Amplitude Dependent Stability and Instability Analysis for Nonlinear Sampled-Data Control Systems

Yoshifumi OKUYAMA and Fumiaki TAKEMORI

Department of Information and Knowledge Engineering
Faculty of Engineering, Tottori University
Tottori, 680-8552 Japan

E-mail: oku@ike.tottori-u.ac.jp

Abstract: The robust stability condition for sampled-data control systems with a sector nonlinearity was presented in our previous paper. Although it is applicable only to the sampled-data control system of a certain class, a usual discrete-time control system can belong to this type of class. This paper analyzes the amplitude dependent behavior of nonlinear sampled-data (i.e., discrete-time) control systems in a frequency domain. First, the robust stability condition which was derived in our previous papers is applied to a sampled-data control system containing a single time-invariant nonlinear element in the forward path. Then, an instability condition for that type of nonlinear feedback system is derived. By considering restricted areas (two sectors) in the nonlinear characterisitic, the amplitude of a sustained oscillation is estimated (whether it is periodic or not), and the relationship between the stable/unstable conditions and the result which is derived from the classic describing function is compared. Numerical examples will be presented to illustrate the theory.

Key words: Nonlinear sampled-data systems; input-output stability; Popov criterion; instability

1. INTRODUCTION

This paper analyzes the amplitude dependent behavior of nonlinear sampled-data control systems in *a frequency domain.* In actuality, a sustained oscillation (whether it is periodic or not) cannot be avoided in the response of nonlinear dynamical systems. Nonetheless, the practical analysis and design method is only a graphical and approximated version for a periodic oscillation in respect to continuous-time systems, that is, *describing function*, in other words, the harmonic balance method(Atherton[1], Vidyasagar[2], Gordillo *et al.*[3]). As for discrete-time system, there is no method in particular to analyze and design such a control system.

In this paper, first, the robust stability condition for nonlinear discrete-time feedback systems (which was derived in our previous paper) is applied to a sampled-data control system containing a single time-invariant nonlinear element in the forward path. Then, an instability condition for that type of nonlinear feedback system is derived as an inverse problem of the robust stability. By considering restricted areas (two sectors) in the nonlinear characteristic, the amplitude of a sustained oscillation is estimated, and the relationship between the stable/unstable conditions and the result which is derived from the classic describing function is compared.

2. EQUIVALENT TRANSFORMATION

In our previous papers (Okuyama and Takemori[4, 5, 6], the robust stability for nonlinear sampled-data control systems was analyzed in the frequency domain as a natural expansion of Popov's criterion for continuous-time systems.

The control system in question is a sampled-data control system with time-invariant nonlinear characteristic $N(\cdot)$ as shown in Fig. 1. Here, $\mathcal{H}$ is the zero-order-hold which is usually performed in AD(DA) conversion and $G(s)$ is the transfer function of the system to be controlled, which is expressed by continuous-time.

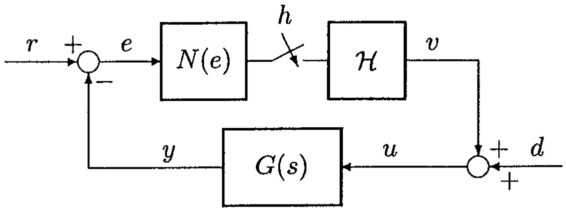

Fig. 1 Nonlinear sampled-data control system.

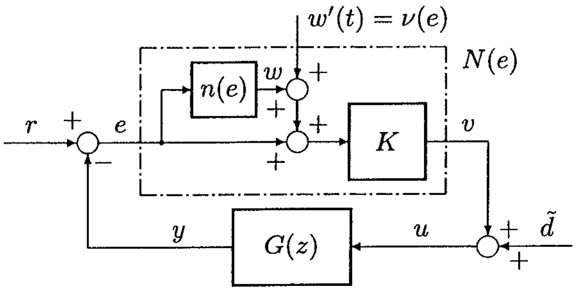

Fig. 2 Equivalent nonlinear discrete-time system.

In addition, nonlinear characteristic $N(\cdot)$ is time-invariant and can be written as

$$N(e) = K(e + n(e) + \nu(e)), \quad 0 < K < \infty \quad (1)$$

$$|w| = |n(e)| \leq \alpha|e|, \quad 0 < \alpha \leq 1 \quad (2)$$

$$|w'| = |\nu(e)| < \infty, \quad (3)$$

where $n(e)$ and $\nu(e)$ are nonlinear terms relative to nominal linearized gain K. The nonlinear term $n(\cdot)$ need not be specially memoryless, but the summation of trapezoidal areas determined by the path of sampling points should be non-negative(Okuyama *et al.*[6]).

