An Off-Axis Circle Criterion for the Stability of Feedback Systems with a Sector Nonlinearity

Yoshifumi OKUYAMA and Fumiaki TAKEMORI

Department of Information and Knowledge Engineering
Faculty of Engineering, Tottori University
Tottori, 680-8552 Japan

E-mail: oku@ike.tottori-u.ac.jp

Abstract: This paper describes a geometrical evaluation of the robust stability in a frequency domain based on the results from our previous papers in which Popov's criterion was expressed in an explicit form. The control system described herein is a feedback system with one time-invariant nonlinear element (a sector nonlinearity) in the forward path. By applying a small gain theorem that concerns L_2 gain in regard to a nonlinear subsystem with a free parameter, a robust stability condition for control systems with time-invariant nonlinearity is presented. Using this concept, we will show a representation of an off-axis circle criterion on a Nyquist diagram, and propose an evaluation method of the stability from the relative position with the vector locus of the open loop frequency response characteristic. This type of a diagram will be available to the design of robust control systems.

Key words: Robust stability; nonlinear control systems; Popov criterion; Nyquist diagram; gain margins

1. INTRODUCTION

A small gain theorem with a L_2 norm is generally applied for the robust stability condition in a frequency domain for control systems with uncertainty and nonlinearity. It can also be applied to time-varying nonlinearity or frequency dependent uncertainty and has been widely used as a design technique for H_∞ robust control systems (Vidyasagar [1], Francis [2]). This concept has been extended in order to solve design problems associated with several uncertainties and/or nonlinearities (Packerd [3]). However, the stability theory in regard to L_2 norm which was proposed by Sandberg [4], consequently, imposes more conservative restrictions on the frequency response characteristics of the linear parts of a control system (Zames [5], Desoer and Vidyasagar [6],Harris and Valenca [7]).

In our previous paper, a robust stability condition for control systems with a time-invariant nonlinearity was given (Okuyama et al. [8, 9, 10]) by applying the small gain theorem to a nonlinear subsystem with a free parameter. It can be considered as an explicit expression of Popov's criterion.

This paper describes the relationship between the robust stability condition and Popov's criterion, and presents a geometrical evaluation of the robust stability in a frequency domain. As a geometrical evaluation method, an off-axis circle criterion on a Nyquist diagram which corresponds to the Hall diagram (M, N circles) will be presented in this paper. Numerical examples show that the diagram will be available to the design of robust control systems.

2. CONTROL SYSTEMS WITH A SECTOR NONLINEARITY

Consider a nonlinear feedback system as shown in Fig. 1. Here, $G(s)$ indicates the time-invariant linear characteristic, the frequency response characteristic

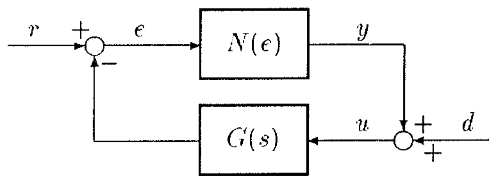

Fig. 1 Nonlinear feedback control system.

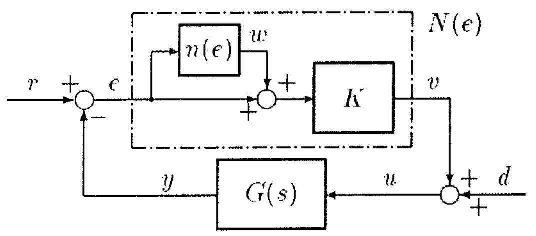

Fig. 2 Equivalent nonlinear control system.

of which is known. Even if $G(s)$ is uncertain, when the band of both the real and imaginary parts of the frequency response characteristics (or the characteristic locus in a worst case) is taken into account, the following discussion is still applicable.

