(\lambda)$ and called the length of λ. And the sum of the parts $|\lambda| = \lambda_1 + \cdots + \lambda_m$ is called the weight of λ. For example, $\lambda = (5441)$ is the partition of 14 with length $\ell(\lambda) = 4$, and denoted by the Ferrers graph



Let $\lambda'_i = \# \{ j : \lambda_j \geq i \}$. Then the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ is called the conjugate of λ. Suppose that the main diagonal of the Ferrers graph of λ consists of r nodes. This r is denoted by $p(\lambda)$. Let $\alpha_i = \lambda_i - i$ be the number of nodes in the i-th row to the right of (i, i), for $1 \leq i \leq p(\lambda)$, and let $\beta_i = \lambda'_i - i$ be the number of nodes in the i-th column of λ below (i, i) for $1 \leq i \leq p(\lambda)$. We have $\alpha_1 > \cdots > \alpha_r \geq 0$ and $\beta_1 > \cdots > \beta_r \geq 0$, and we denote the partition λ by

$$\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r) = (\alpha | \beta).$$

This is called the Frobenius notation. For example the above partition $\lambda = (5441)$ is denoted by $\lambda = (421|310)$ in the Frobenius notation. For a tuple $\alpha = (\alpha_1, \dots, \alpha_r)$ such that $\alpha_1 > \cdots > \alpha_r \geq 0$, we denote the tuple $(\alpha_1 + 1, \dots, \alpha_r + 1)$ by $\alpha + 1$. Further, if a is a nonnegative integer which doesn't coincide with any of α_i's, then let $q(\alpha, a)$ denote the number of α_i's which is bigger than a. So $(\alpha | \alpha + 1)$ denote the partition $(\alpha_1, \dots, \alpha_r | \alpha_1 + 1, \dots, \alpha_r + 1)$ for some r. For exsample, if $\alpha = (310)$ then $q(\alpha, 2) = 1$ and $(\alpha + 1 | \alpha) = (421 | 310)$.

Definition 3.1. Let $\lambda = (\alpha_1, \dots, \alpha_r | \beta_1, \dots, \beta_r)$ be a partition expressde in the Frobenius notation. Let a ans b be nonnegative integers such that $a \neq \alpha_1, \dots, \alpha_r$ and $b \neq \beta_1, \dots \beta_r$. There are some k and l such that $\alpha_k > a > \alpha_{k+1}$ and $\beta_l > b > \beta_{l+1}$. The partition $\lambda \Cup (a|b)$ is defined by

$$\lambda \Cup (a|b) = (\alpha_1, \ldots, \alpha_k, a, \alpha_{k+1}, \ldots, \alpha_r | \beta_1, \ldots, \beta_l, b, \beta_{l+1}, \ldots, \beta_r).$$

For example, $(421|310) \Cup (0|2) = (4210|3210)$.

For a partition $\lambda := (\lambda_1, \cdots \lambda_m)$, put $l = (l_1, \ldots l_m) = \lambda + \delta = (\lambda_1 + m-1, \lambda_2 + m-2, \ldots, \lambda_m)$, where $\delta = (m-1, m-2, \ldots, 0)$. So we have $l_1 > l_2 > \cdots > l_m \geq 0$. Then we set $a_l(x_1, \ldots, x_m) = a_{\lambda+\delta}(x_1, \ldots, x_m)$ to be

$$a_l = a_{\lambda+\delta} = \begin{vmatrix} x_1^{l_1} & \cdots & x_1^{l_m} \\ \vdots & \ddots & \vdots \\ x_m^{l_1} & \cdots & x_m^{l_m} \end{vmatrix}.$$

When $\lambda = 0$, a_δ is Vendermonde's determinant and equal to the product $\Pi_{1 \leq i < j \leq m}(x_i - x_j)$. For a partition $\lambda := (\lambda_1, \cdots, \lambda_m)$, the Schur function $s_\lambda = s_\lambda(x_1, \ldots, x_m)$ corresponding to λ is defined by

$$s_\lambda = a_{\lambda+\delta} / a_\delta.$$

(See Chap. 1, Sec. 3 of [Mc].)

