Kharitonov-like Condition for Characteristic Roots Area of Interval Systems

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Abstract: In actual systems, the physical parameters of plants are uncertain and are accompanied by nonlinearity. The transfer function and the characteristic polynomial should, therefore, be expressed by interval polynomials whether the input-output signals are continuous or discrete time. This paper examines the robust performance of that type of control system, based on the existing area of characteristic roots. In particular, in this paper, a sufficient condition for the roots area which is enclosed by a specified circle on an s-plane will be given by applying the classic Sturm's theorem (division algorithm) to the four corners of a segment polynomial. The result that is obtained by finite calculations in regard to the coefficients of the segment polynomial, can be extended to general interval polynomials with multiple uncertain parameters.

Keywords: Robust performance; interval polynomials; characteristic root area; Sturm theorem; Kharitonov theorem

1. INTRODUCTION

The physical parameters of controlled systems (plants) are uncertain and are accompanied by nonlinearity. The transfer function and the characteristic polynomial should be expressed by interval (polytopic) polynomials whether the input-output signals are continuous or discrete time (Ackermann [1], Barmish [2], Bhattacharyya et al. [3]). With respect to the stability of continuous-time linear system, there is a famous theorem presented by Kharitonov [4].

However, in our previous paper, we applied the classic Sturm's theorem directly to interval polynomials in regard to the existing area of characteristic roots, and derived theorems which correspond to a weak-Kharitonov's theorem based on an assumption for the uncertain systems(Okuyama et al. [5, 6]).

In this paper, by applying Sturm's theorem (a division algorithm) to the four corners of a segment polynomial, we will give a sufficient condition for the characteristic roots area which is enclosed by a specified circle on the s-plane. The concept of finite calculations based on the division algorithm in regard to the coefficients of the segment polynomial will be extended to general interval polynomials with multiple uncertain parameters.

2. INTERVAL POLYNOMIALS

The transfer function of a control system with uncertainty (and nonlinearity) is expressed by interval polynomials. Generally, an interval polynomial of a dynamical system with uncertainties can be written as follows:

$$\tilde{F}(s) = \tilde{a}_0 s^n + \tilde{a}_1 s^{n-1} + \cdots + \tilde{a}_{n-1} s + \tilde{a}_n, \quad (1)$$
$$\tilde{a}_k \in [a_k^-, a_k^+], \quad k = 0, 1, 2, \cdots, n.$$

In general, coefficients of the interval polynomial (1) are not always independent of each other. It can also be written affinely by the following general form with interval coefficients [9, 10]:

$$\tilde{F}(s) = \sum_{k=0}^{n}\left(\sum_{\ell=1}^{m} c_{\ell,k}\tilde{q}_\ell\right) s^{n-k}, \qquad (2)$$

$$= \sum_{\ell=1}^{m}\tilde{q}_\ell\left(\sum_{k=0}^{n} c_{\ell,k}s^{n-k}\right), \qquad (3)$$

$$\tilde{q}_\ell \in [q_\ell^-, q_\ell^+], \quad \ell = 1, 2, \cdots, m,$$

where coefficients $c_{\ell,k}$ are real constants.

The discrimination of the roots area based on the above system with multiple uncertainties becomes a considerably complicated problem. Thus, in this paper, we will analyze the problem by using a set of segment polynomials.

3. SEGMENT POLYNOMIAL

First, the following segment polynomial (i.e., a polynomial with only one interval set coefficient) is considered:

$$\tilde{F}(s) = \tilde{a}_0 s^n + \tilde{a}_1 s^{n-1} + \cdots + \tilde{a}_{n-1}s + \tilde{a}_n, \qquad (4)$$

$$\tilde{a}_h \in [a_h^-, a_h^+], \quad \tilde{a}_k = a_k, \quad k \neq h, \quad h, k = 0, 1, 2, \cdots, n,$$

Then, the following more general form as to Eq. (3) is defined:

$$\tilde{F}(s) = \sum_{\ell=1}^{m}\tilde{q}_\ell\left(\sum_{k=0}^{n} c_{\ell,k}s^{n-k}\right), \qquad (5)$$

$$\tilde{q}_h \in [q_h^-, q_h^+], \quad \tilde{q}_\ell = q_\ell, \quad \ell \neq h, \quad h, \ell = 1, 2, \cdots, m.$$

Here, a_k and q_ℓ (without a mark) indicate fixed coefficients.