Assume that $N(e)$ is the time-invariant zero-memory-type nonlinearity characteristic, which can be written as follows:

$$N(e) = K(e + n(e)), \tag{1}$$

$$|w| = |n(e)| \leq \alpha|e|. \tag{2}$$

where $n(e)$ is a nonlinear term relative to nominal linear gain K, in other words, a multiplicative perturbation expression. Without loss of generality, we also assume $0 \leq \alpha \leq 1$.

By rearranging the nonlinear control system, Fig. 2 can be obtained. For this nonlinear term $n(e)$, we will suppose a subsystem as shown in Fig. 3 (Okuyama et al. [9, 10]). Here, q is a nonnegative free parameter. As is obvious from the figure, the following equation is obtained:

$$e^* = e + q\frac{de}{dt}, \tag{3}$$

$$w^* = w - \alpha q\frac{de}{dt}. \tag{4}$$

Hence, the following lemma can be given.

[Lemma-1] If the following inequality is satisfied:

$$\left\langle w + \alpha e, \frac{de}{dt}\right\rangle_T = \int_{e(0)}^{e(T)} (n(e) + \alpha e)de \geq 0, \tag{5}$$

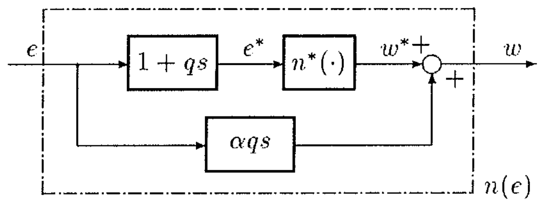

Fig. 3 Nonlinear subsystem.

for any $q \geq 0$,

$$\|w_T^*\| \leq \alpha\|e_T^*\| \tag{6}$$

is obtained. Here, the inner product and the norm is defined as

$$\langle x_1, x_2\rangle_T = \int_0^T x_1(t)x_2(t)dt, \quad \|x\|_T = \sqrt{\int_0^T |x(t)|^2 dt}.$$

When written as $\langle\cdot,\cdot\rangle$ or $\|\cdot\|$, it denotes the case when $T \to \infty$. Moreover, $w_T^*(t)$ and $e_T^*(t)$ denote the truncation functions of $w^*(t)$ and $e^*(t)$ at $t = T$, respectively.

(Proof) From Eqs. (3) and (4), the following equation holds:

$$\alpha^2\|e^*\|_T^2 - \|w^*\|_T^2 = \alpha^2\left\|e + q\frac{de}{dt}\right\|_T^2 - \left\|w - \alpha q\frac{de}{dt}\right\|_T^2$$

$$= \alpha^2\|e\|_T^2 - \|w\|_T^2 + 2\alpha q\left\langle w + \alpha e, \frac{de}{dt}\right\rangle_T$$

Based on sector nonlinear characteristics in Eq. (2) and the premise of this Lemma Eq. (5), i.e.,

$$\|w\|_T^2 \leq \alpha^2\|e\|_T^2,$$

$$\left\langle w + \alpha e, \frac{de}{dt}\right\rangle_T \geq 0,$$

as for $q \geq 0$ and $\alpha > 0$ the following inequaliy:

$$\|w^*\|_T^2 \leq \alpha^2\|e^*\|_T^2, \quad \text{i.e.,} \quad \|w_T^*\|^2 \leq \alpha^2\|e_T^*\|^2$$

can be obtained for any truncation time T. Here, we assume $e(0) = 0$. □

Incidentally, Eq. (6) may be appropriate for $T \to \infty$. The details of this Lemma were described in our previous papers for a continuous-time system[8, 9, 10] and for a discrete time system[11, 13, 14].