Put

$$T = \begin{pmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_m^{n-1} & \cdots & x_m & 1 \end{pmatrix}.$$

From now on, we suppose that the column of T is indexed in the opposite direction, that is, they are $n-1$, $n-2, \ldots, 0$ from left to right. Similary both the row and column indeces of B are in this way. Further assume that both m and n are even. This makes our argument easy. Then it is easy to see that the Schur function is equal to

$$s_\lambda = \det T^{1 \ldots m}_{l_1 \ldots l_m} \ / \ \det T^{1 \ldots m}_{m-1 \ldots 0}.$$

We consider several antisymmetric matrices B in this paper.

First Consider $B=(\beta_{ij})_{i,j=n-1,\dots,0}$ defined by

$$\beta_{k+l+1,k}=s^l t^k \qquad (0\leq k,\ l\leq n-2),$$

where s, t are indeterminates. Let $c(\lambda)=\lambda_1-\lambda_2+\cdots+\lambda_{m-1}-\lambda_m$ be the number of columns of odd length in λ and let $e(\lambda)=\lambda_2+\lambda_4+\cdots+\lambda_m$. Then it is easy to see the following claim.

Praposition 3.1. *For a partition* $\lambda=(\lambda_1,\ \dots,\ \lambda_m)$, *put* $l=\lambda+\delta$. *Then*

$$\mathrm{pf}(B_{l_1\ l_2\ \dots\ l_m})=s^{c(\lambda)}t^{e(\lambda)+n(n-2)/4}.$$

We apply Theorem 2.1 (1) to this B and T and then we substitute $+\infty$ into n. Then it is an easy calculation to obtain Q. The result is

$$Q_{ij}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}s^l t^k\begin{vmatrix} x_i^{k+l+1} & x_i^k \\ x_j^{k+l+1} & x_j^k\end{vmatrix}=\frac{1}{(1-sx_i)(1-sx_j)}\,\frac{x_i-x_j}{1-tx_i x_j}$$

The following lemma gives us a way to calculate the Pfaffian.

Lemma 3.1. *Let n be even integer. Then*

$$\mathrm{pf}\left[\frac{x_i-x_j}{1-tx_i x_j}\right]_{1\leq i,\ j\leq n}=t^{\frac{n(n-2)}{4}}\prod_{1\leq i<j\leq n}\frac{x_i-x_j}{1-tx_i x_j}. \tag{3.1}$$

(See Prop. 2.3 (e) of [Ste].)

Proposition 3.1 and Lemma 3.1 give the proof of the following theorem.

Theorem 3.1.

$$\sum_{\lambda}s^{c(\lambda)}t^{e(\lambda)}s_\lambda(x_1,\dots,x_m)=\prod_{i=1}^{m}\frac{1}{1-sx_i}\prod_{1\leq i<j\leq m}\frac{1}{1-tx_i x_j}. \tag{3.2}$$

Next we take $B=(\beta_{ij})_{i,j=n-1,\dots,0}$ defined by

$$\beta_{2k+2l+1,2k}=t^{2k+l},$$
$$\beta_{k+2l+2,k}=st^{k+l},$$
$$\beta_{2k+2l+2,2k+1}=s^2t^{2k+l},$$

for $k,l \geq 0$. Similarly, as before, let $r(\lambda)$ be the number of rows of odd length in λ and put $h(\lambda) = [\lambda_1/2]+[\lambda_2/2]+\cdots+[\lambda_m/2]$, where $[x]$ stands for the greatest integer which does not exceed x. In this situation we obtain the following claim.

Proposition 3.2. *For a partition* $\lambda=(\lambda_1, \ldots, \lambda_m)$, *put* $l=\lambda+\delta$. *Then*

$$\mathrm{pf}(B_{l_1 l_2 \ldots l_l}) = r^{c(\lambda)} t^{h(\lambda)+n(n-2)/4}.$$

It is also an easy calculation to find

$$Q_{ij} = \frac{(1+sx_i)(1+sx_j)}{(1-tx_i^2)(1-tx_j^2)} \frac{x_i-x_j}{1-tx_ix_j}.$$

From Proposition 3.2 and Lemma 3.1 we obtain the following theorem.