In either case, these segment polynomials can be written as the following form:

$$\tilde{F}(s) = F(s,\lambda) = \lambda F^+(s) + (1-\lambda)F^-(s), \qquad (6)$$

$$\lambda \in [0,\ 1].$$

The polynomials at both ends of Eq. (5) are expressed as follows:

$$F^+(s) = \sum_{\ell=1}^{m}\tilde{q}_\ell\left(\sum_{k=0}^{n} c_{\ell,k}s^{n-k}\right), \qquad (7)$$

$$\tilde{q}_h = q_h^+, \quad \tilde{q}_\ell = q_\ell, \quad \ell \neq h,$$

$$F^-(s) = \sum_{\ell=1}^{m}\tilde{q}_\ell\left(\sum_{k=0}^{n} c_{\ell,k}s^{n-k}\right), \qquad (8)$$

$$\tilde{q}_h = q_h^-, \quad \tilde{q}_\ell = q_\ell, \quad \ell \neq h.$$

As for Eq. (2), the following expression can be given:

$$F^+(s) = \sum_{k=0}^{n} a_k^+ s^{n-k}, \qquad (9)$$

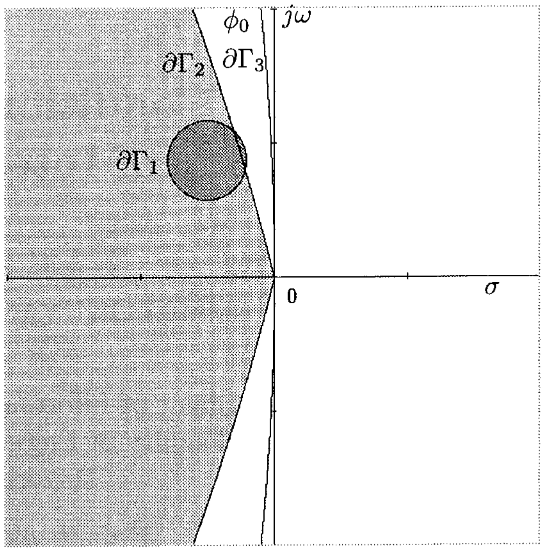

Fig. 1 Sectorial and circular contours and areas.

$$F^-(s) = \sum_{k=0}^{n} a_k^- s^{n-k}, \qquad (10)$$

where

$$a_k^+ = \sum_{\ell=1,\ell\neq h}^{m} c_{\ell,k}q_\ell + c_{h,k}q_h^+, \qquad (11)$$

$$a_k^- = \sum_{\ell=1,\ell\neq h}^{m} c_{\ell,k}q_\ell + c_{h,k}q_h^-. \qquad (12)$$

Here, without loss of generality, we assume $c_{h,k} > 0$.

In these segment polynomials, when considering the algebraic equation $\tilde{F}(s) = 0$, segments of the characteristic root locus can be drawn on the s-plane. On the other hand, when considering mapping $\tilde{F}(s)$ for a contour $s \in \partial\Gamma$ as shown in Fig. 1, a set of line segments will be drawn on a complex F-plane as shown in Fig. 2.

4. CIRCULAR CONTOUR

In this paper, we examine a sufficient condition for the number of characteristic roots in a specified area on the s-plane. As a specified area, we consider a contour $\partial\Gamma$ (boundary of area Γ) on the s-plane, e.g., a circular contour

$$s = \rho e^{j\theta} + \sigma_0 + j\omega_0, \qquad (13)$$

$$\theta : -\pi \rightarrow \pi.$$

as shown in Fig. 1. Here, ρ, (σ_0, ω_0) and θ are the radius the center and the angle of rotation for the specified circle, respectively.

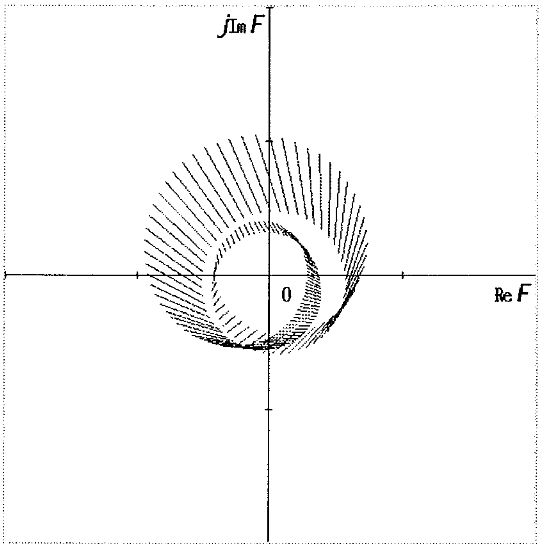

Fig. 2 Locus of line segments.