3. ROBUST STABILITY CONDITION

By placing nonlinear subsystem $n(e)$ of Fig. 3 inside nonlinear part $n(e)$ of the control system in Fig. 2, the loop transfer function from w^* to e^* can be expressed as follows:

$$H(\alpha, q, s) = \frac{(1 + qs)KG(s)}{1 + (1 + \alpha qs)KG(s)}. \tag{7}$$

Hence, by applying the small gain theorem in regard to L_2 gains, the following robust stability condition can be obtained:

$$\left|\frac{(1+jq\omega)KG(j\omega)}{1+(1+j\alpha q\omega)KG(j\omega)}\right| < \frac{1}{\alpha}. \quad (8)$$

If the open loop transfer function $KG(j\omega)$ is expressed as

$$KG(j\omega) = U(\omega) + jV(\omega), \quad (9)$$

Eq. (8) is also written as

$$\left|\frac{(1+jq\omega)(U(\omega)+jV(\omega))}{1+(1+j\alpha q\omega)(U(\omega)+jV(\omega))}\right| < \frac{1}{\alpha}. \quad (10)$$

The robust stability condition for (10) can be rewritten as the following theorem.

[Theorem-1] For any $q \geq 0$, if nonlinear sector α satisfies the following inequality, the control system of Fig. 1 is robust stable:

$$\xi(q,\omega) = \frac{U^2+V^2}{-q\omega V + \sqrt{q^2\omega^2V^2+(U^2+V^2)\{(1+U)^2+V^2\}}} < \frac{1}{\alpha}, \quad \forall\omega. \quad (11)$$

(Proof) From the square of both sides of inequality (10),

$$\alpha^2(1+q^2\omega^2)(U^2+V^2) < (1+U-\alpha q\omega V)^2 + (V+\alpha q\omega U)^2.$$

The following quadratic inequality is obtained:

$$\alpha^2(U^2+V^2) + 2\alpha q\omega V - \{(1+U)^2+V^2\} < 0 \quad (12)$$

Consequently, as a solution of inequality (12)

$$\alpha < \frac{-q\omega V + \sqrt{q^2\omega^2V^2+(U^2+V^2)\{(1+U)^2+V^2\}}}{U^2+V^2}$$

is given. □

It can be shown that Eq. (11) in Theorem-1 is equivalent to Popov's criterion and contains the circle criterion for nonlinear time-varying systems in a special case.

Obviously, inequality (8) is rewritten as follows:

$$\left|\frac{\alpha\hat{H}(\alpha,q,j\omega)}{1+\alpha\hat{H}(\alpha,q,j\omega)}\right| < 1, \quad (13)$$

where

$$\hat{H}(\alpha,q,j\omega) = \frac{(1+jq\omega)KG(j\omega)}{1+(1-\alpha)KG(j\omega)}.$$

From this inequality, we can obtain

$$2\alpha\cdot\Re\{\hat{H}(\alpha,q,j\omega)\} + 1 > 0. \quad (14)$$

Thus, we can give the following robust stability condition:

$$\Re\left\{\frac{1+(1+\alpha)KG(j\omega)+2j\alpha q\omega KG(j\omega)}{1+(1-\alpha)KG(j\omega)}\right\} > 0, \quad (15)$$

which is equivalent to (11).

If we can determine $\alpha = 1$ in regard to the system and the nonlinear characteristic is expressed as

$$0 \leq N(e)e \leq K_m e^2, \quad K_m = 2K, \quad (16)$$

inequalities (14) and (15) can be written simply as

$$\Re\left\{\frac{1}{K_m} + (1+jq\omega)G(j\omega)\right\} > 0. \quad (17)$$

Inequality (17) corresponds to a well known expression of Popov's criterion.

As is obvious when $q = 0$, the left side of Eq. (11) becomes the absolute value of complementary sensitivity function $T(j\omega)$. Therefore, the condition can be written as

$$\xi(0,\omega) = \frac{\sqrt{U^2+V^2}}{\sqrt{(1+U)^2+V^2}} = |T(j\omega)| < \frac{1}{\alpha}. \quad (18)$$

On the other hand, from Eq. (15)

$$\Re\left\{\frac{1+(1+\alpha)KG(j\omega)}{1+(1-\alpha)KG(j\omega)}\right\} > 0 \quad (19)$$

is obtained. Eqs. (18) and (19) correspond to the circle criterion for nonlinear time-varying systems.