By rearranging the nonlinear sampled-data control system, Fig. 2 can be obtained, where $G(z)$ is the z-transform of $G(s)$ together with zero-order-hold $\mathcal{H}$. In the stability analysis, it is sufficient to consider only nonlinear term $n(e)$, because nonlinear term $\nu(e)$ can be treated as a disturbance signal as shown in this figure.

Consider new sequences $e_m^*(k)$ and $w_m^*(k)$ ($k = 1, 2 \cdots, N$) which satisfy the following equation:

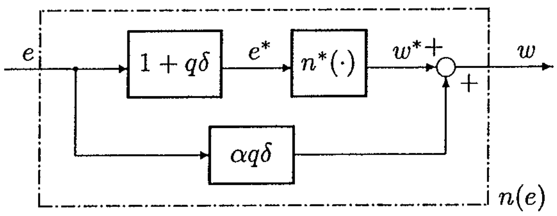

Fig. 3 Nonlinear subsystem.

$$e_m^*(k) = e_m(k) + q \cdot \frac{\Delta e(k)}{h}, \quad (4)$$

$$w_m^*(k) = w_m(k) - \alpha q \cdot \frac{\Delta e(k)}{h}, \quad (5)$$

where q is a non-negative number, $e_m(k)$ and $w_m(k)$ are neutral points of sequences $e(k)$ and $w(k)$, i.e.,

$$e_m(k) = \frac{e(k) + e(k-1)}{2}, \quad w_m(k) = \frac{w(k) + w(k-1)}{2} \quad (6)$$

and $\Delta e(k) = e(k) - e(k-1)$ is the backward difference of error. The relationship between these equations is shown by the block diagram in Fig. 3. In this figure, δ is defined as

$$\delta(z) := \frac{2}{h} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}. \quad (7)$$

Eq. (7) corresponds to the bilinear transformation approximation between z and δ when relating δ to Laplace transform variable s for a continuous-time system. Then, the loop transfer function from w^* to e^* can be given by $F(\alpha, q, z)$ as shown in Fig. 4. Here,

$$F(\alpha, q, z) = \frac{(1 + q\delta(z))KG(z)}{1 + (1 + \alpha q\delta(z))KG(z)}, \quad (8)$$

and r', d' are transformed exogenous inputs.

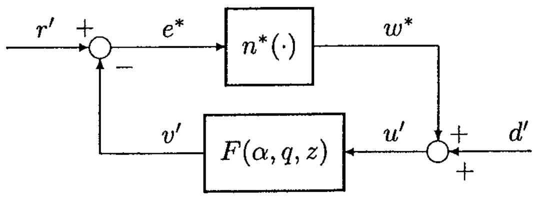

Fig. 4 Equivalent closed loop system.

3. PRELIMINARIES

Let us define a new nonlinear function for $n(\cdot)$ of Eq. (2) such as $f(e) := n(e) + \alpha e$. This function belongs to the first and third quadrants. Considering the equivalent linear characteristic which varies with discrete-time $k = 1, 2, \cdots$, it can be written as

$$0 \leq \frac{f(e(k))}{e(k)} \leq 2\alpha. \tag{9}$$

Here, $\gamma(k) := f(e(k))/e(k)$ can be defined. When this type of $\gamma(k)$ is used, sector inequality (2) is expressed as $w(k) = n(e(k)) = (\gamma(k) - \alpha)e(k)$.

The following assumption will be provided in regard to the nonlinear characteristics to avoid the difficult problems that are peculiar to nonlinear sampled-data control systems (Kalman[7]).

[Assumption-1] Error sequence $e(k)$ passes the origin. Specifically, the relationship $\gamma(k-1) = \gamma(k)$ is maintained whenever $e(k-1)e(k) < 0$. Therefore, the line between coordinates $(e(k-1), f(e(k-1))$ and $(e(k), f(e(k)))$ by linear interpolation also passes the origin. □

This assumption is not too inaccessible. If the sampling period is shorter than the transient response of the system, variations of error $\Delta e(k)$ are also expected to be small when the sequence passes the origin. Hence, Assumption-1 will be satisfied. Even if the sampling period is relatively long, it will be satisfied when nonlinear characteristics are gentle around the origin. Therefore, the above covers a considerably wide range of problems.

Based on the above premise, the following properties can be shown.