Any point s on circular contour (13) can also be written by the following rational function of real variable α:

$$s = \frac{u + jv\alpha}{1 - j\alpha}, \tag{14}$$

where

$$u = \rho + \sigma_0 + j\omega_0,$$
$$v = \rho - \sigma_0 - j\omega_0$$

Here, α can be considered correspondingly as follows:

$$\alpha = \tan(\theta/2). \tag{15}$$

Obviously, the relationship between variable θ and variable α is expressed as:

$$\theta = -\pi; \quad \alpha = -\infty,$$
$$\theta = 0; \quad \alpha = 0,$$
$$\theta = +\pi; \quad \alpha = +\infty.$$

Circular contours (13) and (14) includes the following extreme case.

(a) In Eqs. (13) and (14), when we consider

$$\rho = R, \quad \sigma_0 = -R \quad (\text{or} \quad \sigma_0 = R), \quad \omega_0 = \epsilon$$

and $R \to \infty$, $\epsilon \to 0$, the specified area becomes a left half (or right half) s-plane ($\partial\Gamma_3$ as shown in Fig. 1). In this case, a part of the circular contour approaches the imaginary axis on the s-plane, provided that variable ω is considered in a certain limited range $\omega_c < \omega < \omega_c$, where ω_c is a cutoff frequency. Thus, variable α becomes a small number, i.e.,

$$\alpha = \frac{\omega}{2R} \ll 1.$$

(b) Similarly, when the parameters are chosen as

$$\rho = R, \quad \sigma_0 = -R, \quad \omega_0 = R\tan\phi_0, \quad R \to \infty,$$

the specified area becomes a problem of a sectorial area in the left half s-plane ($\partial\Gamma_2$ as shown in Fig. 1).

As mentioned above, since Eqs. (13) and (14) can include most of the problems, the following shows the discrimination of the number of roots in a circular area.

When applying the above transformation (14) to Eq. (5), the following numerator polynomials for real variable α can be obtained:

$$\tilde{\Phi}(j\alpha) = (1 - j\alpha)^n \tilde{F}(s)$$
$$= \sum_{\ell=1}^{m} \tilde{q}_\ell \left(\sum_{k=0}^{n} c_{\ell,k}(u + jv\alpha)^{n-k}(1 - j\alpha)^k \right) \tag{16}$$

Here, in the case of segment polynomials, these coefficients are related from Eqs. (11) and (12) as follows:

$$\tilde{a}_k = [a_k^-,\ a_k^+] = \sum_{\ell=1,\ell\neq h}^{m} c_{\ell,k} q_\ell + c_{h,k}[q_h^-,\ q_h^+]. \tag{17}$$

Since Eq. (16) are polynomials with complex coefficients, they can be written by the following form:

$$\tilde{\Phi}(j\alpha) = (1 - j\alpha)^n \tilde{F}(s) = \tilde{P}(\alpha) + j\tilde{Q}(\alpha), \tag{18}$$

where

$$\tilde{P}(\alpha) = \tilde{a}_{0,0}\alpha^n + \cdots + \tilde{a}_{0,n-1}\alpha + \tilde{a}_{0,n}, \tag{19}$$
$$\tilde{Q}(\alpha) = \tilde{b}_{0,0}\alpha^n + \cdots + \tilde{b}_{0,n-1}\alpha + \tilde{b}_{0,n}. \tag{20}$$

The coefficients in Eqs. (19) and (20) can be determined as follows:

$$\tilde{a}_{0,\nu} + j\tilde{b}_{0,\nu}$$
$$= \sum_{k=0}^{n} \sum_{l=k-\nu}^{n-\nu} \tilde{a}_k \binom{n-k}{n-\nu-l} \binom{k}{l} (-1)^l j^{n-\nu} u^{\nu-k+l} v^{n-\nu-l},$$
$$\nu = 0, 1, \cdots, n,$$

where

$$\binom{k}{l} = {}_kC_l$$

denotes the combination sign.