Theorem 3.2.

$$\sum_{\lambda} s^{r(\lambda)} t^{h(\lambda)} s_\lambda(x_1, \ldots, x_m) = \prod_{i=1}^{m} \frac{1+sx_i}{1-tx_i^2} \prod_{1 \leq i < j \leq m} \frac{1}{1-tx_i x_j}. \tag{3.3}$$

In the rest of this section we list up some antisymmetric matrices B. But, at this point, we can't calculate $\mathrm{pf}(Q)$ corresponding to these B's.

(1) Let $B=(\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij} = \begin{cases} 1 & \text{if } i=2k+1 \text{ and } j=2k \text{ for } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\mathrm{pf}(B_{l_1 \ldots l_m}) = \begin{cases} 1 & \text{if } \lambda \text{ and } \lambda' \text{ are both even partitions,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$Q_{ij} = \frac{x_i - x_j}{1-x_i^2 x_j^2}.$$

This leads to

$$\sum_{\lambda} s_\lambda(x_1, \dots, x_m) = \mathrm{pf}\left[\frac{x_i - x_j}{1 - x_i^2 x_j^2}\right]_{1 \le i < j \le m} \Big/ \prod_{1 \le i < j \le m} (x_i - x_j), \tag{3.4}$$

where the sum ranges all partitions λ such that the both of λ and λ' are even partitions.

(2) Let $B = (\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij} = \begin{cases} 1 & \text{if } i > j \geq 0 \text{ and } i = 2k \text{ for some } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Q_{ij} = \frac{1}{(1-x_i)(1-x_j)} \frac{x_i - x_j}{1 - x_i^2 x_j^2}.$$

This leads to

$$\sum_{\lambda} s_\lambda(x_1, \dots, x_m) = \mathrm{pf}\left[\frac{x_i - x_j}{1 - x_i^2 x_j^2}\right]_{1 \le i, j \le m} \Big/ \prod_{i=1}^{m} (1 - x_i) \prod_{1 \le i < j \le m} (x_i - x_j). \tag{3.5}$$

where the sum ranges all partitions λ such that $\lambda_2, \lambda_4, \dots, \lambda_m$ are all even.

(3) Let $B = (\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij} = \begin{cases} 1 & \text{if } i > j \geq 0 \text{ and } j = 2k+1 \text{ for some } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Q_{ij} = \frac{1}{(1-x_i^2)(1-x_j^2)} \frac{(x_i - x_j)\{1 + x_i x_j (1 + x_i + x_j)\}}{1 - x_i^2 x_j^2}.$$

This leads to

$$\begin{aligned} &\sum_{\lambda} s_\lambda(x_1, \dots, x_m) \\ &= \mathrm{pf}\left[\frac{(x_i - x_j)\{1 + x_i x_j (1 + x_i + x_j)\}}{1 - x_i^2 x_j^2}\right]_{1 \le i, j \le m} \Big/ \prod_{i=1}^{m} (1 - x_i) \prod_{1 \le i < j \le m} (x_i - x_j), \end{aligned} \tag{3.6}$$

where the sum ranges all partitions λ such that $\lambda_1, \lambda_3, \ldots, \lambda_{m-1}$ are all even.

(4) Let $B=(\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij}=\begin{cases} 1 & \text{if } i>j\geq i-2 \text{ and } i=2k \text{ for } k\geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Q_{ij}=\frac{(x_i-x_j)(1+x_i+x_j)}{1-x_i^2x_j^2}.$$

This leads to

$$\text{(3.7)} \qquad \sum_{\lambda} s_\lambda(x_1,\ldots,x_m)=\mathrm{pf}\left[\frac{(x_i-x_j)(1+x_i+x_j)}{1-x_i^2x_j^2}\right]_{1\leq i,j\leq m} \Big/ \prod_{1\leq i<j\leq m}(x_i-x_j),$$

where the sum ranges all partitions λ such that $\lambda_2, \lambda_4, \ldots, \lambda_m$ are all even and ($\lambda_{2k-1}=\lambda_{2k}$ or $\lambda_{2k-1}=\lambda_{2k}+1$).