[Lemma-1] For a positive integer N (the number of steps), the following inequality holds:

$$\|w_m(k)\|_N \leq \alpha \|e_m(k)\|_N. \tag{10}$$

(Proof) The proof is omitted (See e.g., [5]). □

[Lemma-2] If the following inequality is satisfied in regard to the inner product of the neutral points of $f(e)$ and the backward difference sequence of error:

$$\langle\ w_m(k) + \alpha e_m(k), \Delta e(k)\ \rangle_N \geq 0, \tag{11}$$

the following inequality holds:

$$\|w_m^*(k)\|_N \leq \alpha \|e_m^*(k)\|_N. \tag{12}$$

for any $q \geq 0$. Here, the norm and the inner product are defined as:

$$\|x(k)\|_N := \left(\sum_{k=1}^{N} |x(k)|^2\right)^{1/2},$$

$$\langle\ x_1(k), x_2(k)\ \rangle_N := \sum_{k=1}^{N} x_1(k)x_2(k).$$

(Proof) Based on Eq. (4) and (5),

$$\begin{aligned}
&\alpha^2\|e_m^*(k)\|_N^2 - \|w_m^*(k)\|_N^2 \\
&= \alpha^2 \left\| e_m(k) + q\frac{\Delta e(k)}{h} \right\|_N^2 - \left\| w_m - \alpha q \frac{\Delta e(k)}{h} \right\|_N^2 \\
&= \alpha^2\|e_m(k)\|_N^2 - \|w_m(k)\|_N^2 \\
&\qquad + \frac{2\alpha q}{h}\langle w_m(k) + \alpha e_m(k), \Delta e(k)\rangle_N.
\end{aligned}$$

Therefore, from inequalities (10) and (11), inequality (12) is obtained. □

The left side of Eq. (11) can be expressed in terms of nonlinear function $f(\cdot)$.

[Lemma-3] For any step N, the following equation is valid:

$$\begin{aligned}
&\langle\ w_m(k) + \alpha e_m(k), \Delta e(k)\ \rangle_N \\
&= \frac{1}{2}\sum_{k=1}^{N}(f(e(k)) + f(e(k-1)))\Delta e(k). \quad (13)
\end{aligned}$$

(Proof) The proof is obvious from definition (6). □

If we use $\sigma(N)$ for the right side of Eq. (13), we can show that $\sigma(N)$ is the total area of the trapezoid formed by sampling point $(f(e(k-1)), f(e(k)))$ on nonlinear curve $f(e)$ and error step width $\Delta e(k)$. The total area of trapezoid $\sigma(N)$ can be rewritten by the following.

[Lemma-4] For any step N,

$$\sigma(N) = \frac{1}{2}(f(e(N))e(N) - f(e(0))e(0)) + \epsilon(N), \tag{14}$$

where

$$\begin{aligned}
\epsilon(N) &= \frac{1}{2}\sum_{k=1}^{N}(f(e(k-1))e(k) - f(e(k))e(k-1)) \\
&= \frac{1}{2}\sum_{k=1}^{N} f_0(k)\cdot\Delta e(k). \quad (15)
\end{aligned}$$

Here, $f_0(k)$ is an intercept at which the straight line passing sample points p_k and p_{k-1} on the nonlinear function $f(e)$ intersects the vertical axis.

(Proof) The proof is omitted (See e.g., [5]). □

4. CLASSES OF SAMPLED-DATA CONTROL SYSTEMS

[Assumption-2] The total area of a trapezoid, allowing for signs of coordinate $(e(k), f(e(k)))$, $(k = 0, 1, 2, \cdots, N)$ which traces a nonlinear curve is always non-negative, i.e.,

$$\sigma(N) \geq 0, \tag{16}$$

regardless of the transient response. □

Although this Assumption seems to be too inaccessible, some of the following sampled-data systems can satisfy it.

(1) Nonlinear sampled-data systems of which point $(e(k), f(e(k)))$ traces the same points on the nonlinear curve (depending on if error $e(k)$ tends to increase or decrease) belongs to Class $\mathbf{S}_c$.

The response of sampled-data systems of which point $(e(k), f(e(k)))$ exactly traces the same point on the nonlinear curve seldom occurs. In general, the fulfillment of **(1)** can be expected from systems that are similar to continuous-time ones, which are characterized by a very short sampling period h or a very slow response which is regarded as $\Delta e(k) \to 0$.

(2) Nonlinear sampled-data systems (which satisfy $\epsilon(N) = 0$ at any step N, i.e., $f_0(k) = 0$ $(k = 1, 2, \cdots, N)$) are classified into Class $\mathbf{S}_l$.

The condition $f_0(k) = 0$ is established when $\gamma(k)$ of Eq. (3.) becomes a positive constant γ. In other words, nonlinear sampled-data systems which belong to $\mathbf{S}_l$ refer to linear sampled-data systems.

(3) Nonlinear sampled-data systems (which satisfy $\epsilon(N) \geq 0$, at any step N, i.e., $f_0(k) \cdot \Delta e(k) \geq 0$ $(k = 1, 2, \cdots, N)$) are classified into Class $\mathbf{S}_r$.