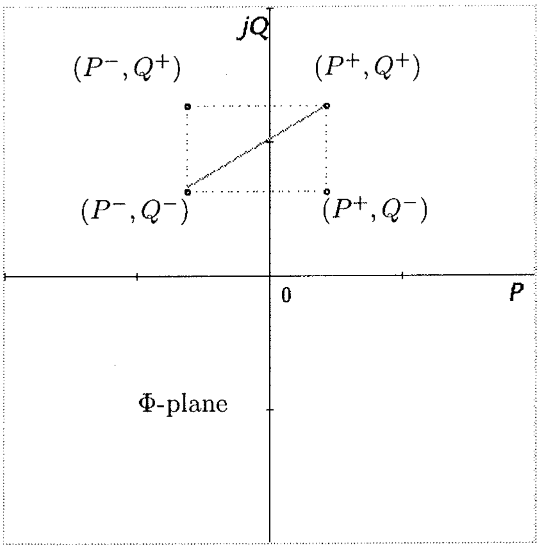

Fig. 3 Four corner points and rectangle.

5. FOUR CORNER POINTS PROBLEM

In case of segment polynomials, by using expression (6) Eq. (18) is rewritten as follows:

$$\begin{aligned}\tilde{\Phi}(j\alpha) &= \Phi(j\alpha,\lambda) = (1-j\alpha)^n(\lambda F^+(s) + (1-\lambda)F^-(s)) \\ &= (\lambda P^+ + (1-\lambda)P^-) + j(\lambda Q^+ + (1-\lambda)Q^-). \qquad (21)\end{aligned}$$

The real and imaginary parts of Eq. (21) correspond to those of Eq. (18), i.e.,

$$\tilde{P}(\alpha) = \lambda P^+(\alpha) + (1-\lambda)P^-(\alpha),$$
$$\tilde{Q}(\alpha) = \lambda Q^+(\alpha) + (1-\lambda)Q^-(\alpha).$$

Here, the extreme polynomials are expressed as follows:

$$\begin{aligned}P^+(\alpha) &= a_{0,0}^+\alpha^n + \cdots + a_{0,n-1}^+\alpha + a_{0,n}^+, \\ Q^+(\alpha) &= b_{0,0}^+\alpha^n + \cdots + b_{0,n-1}^+\alpha + b_{0,n}^+, \qquad (22)\end{aligned}$$

and

$$\begin{aligned}P^-(\alpha) &= a_{0,0}^-\alpha^n + \cdots + a_{0,n-1}^-\alpha + a_{0,n}^-, \\ Q^-(\alpha) &= b_{0,0}^-\alpha^n + \cdots + b_{0,n-1}^-\alpha + b_{0,n}^-. \qquad (23)\end{aligned}$$

Thus, the following four corner points (vertices) can be given, and a rectangle together with a line segment (edge) can be drawn in the Φ-plane as shown in Fig. 3:

$$\begin{aligned}\mathbf{V}_1 &= (P^+,\ Q^+), \\ \mathbf{V}_2 &= (P^-,\ Q^-), \\ \mathbf{V}_3 &= (P^-,\ Q^+), \\ \mathbf{V}_4 &= (P^+,\ Q^-),\end{aligned}$$

where the latter two points are additional ones.

Then, we can define the following four pairs of polynomials:

$$\begin{aligned}P^{(i)}(\alpha) &= a_{0,0}^{(i)}\alpha^n + \cdots + a_{0,n-1}^{(i)}\alpha + a_{0,n}^{(i)}, \qquad (24) \\ Q^{(i)}(\alpha) &= b_{0,0}^{(i)}\alpha^n + \cdots + b_{0,n-1}^{(i)}\alpha + b_{0,n}^{(i)}, \qquad (25) \\ &i = 1,2,3,4,\end{aligned}$$

where

$$\begin{aligned}P^{(1)}(\alpha) &= P^+(\alpha), & Q^{(1)}(\alpha) &= Q^+(\alpha), \\ P^{(2)}(\alpha) &= P^-(\alpha), & Q^{(2)}(\alpha) &= Q^-(\alpha), \\ P^{(3)}(\alpha) &= P^-(\alpha), & Q^{(3)}(\alpha) &= Q^+(\alpha), \\ P^{(4)}(\alpha) &= P^+(\alpha), & Q^{(4)}(\alpha) &= Q^-(\alpha).\end{aligned}$$

As an expression of polynomials with complex coefficients, they can be given as follows:

$$\Phi^{(i)}(j\alpha) = P^{(i)}(\alpha) + jQ^{(i)}(\alpha), \qquad (26)$$
$$i = 1,2,3,4,$$

(Note that as for the edges in the F-plane, two additional polynomials with constant coefficients cannot be determined in general.)