(5) Let $B=(\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij}=\begin{cases} 1 & \text{if } j+2\geq i>j\geq 0 \text{ and } j=2k+1 \text{ for } k\geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Q_{ij}=\frac{(x_i-x_j)\{1+x_ix_j(x_i+x_j)\}}{1-x_i^2x_j^2}.$$

This leads to

(3.8)

$$\sum_{\lambda} s_\lambda(x_1,\ldots,x_m)=\mathrm{pf}\left[\frac{(x_i-x_j)\{1+x_ix_j(x_i+x_j)\}}{1-x_i^2x_j^2}\right]_{1\leq i,j\leq m} \Big/ \prod_{1\leq i<j\leq m}(x_i-x_j),$$

where the sum ranges all partitions λ such that $\lambda_1, \lambda_3, \ldots, \lambda_{m-1}$ are all even and ($\lambda_{2k-1}=\lambda_{2k}$ or $\lambda_{2k-1}=\lambda_{2k}+1$).

(6) Let $B=(\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij}=\begin{cases} 1 & \text{if } j=i-1\geq 0 \text{ or } (j=i-2 \text{ and } i \text{ is even}), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Q_{ij}=(1+x_i)(1+x_j)\frac{x_i-x_j}{1-x_i^2x_j^2}.$$

This leads to

$$\sum_{\lambda} s_\lambda(x_1,\dots,x_m)=\prod_{i=1}^{m}(1+x_i)\operatorname{pf}\left[\frac{x_i-x_j}{1-x_i^2x_j^2}\right]_{1\leq i,j\leq m}\Big/\prod_{1\leq i<j\leq m}(x_i-x_j), \tag{3.9}$$

where the sum ranges all partitions λ such that $\lambda_{2k-1}=\lambda_{2k}$ or ($\lambda_{2k-1}=\lambda_{2k}+1$ and λ_{2k} is even).

(7) Let $B=(\beta_{ij})$ be the antisymmetric matrix given by

$$\beta_{ij}=\begin{cases} 1 & \text{if } j=i-1\geq 0 \text{ or } (j=i-2 \text{ and } i \text{ is odd}), \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$Q_{ij}=\frac{(x_i-x_j)\{1+x_ix_j(1+x_i+x_j)\}}{1-x_i^2x_j^2}.$$

This leads to

(3.10)

$$\sum_{\lambda} s_\lambda(x_1,\dots,x_m)=\operatorname{pf}\left[\frac{(x_i-x_j)\{1+x_ix_j(1+x_i+x_j)\}}{1-x_i^2x_j^2}\right]_{1\leq i,j\leq m}\Big/\prod_{1\leq i<j\leq m}(x_i-x_j),$$

where the sum ranges all partitions λ such that $\lambda_{2k-1}=\lambda_{2k}$ or ($\lambda_{2k-1}=\lambda_{2k}+1$ and λ_{2k} is odd).

4. Another Type of Littlewood's Formulas

In this section we give a proof of another type of the Littlewood type formulas by using our minor summation formula. These kinds of identities can be seen in Sectiton 5, Ex. 9 of [Mc]. The advantage of our proof is that its deformation leads to other kinds of identities on Schur functions which are new. The follong lemma play an important role in this section.

Lemma 4.1. *Let m be a positive integer. Then*

$$\mathrm{pf}\left[\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j}\right]_{1\leq i,j\leq m}=\prod_{1\leq i<j\leq m}(x_i-x_j)(1-x_i x_j). \tag{4.1}$$

Fix

$$T=\begin{pmatrix} x_1^{4m-2} & \cdots & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{2m}^{4m-2} & \cdots & x_{2m} & 1 \end{pmatrix}.$$

First let $B=(\beta_{ij})_{i,\ j=4m-2,\dots,0}$ be the antisymmetric matrix of size $(4m-1)$ given by

$$\beta_{2m-1-k+l,k+l}=\begin{cases} 1 & \text{for } 0\leq k\leq m \text{ and } 0\leq l\leq m-1, \\ -1 & \text{for } 0\leq k\leq m \text{ and } m\leq l\leq 2m-1, \\ 0 & \text{otherwise.} \end{cases}$$

It is an easy calculation to make it sure that

$$Q_{ij}=\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j}.$$

The following proposition and Lemma 4.1 give us the following well-known formula which is found in Sectiton 5, Ex. 9 of [Mc].