The fulfillment of **(3)** is expected from systems where response $(e(k), f(e(k)))$ on a nonlinear curve turns in a clockwise direction. The systems in Class $\mathbf{S}_r$ naturally contain the above-mentioned systems of Class $\mathbf{S}_l$ which satisfies $\epsilon(N) = 0$.

5. ROBUST STABILITY FOR DISCRETE-TIME SYSTEMS

As was described in our previous paper (e.g., Okuyama and Takemori [5, 8]), (when using the subsystem in Fig. 3 instead of nonlinear element $n(\cdot)$ in Fig. 2), the robust stability condition for the above system can be given by using a small gain theorem in regard to the closed loop system as shown in Fig. 4.

[Theorem-1] If there exists a $q \geq 0$ in which the sector parameter α in regard to nonlinear term $n(\cdot)$ satisfies the following inequality, then the nonlinear sampled-data control system shown in Fig. 1 (equivalent to Fig. 2) is robust stable in the ℓ_2 sense:

$$\frac{U^2 + V^2}{-q\Omega V + \sqrt{q^2\Omega^2 V^2 + (U^2 + V^2)\{(1 + U)^2 + V^2\}}}$$

$$:= \xi(q, \omega) < \frac{1}{\alpha}, \quad \forall \omega \in [0, \omega_c]. \tag{17}$$

Here, $\Omega(\omega)$ is the distorted frequency of ω, and is given as

$$\delta(e^{j\omega h}) = j\Omega(\omega) = j\frac{2}{h}\tan\left(\frac{\omega h}{2}\right)$$

from Eq. (7), and ω_c is a cut-off frequency which is the range satisfying *Shannon's sampling theorem.* Moreover, U and V are the real and the imaginary parts of $KG(e^{j\omega h})$, respectively.

(Proof) The proof is obtained from inequality

$$|F(\alpha, q, e^{j\omega h})| = \left|\frac{(1 + jq\Omega(\omega))KG(e^{j\omega h})}{1 + (1 + j\alpha q\Omega(\omega))KG(e^{j\omega h})}\right| < \frac{1}{\alpha}. \tag{18}$$

based on Eq. (8). □

Theorem-1 corresponds to a discrete-time version of Popov's criterion (Netushil ed.[9], Desoer and Vidyasagar[10]). Since inequality (17) in Theorem-1 is for all ω considered and a certain q, the condition results in the following min-max problem:

$$\xi(q_0, \omega_0) = \min_q \max_\omega \xi(q, \omega) < \frac{1}{\alpha}. \tag{19}$$

That is, if inequality (19) is satisfied, the nonlinear sampled-data system as shown in Fig. 1 is stable when the nominal linear sampled-data system with gain K is stable.

6. INSTABILITY CONDITION

On the contrary, in this section, the instability problem of the nonlinear discrete-time system is examined, when the nominal system with gain K is unstable(Desoer and Vidyasagar [10], Harris and Valenca[11]). Consider the frequency transfer function $F(\alpha, q, e^{j\omega h})$ to be a linear causal operator $\mathcal{F}$ in an ℓ_2 space, i.e., $\mathcal{F}: \ell_2 \to \ell_2$. In addition, $\mathcal{F}$ is assumed to be unstable in the sense that

$$\mathcal{U} = \{u'_m \in \ell_2 \mid v'_m = \mathcal{F}u'_m \in \ell_2\} \tag{20}$$

is not all of ℓ_2. Obviously, $\mathcal{U}$ is a set of stabilizable inputs u'_m (which is a subspace of ℓ_2). Here, u'_m and v'_m are neutral points of sequences $u'(k)$ and $v'(k)$, respectively. (In order to avoid complicated expressions, step k will be abbreviated hereafter.)

Since $\mathcal{U}$ is not all of ℓ_2, the orthogonal subspace of it, $\mathcal{U}^\perp$, is nontrivial in the ℓ_2 space. If exogenous input d'_m exists in the orthogonal subspace (i.e., $d'_m \in \mathcal{U}^\perp$), $\langle u'_m, d'_m\rangle_N = 0$ must hold.

In such a case, from the relation $w^* = u' - d'$, the following holds:

$$\|w^*_m\|^2_N = \|u'_m\|^2_N - 2\langle u'_m, d'_m\rangle_N + \|d'_m\|^2_N,$$
$$= \|u'_m\|^2_N + \|d'_m\|^2_N.$$

Hence,

$$\|w^*_m(k)\|_N \geq \|u'_m(k)\|_N. \quad (21)$$

Furthermore, when considering e^* as a stabilizable input, the following set is given:

$$\mathcal{E} = \{e^*_m \in \ell_2 \mid v'_m = \mathcal{F}(w^*_m + d'_m) \in \ell_2\} \quad (22)$$

Since $\mathcal{E}$ is similarly not all of ℓ_2, the orthogonal subspace of it, $\mathcal{E}^\perp$, is nontrivial in the ℓ_2 space. If exogenous input r'_m exists in the orthogonal subspace (i.e., $r'_m \in \mathcal{E}^\perp$), $\langle r'_m, e^*_m\rangle_N = 0$ must hold.