Here, we can see that the following lemma holds by using Sturm's theorem.

[Lemma-1] When coefficient ratios

$$\frac{b_{0,0}^{(i)}}{a_{1,1}^{(i)}},\ \frac{b_{1,1}^{(i)}}{a_{2,2}^{(i)}},\ \cdots,\ \frac{b_{n-1,n-1}^{(i)}}{a_{n,n}^{(i)}} \qquad (27)$$

are calculated for an extreme polynomial $\Phi^{(i)}(\alpha)$ ($i = 1,2$) (e.g., $\Phi^{(1)}(\alpha)$), the number of ratios (27) to be negative μ is equal to the number of characteristic roots for the polynomial in the specified circle. In these ratios (27), $a_{q,q}^{(i)}$, $b_{q-1,q-1}^{(i)}$ ($q = 1,2,\cdots,n$) are caluculated by using a division algorithm .

(Proof) This lemma is a necessary and sufficient condition in regard to the existing area of characterisitic roots for the fixed polynomial. In our previous paper (Okuyama et al. [6, 7]), the proof will be given systematically by using Sturm's theorem. □

Based on the above premise, the following theorem is obtained as to the above four pairs of polynomials:

[Theorem-1] If the number of ratios (27) to be negative is not changed for the four corner polynomials, the control system that is characterized by segment polynomial Eq. (4) (or (5)) has a robust performance in regard to the invariance of the number of characteristic roots in the specified circle. When considering

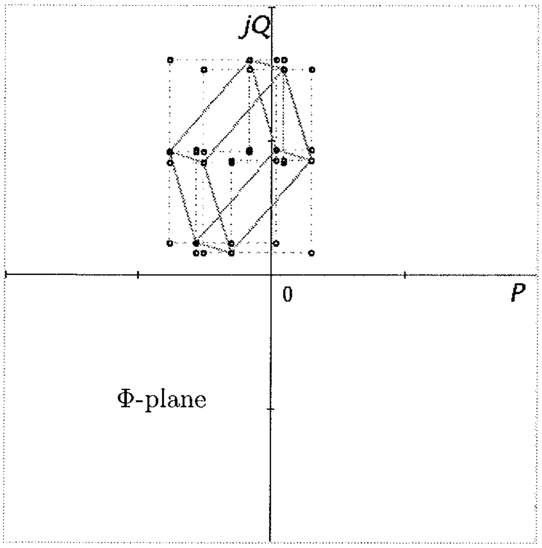

Fig. 4 Parallelotope and rectangles.

only one root (i.e., $\mu = 1$), for instance, a dominant root, the circle $\partial\Gamma$ (i.e., disc Γ) gives a sufficient condition for the characterisic root area of the control system with an uncertain (interval set) parameter.

(Proof) This theorem is a sufficient condition in regard to the existing area of characterisitic roots for the segment polynomial. The proof is obvious from the zero exclusion of the Kalitonov-like rectangle that is composed of the four corner points (24) and (25). That is, any edge of the rectangle does not pierce the origin. As a natural consequence, the line segment in the Φ-plane and also in the F-plane does not pierce the origin. □

6. MULTIPLE UNCERTAINTIES

Theorem-1 can also be applied to control systems with multiple uncertainties, the characteristic polynomials of which are written as shown in Eqs (1) and (2), in general. When complex variable s is fixed (frozen), a view of (hyper)polyhedron (a parallelotope) is drawn on the s-plane in regard to the polynomial, e.g., (2) as shown in Fig 4.

As for polynomials expressed by Eqs. (2) and (3) the number of vertices is 2^m, and the number of edges becomes $m \cdot 2^{m-1}$. Obviously, the number of additional vertces is given by $2 \times m \cdot 2^{m-1}$. Thus, the number of total vertices which should be checked for interval polynomials (2) and (3) is given by

$$2m + m \cdot 2^m = (m+1) \cdot 2^m. \qquad (28)$$

Based on the above premise, the following theorem is derived as to the above number (28) of corner polynomials:

[Theorem-2] If the number of ratios (27) to be negative is not changed for all the corner polynomials (28), the control system that is characterized by interval polynomial Eq. (2) (consequently, Eq. (1)) has a robust performance in regard to the invariance of the number of characteristic roots in the specified circle. When considering only one root (i.e., $\mu = 1$), for instance, a dominant root, the circle $\partial\Gamma$ (i.e., disc Γ) gives a sufficient condition for the characterisic root area of the control system with uncertain (interval set) parameters.