Proposition 4.1. *For a partition $\lambda=(\lambda_1,\dots,\lambda_{2m})$, put $l=\lambda+\delta$. Then*

$$\mathrm{pf}(B_{l_1 l_2 \ldots l_{2m}}) = \begin{cases} (-1)^{|\lambda|/2} & \textit{if } \lambda = (\alpha \mid \alpha+1), \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 4.1. *Let m be a positive integer.*

(4.2) $$\sum_{\lambda=(\alpha|\alpha+1)} (-1)^{|\lambda|/2} s_\lambda(x_1, \ldots, x_m) = \prod_{1\le i<j\le m} (1 - x_i x_j)$$

In the proof of the above theorem we have to take another B to prove it when m is odd. But we ommit the detailed discussion here.

Lemma 4.2. *Let m be a positive integer. Then*

(4.3) $$\mathrm{pf}\left[\frac{(x_i^m - x_j^m)^2}{x_i - x_j} \frac{\{1-(-x_i x_j)^m\}^2}{1 + x_i x_j}\right]_{1\le i,j\le m} = \prod_{1\le i<j\le m} (x_i - x_j)(1 + x_i x_j).$$

We give the symmetric matrix $B = (\beta_{ij})$ of size $(4m-1)$ which have the above antisymmetric matrix as the $Q = (Q_{ij})$.

$$\beta_{2m-1-k+l,k+1} = \begin{cases} (-1)^l & \text{for } 0\le k \le m \text{ and } 0 \le l \le m-1, \\ -(-1)^l & \text{for } 0 \le k \le m \text{ and } m \le l \le 2m-1, \\ 0 & \text{otherwise.} \end{cases}$$

A similar argument shows the following theorem. But the reader who is familiar with Schur functions notice that the following identity is obtained immediately from Theorem 4.1 by substituting $\sqrt{-1}x_i$ into x_i. But it's still worth mentioning the above B since there is a possibility that its deformation leads to other identities which are unkown.

Theorem 4.2. *Let m be a positive integer.*

(4.4) $$\sum_{\lambda=(\alpha|\alpha+1)} s_\lambda(x_1, \ldots, x_m) = \prod_{1\le i<j\le m} (1 + x_i x_j).$$

Definition 4.1. Let m be a positive integer and let $B = (\beta_{ij})_{0\le i,j\le m}$ be an antisymmetric matrix. B is said to be *symmetrically proportional* if the $(m-i)$-th row is proportional to the i-th row for all $0 \le i \le m$.

Further B is called *row-symmetric* if the $(m-i)$-th row is equal to the i-th row for all $0 \leq i \leq m$, and B is called *row-antisymmetric* if the $(m-i)$-th row is opposite in sign to the i-th row for all $0 \leq i \leq m$.

This notion has importance since it makes us easy to find the subpfaffians $\mathrm{pf}(B_{l_1 \ldots l_m})$ for given B. At this point it is not so easy for us to find $\mathrm{pf}(B_{l_1 \ldots l_m})$ for not symmetrically proportional (diagonal-) antisymmetric matrices. From now on we assume that B is always supposed to be (diagonal-) antisymmetric matrix in ordinary means without mentioning it.

Now fix

$$T = \begin{pmatrix} x_1^{4m-1} & \cdots & x_1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{2m}^{4m-1} & \cdots & x_{2m} & 1 \end{pmatrix}.$$

Let $\mathrm{B} = (\beta_{i,j}(a))$ be the matrix of size $4m$ defined by

$$(1+ax_i)(1+ax_j)\frac{(x_i^m - x_j^m)^2}{x_i - x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j} = \sum_{k>l\geq 0} \beta_{kl} \begin{vmatrix} x_i^k & x_j^l \\ x_j^k & x_j^l \end{vmatrix},$$

where a is a constant. Then it is easy to see that $B = (\beta_{kl}(a))$ is symmetrically proportional if and only if $a = \pm 1$, and, further, if $a=1$, then B become row-symmetric, on the other hand, if $a=-1$, then B become row-antisymmetric. The result in the case of $a=1$ is easily derived from the $a=-1$ case, so here we treat only this case.