From the relation $v' = r' - e^*$, the following holds:

$$\|v'_m\|^2_N = \|r'_m\|^2_N - 2\langle r'_m, e^*_m\rangle_T + \|e^*_m\|^2_T,$$
$$= \|r'_m\|^2_N + \|e^*_m\|^2_N.$$

Hence,

$$\|v'_m(k)\|_N \geq \|e^*_m(k)\|_N. \quad (23)$$

By using inequalities (12), (21) and (23), the following relation can be obtained:

$$\|u'_m(k)\|_N \leq \alpha\|v'_m(k)\|_N. \quad (24)$$

Then, inequality (24) can be rewritten as follows:

$$\|u'_m(k)\|_N \leq \alpha \sup_\omega |F(q, \alpha, e^{j\omega h})| \cdot \|u'_m(k)\|_N. \quad (25)$$

However, if a small gain theorem, i.e.,

$$\sup_\omega |F(q, \alpha, e^{j\omega h})| < \frac{1}{\alpha} \quad (26)$$

is satisfied for any $q \geq 0$, the above inequality (25) is contradicted for $N \to \infty$. Thus, the following should be written:

$$u'_m \notin \ell_2 \text{ and } v'_m \notin \ell_2. \quad (27)$$

It is obvious that the nonlinear discrete-time feedback system is unstable.

With respect to such an instability problem, the following theorem can be given.

[Theorem-2] If a small gain theorem (26) in regard to the closed loop system as shown in Fig. 4 is satisfied (i.e., inequalities (17), (19) are satisfied), the nonlinear sampled-data control system shown in Fig. 1 is unstable when the nominal linear discrete-time system with gain K is unstable.

(Proof) The proof would be obvious from the above derivation process. □

7. DESCRIBING FUNCTION

A method of the amplitude dependent stability analysis for actual higher order nonlinear systems is harmonic balance, i.e., describing function. Although the analysis is based on an approximation in the Fourier series expansion, it is still a useful method for designing a nonlinear feedback system. In complex numbers, the describing function is defined as

$$N(A) = \frac{U_1}{A} \cdot e^{j\phi_1},$$

where A is the amplitude of input signal to the nonlinear function,

$$U_1 = \sqrt{a_1^2 + b_1^2}$$

and

$$\phi_1 = -\tan^{-1}\frac{b_1}{a_1}.$$

When considering the above in a discrete-time domain, the following expression can be given:

$$a_1 = \frac{\Delta\theta}{2\pi}\sum_{\theta=-\pi}^{\pi}(u(\theta)\cos\theta + u(\theta+\Delta\theta)\cos(\theta+\Delta\theta)),$$
$$b_1 = \frac{\Delta\theta}{2\pi}\sum_{\theta=-\pi}^{\pi}(u(\theta)\sin\theta + u(\theta+\Delta\theta)\sin(\theta+\Delta\theta)).$$

Here, $\theta = k\omega h$ and $\Delta\theta = \omega h$. By using these equations, describing function, e.g., a_1 can be calculated numerically.

8. NUMERICAL EXAMPLES

[Example-1] Consider the following controlled system:

$$G(s) = \frac{(s+6)}{s(s+1)(s+2)}. \quad (28)$$

The stability region for linear gain K can be given by $0 < K < 1.20$ when the sampling period is $h = 0.2$. In this example, we suppose that the nonlinear characteristic can be given by the following *sigmoid type function*:

$$N(e) = \frac{4}{\pi} \cdot \tan^{-1}(2e(t)). \quad (29)$$

When choosing the nominal gain $K = 0.8$, we can obtain $\min_q \xi(q, \omega_0) = \xi(q_0, \omega_0) = 1.98$ and $\alpha < 0.505$ from Eqs. (17) and (19) in Theorem-1 (i.e., the upper bound of stability region becomes $K^+ = 1.20$). In this case, the Aizerman conjecture for discrete-time system(Okuyama and Takemori[12]) is valid.