(Proof) This theorem is a sufficient condition in regard to the existing area of characterisitic roots for the interval polynomial. The proof is obvious from the result in Theorem-1 in which any edge of the rectangles does not pierce the origin. Consequently, as mentioned in the proof of Theorem-1, any edge of the parallelotope (a set of line segments) in the Φ-plane and in the F-plane does not pierce the origin (Bartlett et al. [8]). □

Although the above results (Theorem-1,2) are only a sufficient condition (i.e., sufficient for edge theorems), the discrimination method proposed in this paper will be useful in the design of robust control systems. The realization robust performance via model reference feedback was presented in our previous paper (Takemori and Okuyama [9], Okuyama and Takemori [10]).

7. NUMERICAL EXAMPLE

[Example] Consider the following characteristic equation expressed by interval polynomial:

$$\tilde{F}(s) = s^3 + 2s^2 + 2s + 1 + [-0.1, 0.1](s^3 + 2s^2) + [-0.2, 0.2]s + [-0.1, 0.1] = 0. \qquad (29)$$

When a circle with a center of $(-0.5, 0.9j)$ and a radius of $r = 0.3$ was specified as shown in Fig. 5, the parallelotopes and rectangles with $(m+1) \cdot 2^m = 32$ corners are drawn in the Φ-plane as shown in Fig. 6. The series of corner 32 points (sets of rectangles) slightly excludes the origin. The number of roots in the specified circle did not change. That is, the robust performance was guaranteed in regard to the existing area of dominant root for the interval system.

When considering a sectorial area as shown in Fig. 5, $\mu = 3$ was obtained as to the 32 corner polynomials.

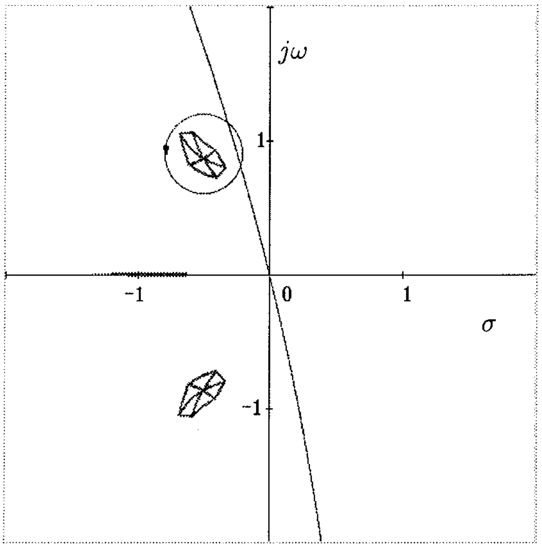

Fig. 5 Circles and root areas.

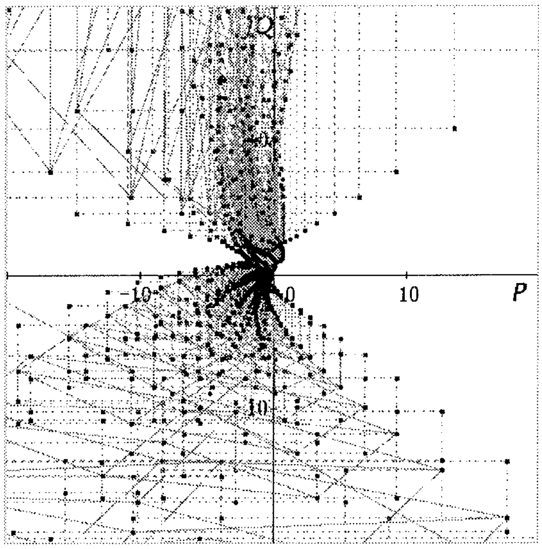

Fig. 6 Parallelotopes, four corners and rectangles.

It could be shown that the robust stability of the system characterized by Eq. (29) is guaranteed.

8. CONCLUSIONS

In this paper, by applying Sturm's theorem (division algorithm) to the four corners of a segment polynomial, we gave a sufficient condition for the characteristic roots area which is enclosed by a specified circle on the s-plane. The concept of finite calculations based on the division algorithm in regard to the coefficients of the segment polynomial was extended to general interval polynomials with multiple uncertain parameters. This four points (Kharitonov-like rectangle) condition proposed in this paper will be useful for Computer Aided Control Systems Design.

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