Let $B = (\beta_{kl})_{k,\, l=4m-1, \ldots, 0}$ be the row-symmetric matrix defined by

$$\beta_{kl} = \begin{cases} 1 & \text{if } (k, l) = (2m-1-p, p) \text{ for } 0 \leq p \leq m-1, \\ (-1)^{k+l-1} 2 & \text{if } k+l \geq 2m,\ l-k \leq 1 \text{ and } l < m, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that these data determine all the entries of B since B is diagonal-antisymmetric and row-antisymmetric. Then it is an easy calculation to show that

$$Q_{ij} = (1-x_i)(1-x_j)\frac{(x_i^m - x_j^m)^2}{x_i - x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j}.$$

It is not so hard to see the following proposition.

Proposition 4.2. *For a partition* $\lambda=(\lambda_1,\ldots,\lambda_{2m})$, *put* $l=\lambda+\delta$. *Then*

$$\mathrm{pf}(B_{l_1 l_2 \ldots l_{2m}})=\begin{cases}(-1)^{(|\lambda|+p(\lambda))/2} & \text{if } \lambda=(\alpha\,|\,\alpha),\\ 0 & \text{otherwise.}\end{cases}$$

This shows the following well-known formula.

Theorem 4.3. *Let* m *be a positive integer.*

$$\sum_{\lambda=(\alpha|\alpha)}(-1)^{(|\lambda|+p(\lambda))/2}s_\lambda(x_1,\ldots,x_m)=\prod_{i=1}^{m}(1-x_i)\prod_{1\le i<j\le m}(1-x_i x_j) \tag{4.5}$$

Let us consider another case. Let $B=(\beta_{i,j}(a))$ be the antisymmetric matrix defined by

$$(1+ax_i)(1+ax_j)\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\frac{\{1-(-x_i x_j)^m\}^2}{1+x_i x_j}=\sum_{k>l\ge 0}\beta_{kl}\begin{vmatrix}x_i^k & x_i^l\\ x_i^k & x_j^l\end{vmatrix},$$

where a is a constant. Then it is easy to see that there is no real number a such that $B=(\beta_{kl}(a))$ become symmetrically proportional.

5. Further Identities

In this section we extend the methods in the former section and find further identities on Schur functions. We use the methods we exploited in the former section.

In this section we fix

$$T=\begin{pmatrix}x_1^{4m} & \cdots & x_1 & 1\\ \vdots & \ddots & \vdots & \vdots\\ x_{2m}^{4m} & \cdots & x_{2m} & 1\end{pmatrix}.$$

Let $B=(\beta_{i,j}(a,b))$ be the matrix of size $(4m+1)$ defined by

$$(1+ax_i+bx_i^2)(1+ax_j+bx_j^2)\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j}=\sum_{k>l\ge 0}\beta_{kl}\begin{vmatrix}x_i^k & x_i^l\\ x_j^k & x_j^l\end{vmatrix},$$

where a and b are constants.

Proposition 5.1. *Then $B=(\beta_{kl}(a, b))$ is symmetricarry proportional if and only if B is row-symmetric or row-antisymmetric. Funther, in the case of $b=1$, B becomes row-symmetric, on the other hand, in the case of $a=0$ and $b=-1$, B becomes row-antisymmetric.*

First we consider the classical well-known case that is $a=0$ and $b=-1$. In this case, define a row-antisymmmetric matrix $B=(\beta_{kl})$ of size $(4m+1)$ by

$$\beta_{kl}=\begin{cases} 1 & \text{if}(k, l)=(2m-1-p,\ p) \text{ for } 0\leq p\leq m-1, \\ -1 & \text{if}(k, l)=(2m-p,\ p+1) \text{ for } 1\leq q\leq m-2, \\ 1 & \text{if } (k, l)=(m+1,\ m), \\ 2 & \text{if}(k, l)=(q+m+1,\ q+m) \text{ for } 1\leq q\leq m-2, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is an easy calculation to find that

$$Q_{ij}=(1-x_i^2)(1-x_j^2)\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j}.$$

We use our routine procedure to obtain the following theorem, but here we ommit the detailed proof.