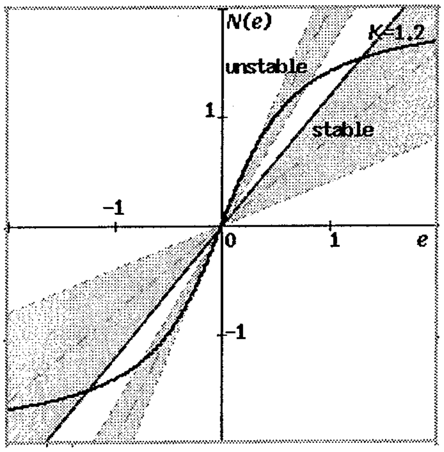

Fig. 5 Nonlinear characteristic and stable/unstable sectors for Example-1 ($h = 0.2$, $K = 0.8$ and upper bound $K^+ = 1.20$, $K = 2.0$ and lower bound $K^- = 1.64$).

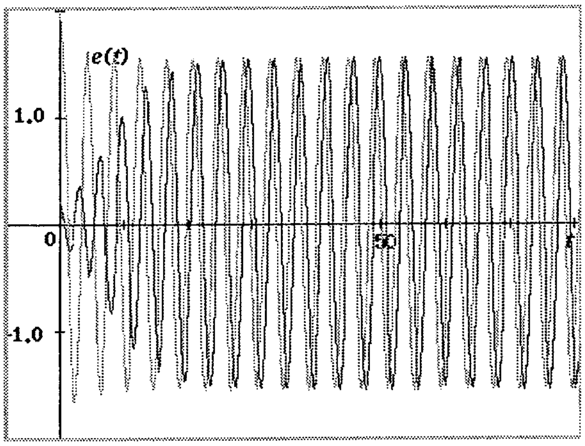

Fig. 6 Error sequence responses for Example-1 ($r = e(0) = 0.2$, $r = e(0) = 2.0$).

On the other hand, when choosing the nominal gain $K = 2.0$, unstable sector $\alpha < 0.179$ can be obtained (i.e., the lower bound of instability region becomes $K^- = 1.64$). There is an area between stable and unstable sectors, which cannot be defined. However, we can predict and estimate a stable (pseudo)periodic oscillation which corresponds to a stable limit cycle for a continuous-time system in a state space.

Figure 6 shows time responses of the nonlinear discrete-time control system. The amplitude of sustained oscillation can approximately be estimated from stable/unstable sectors shown in Fig. 5 and from describing function shown in Fig. 7.

[Example-2] Figure 8 shows the following nonlinear characteristic and stable/unstable sectors:

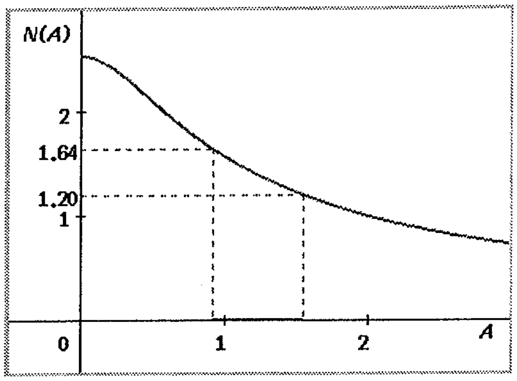

Fig. 7 Describing function for Example-1.

$$N(e) = \frac{4}{\pi} \cdot \tan^{-1}(4e^3(t)). \qquad (30)$$

In this case, the nominal gains were chosen as $K = 0.7$ and $K = 1.7$.

When considering the same controlled system as shown in Example-1, a stable sector is given as $\alpha < 0.72$ (i.e., the upper bound of stability region $K^+ = 1.20$). On the other hand, an unstable sector is given as $\alpha < 0.125$ (i.e., the lower bound of instability region $K^- = 1.49$).

Figure 9 shows time responses of the nonlinear discrete-time control system. In this example, unstable and stable (pseudo)periodic behaviors will be seen in the responses, which corresponds to unstable and stable limit cycles for a continuous-time system in a state space. The describing function of nonlinear characteristic (30) is drawn as shown in Fig. 10. The amplitude of sustained oscillations can be compared with those in Fig. 8 and Fig. 10.

[Example-3] Consider the following controlled system:

$$G(s) = \frac{2.5(s+0.5)}{s(s+2)(s-1)}. \qquad (31)$$

In this example, the stability region can be given by $K > 1.91$ when the sampling period is $h = 0.05$.

When choosing the nominal gain $K = 3.0$, we can obtain $\min_q \xi(q, \omega_0) = \xi(q_0, \omega_0) = 2.75$ and $\alpha < 0.36$ from Eqs. (17) and (19) (i.e., the lower bound of stability region becomes $K^- = 1.91$). On the other hand, when choosing the nominal gain $K = 1.0$, unstable sector $\alpha < 0.21$ can be obtained (i.e., the upper bound of instability region becomes $K^+ = 1.21$). The Aizerman conjecture for discrete-time system is valid

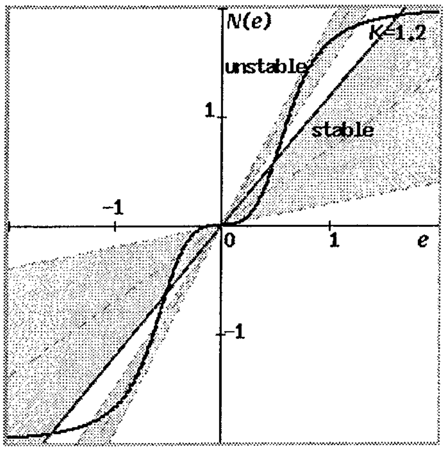

Fig. 8 Nonlinear characteristic and stable/unstable sectors for Example-2 ($h = 0.2$, $K = 0.7$ and upper bound $K^+ = 1.20$, $K = 1.7$ and lower bound $K^- = 1.49$).

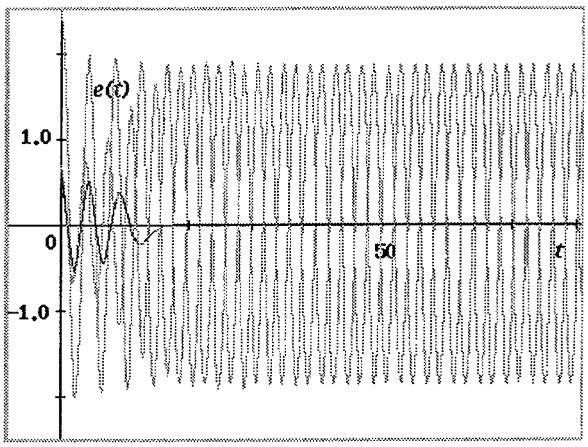

Fig. 9 Error sequence responses for Example-2 ($r = 0.6$, $r = 0.7$, $r = 2.5$).

also in this case. (Counter examples for the Aizerman conjecture were shown our previous papers[4, 12]).

Here, we suppose that the nonlinear characteristic can be given by the following function:

$$N(e) = 0.5e(t) + \tan^{-1}(10e(t)). \qquad (32)$$

Figure 11 shows a nonlinear characteristic and stable/unstable sectors. As is obvious from the figure, there is a considerable size of undefined area between the stable/unstable sectors. However, we can also predict and estimate a stable (pseudo)periodic oscillation which corresponds to a stable limit cycle for a continuous-time system in a state space.

Figure 13 shows the describing function of nonlinear characteristic (32). In this case, describing function $N(A)$ does not reach the lower bound of stability region $K^- = 1.91$ (which corresponds to a necessary and sufficient condition for linear system), because describing function is only an approximation. Nonetheless, the amplitude of sustained oscillation would be compared with those in Fig. 11 and Fig. 13.

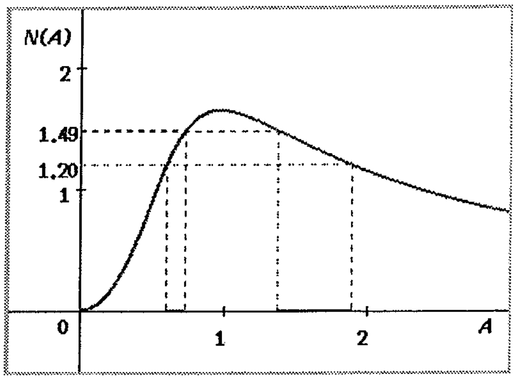

Fig. 10 Describing function for Example-2.

9. CONCLUSIONS

This paper analyzed the amplitude dependent behavior of nonlinear sampled-data (i.e., discrete-time) control systems in a frequency domain. First, the robust stability condition which was derived in our previous papers was applied to a sampled-data control system containing a single time-invariant nonlinear element in the forward path. Then, an instability condition for that type of nonlinear feedback system was derived. By considering restricted areas (two sectors) in the nonlinear characteristic, we could predict and estimate the existence of a sustained oscillation whether it is periodic or not. This concept will be extended to the multi-loop nonlinear discrete-time feedback systems(Okuyama and Takemori[13]).

REFERENCES

[1] Atherton, D. P., Nonlinear Control Engineering, Describing Function Analysis and Design, Van Nostrand Reinhold, 1975.

[2] Vidyasagar, M., Nonlinear Systems Analysis, Prentice-Hall, 1978 & 1993.

[3] Gordillo, F., J. Aracil and A. Ollero "Robust Stability Analysis of MIMO Systems with Asymmetric Nonlinearities," Proc. of the 14th IFAC World Congress, pp. 123-127, 1999.