Proposition 5.2. *For a partition $\lambda=(\lambda_1, \ldots, \lambda_{2m})$, put $l=\lambda+\delta$. Then*

$$\mathrm{pf}(B_{l_1 l_2 \ldots l_{2m}})=\begin{cases} (-1)^{|\lambda|/2} & \text{if } \lambda=(\alpha+1|\alpha) \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 5.1. *Let m be a positive integer.*

(5.1)
$$\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{|\lambda|/2}s_\lambda(x_1, \ldots, x_m)=\prod_{i=1}^{m}(1-x_i^2)\prod_{1\leq i<j\leq m}(1-x_i x_j)$$

This is a classical result which is already known. Next we consider an unknown case which will give us new indentities. Then we extend these results to more general case.

Let $B=(\beta_{k,l})$ be the row-symmetric matrix of size $(4m+1)$ which is given by

$$\beta_{kl}=\begin{cases} 1 & \text{if}(k,\ l)=(2m-1-p,\ p) \text{ for } 0\leq p\leq m-1,\\ -2 & \text{if}(k,\ l)=(2m,\ 0),\\ -4 & \text{if}(k,\ l)=(2m-p,\ p) \text{ for } 1\leq p\leq m-1,\\ 7 & \text{if}(k,\ l)=(2m-p,\ p+1) \text{ for } 1\leq q\leq m-2,\\ 5 & \text{if}(k,\ l)=(m+1,\ m),\\ (-1)^{k+l-1}8 & \text{if } k+l\geq 2m+2,\ l-k\geq 2 \text{ and } l\leq m,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that these data determine all the entries of B since B is diagonal-antisymmetric and row-symmetric. Again we calculate Q_{ij} and we obtain

$$Q_{ij}=(1-x_i)^2(1-x_j)^2\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j}.$$

We also obtain the following proposition.

Proposition 5.3. *For a partition* $\lambda=(\lambda_1,\ \ldots,\ \lambda_{2m})$, *put* $l=\lambda+\delta$. *Then*

$$\operatorname{pf}(B_{l_1\ l_2\ldots l_{2m}})=\begin{cases}(-1)^{|\lambda|/2+p(\lambda)} & \text{if } \lambda=(\alpha+1|\alpha),\\ (-1)^{|\mu|/2+k+q(\alpha,k-1)} & \text{if } \lambda=\mu\cup(0|k-1) \text{ for some } 1\leq k\leq m\\ & \text{and } \mu=(\alpha+1|\alpha) \text{ such that } \alpha\not\ni k-1,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 5.2. *Let* m *be a positive integer.*

(5.2)
$$\sum_{\lambda=(\alpha+1|\alpha)}(-1)^{|\lambda|/2+p(\lambda)}s_\lambda(x_1,\ldots,x_m)$$
$$+2\sum_{k=1}^{m}\sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha\not\ni k-1}}(-1)^{|\lambda|/2+k+p(\alpha,k-1)}s_{\lambda\uplus(0|k-1)}(x_1,\ldots,x_m)$$
$$=\prod_{i=1}^{m}(1-x_i)^2\prod_{1\leq i<j\leq m}(1-x_i x_j)$$

Here we give another example. A similar argument shows us the following formula.

Theorem 5.3. *Let m be a positive integer.*

(5.3)

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{|\lambda|/2+p(\lambda)} s_{\lambda}(x_1,\dots,x_m) + 2\sum_{k=1}^{[m/2]} \sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha \not\ni 2k-1}} (-1)^{|\lambda|/2+k+p(\alpha,k-1)} s_{\lambda \uplus (0|2k-1)}(x_1,\dots,x_m) = \prod_{i=1}^{m} (1+x_i^2) \prod_{1\le i<j\le m} (1-x_i x_j).$$

In the proof of Proposition 5.2 the symmetrical proportionality play a crucial role. At this point we can't give the general form of such formula without assuming this condition. Now we are in the position to consider the general row-symmetric case. Surprisingly this consideration gives us a remarkable relation between Schur functions and Chebyshev polynomials.