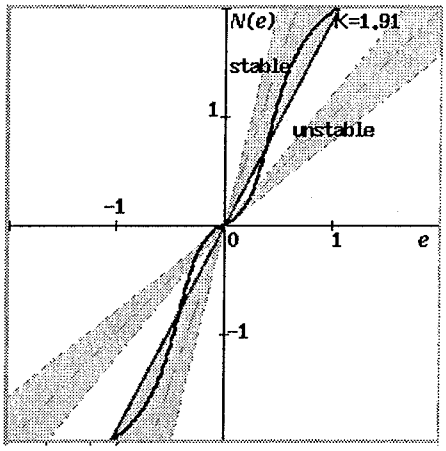

Fig. 11 Nonlinear characteristic and stable /unstable sectors for Example-3 ($h = 0.05$, $K = 1.0$ and upper bound $K^+ = 1.21$, $K = 3.0$ and lower bound $K^- = 1.91$).

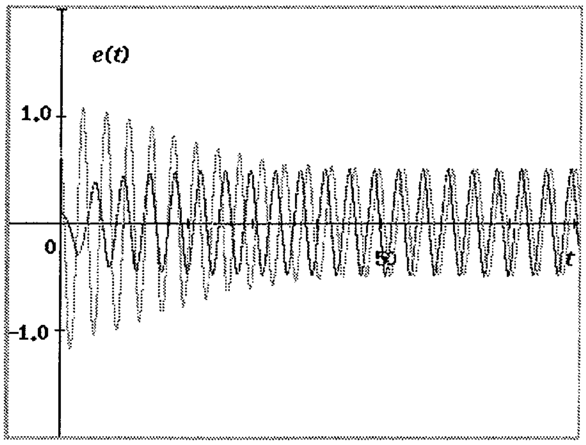

Fig. 12 Error sequence responses for Example-3 ($r = 0.1$, $r = 0.7$).

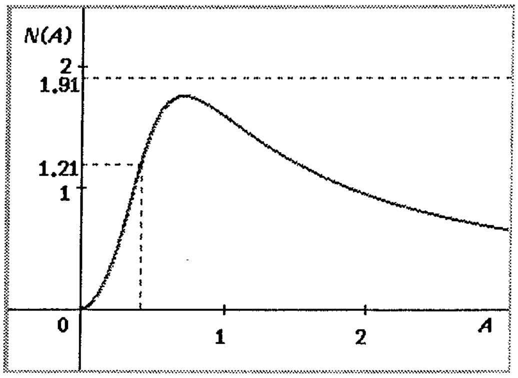

Fig. 13 Describing function for Example-3.

[4] Okuyama, Y., and F. Takemori, "Robust Stability Evaluation of Sampled-Data Control Systems with a Sector Nonlinearity," Proc. of 13th IFAC World Congress (Eds. J. J. Gertler *et al.*), Pergamon Press, Vol. H, pp. 41-46, 1996.

[5] Okuyama, Y. and F. Takemori, "Robust Stability Evaluation for Sampled-Data Control Systems with a Sector Nonlinearity in a Gain-Phase Plane," Int. J. of Robust and Nonlinear Control, J. Wiley & Sons, Vol. 9, No. 1, pp. 15-32, 1999.

[6] Okuyama, Y., and F. Takemori, "On the Robust Stabilization of Sampled-Data Control Systems with a Sector Nonlinearity," Nonlinear Control Systems Design 1998 (Eds. H. J. C. Huijbert *et al.*), Pergamon Press, pp. 279-284, 1998.

[7] Kalman, R. E., "Nonlinear Aspects of Sampled-Data Control Systems," Proc. of the Symp. on Nonlinear Circuit Analysis, pp. 273-313, 1957.

[8] Okuyama, Y., and F. Takemori, "Robust Stability Analysis for Non-Linear Sampled-Data Control Systems in a Frequency Domain," European Journal of Control, HERMES Science Publishing, 2002 (to appear).

[9] Netushil, A. (ed.), Theory of Automatic Control, MIR Publishers, Moscow, 1978.

[10] Desoer, C. A. and M. Vidyasagar, Feedback System: Input-Output Properties, Academic Press, 1975.

[11] Harris, C. J. and J. M. E. Valenca, The Stability of Input-Output Dynamical Systems, Academic Press, 1983.

[12] Okuyama, Y. and F. Takemori, "Robust Stability Analysis for Nonlinear Sampled-Data Control Systems and the Aizerman Conjecture," Proc. of the 37th IEEE International Conference on Decision and Control, pp. 849-852, 1998.

[13] Okuyama, Y. and F. Takemori, "Amplitude Dependent Stability Analysis for Non-Linear Sampled-Data Control Systems in a Frequency Domain," Proc. of the 5th IFAC Symp. on Nonlinear Control Systems Design, St. Petersburg, Russia, pp. 749-754, 2001 (Pergamon, in press).

(Received September 28, 2001)