Let $B=(\beta_{kl}(a))_{0\le k,\ l\le 4m}$ be the matrix of size $4m+1$ defined by

$$(1+ax_i+x_i^2)(1+ax_j+x_j^2)\,\frac{(x_i^m-x_j^m)^2}{x_i-x_j}\,\frac{(1-x_i^m x_j^m)^2}{1-x_i x_j} = \sum_{k>l\ge 0} \beta_{kl}(a) \begin{vmatrix} x_i^k & x_i^l \\ x_j^k & x_j^l \end{vmatrix},$$

The Chebyshev polynomials of the first kind $T_k(x)$ are defined by

$$\mathrm{T}_k(x)=\cos\ (n \arccos\ x).$$

The first a few terms of this polynomials are

$$T_0(x)=1,$$
$$T_1(x)=x,$$
$$T_2(x)=2x^2-1,$$
$$T_3(x)=4x^3-3x,$$

and satisfy the recurrence formula

$$T_{k+1}(x)-2xT_k(x)+T_{k-1}(x)=0.$$

We calculate $B_{l_1\dots l_m}$ and obtain the following result.

Theorem 5.4. *Let m be a positive integer.*

(5.4)

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{|\lambda|/2+p(\lambda)} s_\lambda(x_1,\ldots,x_m)$$
$$+2\sum_{k=1}^{m} T_k(a) \sum_{\substack{\lambda=(\alpha+1|\alpha)\\ \alpha\not\ni k-1}} (-1)^{|\lambda|/2+k+p(\alpha,k-1)} s_{\lambda\uplus(0|k-1)}(x_1,\ldots,x_m)$$
$$=\prod_{i=1}^{m}(1+2ax_i+x_i^2)\prod_{1\le i<j\le m}(1-x_i x_j),$$

where $T_k(x)$ is the Chebyshev polynomial of the first kind.

The special values of Chebyshev polynomials give us the preceding identities. For example

$$T_k(1)=1 \qquad T_k(-1)=(-1)^k$$
$$T_{2k}(0)=(-1)^k \qquad T_{2k-1}(0)=0.$$

References

[As] K. Asai, *A Jacobi-Trudi identity for partially strict skew waved plane partitions and the number of cyclically symmetric (2n,2n,2n)-self-complementry partitions*, Extended abstract of Conference on Commutative Algebra and Combinatorics, 1990, pp.1−8.

[I] M. Ishikawa, *A remark on totally symmetric self-complementary plane partitions, preprint.*

[IOW] M. Ishikawa, S. Okada and M. Wakayama, *Minor summation of Pfaffians and generalized Littlewooa type formnlas*, in preparation.

[IW] M. Ishikawa and M. Wakayama, *Minor summation formula of Pfaffians*, preprint.

[Kr] C. Krattenthaler, *On bideterminantal formnlas for characters of classical groups*, preprint.

[KT] K. Koike and I. Terada, *Littlewood's formnlas and their application to representations of classical groups*, Advanced Studies in Pure Math.11(1987), 147−160.

[Ma] I.G. Macdnald, *Symmetric functions and Hall Polynomials*, Oxford University Press, 1979.

[Ok] S. Okada, *On the generating functions for certain classes of plane partitions*, J. Combin. Theorey Ser.A 51(1989), 1−23.

[Pr1] R.A. Proctor, *Odd symplectic groups*, Invent. Math. 92 (1988),307−332.

[Pr2] ———, *Young Tableaux, Gelfand Patterns, and Branching Rules for Classical Groups*, preprint.

[St1] R.P. Stanley, *Ordered structures and partitions*, vol.119, Mem.Amer.Math.Soc., 1972.

[St2] ———, *Enumerative Combinatorics, Vol.I*, Wadsworth, Monterey CA, 1986.

[Ste] J. Stembridge, *Nonintersecting paths and pfaffians*, Adv.in Math. 83(1990),96−131.

Masao Isikawa, Department of Mathematics, Faculty of General Education, University of Tottori, Koyama, Tottori 680, Japan

E-mail address: m-ishika @ tansei. cc. u-tokyo. ac. jp