Theorem-1 is an explicit expression of Popov's criterion, and can be interpreted as follows.

Eq. (11) in Theorem-1 is for all ω considered and a certain q. Therefore, if a min-max of $\xi(q,\omega)$ is obtainable, then Eq. (11) in Theorem-1 can be rewritten as

$$M_0 = \xi(q_0,\omega_0) = \min_q \max_\omega \xi(q,\omega) < \frac{1}{\alpha}, \quad (20)$$

that is, if Eq. (20) is satisfied, the nonlinear control system as shown in Fig. 1 is robust stable (When the nominal linear control system with gain K as shown in Eq. (1) is stable, the nonlinear control system with a sector nonlinearity is also L_2 stable).

4. OFF-AXIS CIRCLES

As is obvious, the following curve on the complex plane (U-V plane):

$$\xi(0,\omega) = M, \quad M = \text{const.} \quad (21)$$

in Eq. (18), corresponds to a M-circle in the Hall diagram. Therefore, the following curve based on Eq. (11):

$$\xi(q,\omega) = M \tag{22}$$

becomes the modified contour of the M-circle. Hereafter, we will assume $M \geq 1$, because $\alpha \leq 1$ in Eqs. (11) and (20).

The modified contours are given by the following lemma (Okuyama and Takemori [14]).

[Lemma-2] When $M > 1$, the modified contours of the M-circle are written as

$$\left(U + \frac{M^2}{M^2-1}\right)^2 + (V-\gamma)^2 = \frac{M^2}{(M^2-1)^2} + \gamma^2, \tag{23}$$

where $\gamma = \dfrac{q\omega M}{M^2-1} \geq 0$. When $M = 1$,

$$2U + 1 = kV, \qquad k = \frac{2q\omega}{M} \geq 0. \tag{24}$$

In these equations (23) and (24), we will assume that γ and k are constant parameters.

(Proof) From Eqs. (11) and (22),

$$(M^2-1)U^2 + 2M^2U + (M^2-1)V^2 + M^2 - 2Mq\omega V = 0. \tag{25}$$

Obviously, when $M = 1$,

$$2U + 1 = \frac{2q\omega}{M} \cdot V$$

is obtained.

On the other hand, when $M > 1$, from Eq. (25),

$$U^2 + \frac{2M^2}{M^2-1}U + V^2 - \frac{2Mq\omega}{M^2-1}V + \frac{M^2}{M^2-1} = 0,$$

then

$$\left(U + \frac{M^2}{M^2-1}\right)^2 + \left(V - \frac{q\omega M}{M^2-1}\right)^2 = \frac{M^2}{(M^2-1)^2} + \left(\frac{q\omega M}{M^2-1}\right)^2.$$

If we use $\gamma = \dfrac{q\omega M}{M^2-1}$, Eq. (23) can be obtained. That is, off-axis circles with their center at $\left(\dfrac{-M^2}{M^2-1}, \gamma\right)$ and with radius of $\sqrt{\dfrac{M^2}{(M^2-1)^2} + \gamma^2}$ are obtained. □

The following theorem is given based on the abovementioned premise.

[Theorem-2] If vector locus $KG(j\omega) = U(\omega) + jV(\omega)$ exists in the following area determined by $q = q_0$:

$$\xi(q_0,\omega) \leq M_0 < \frac{1}{\alpha}, \qquad \forall\omega, \tag{26}$$

the nonlinear control system as shown in Fig. 1 is robust stable.

(Proof) Obviously, as for a certain vector locus $U(\omega)$-$V(\omega)$ of open loop system $KG(j\omega)$,

$$\xi(q,\omega) \leq \xi(q,\omega_0), \qquad \forall\omega \tag{27}$$

is valid in general, because the right side of this inequality is a peak value for angular frequency ω. Furthermore, ω_0 is a peak frequency. Here, we should note that ω_0 is not always determined as only one frequency, and may be only a smooth (differentiable) point of the frequency range depending on the q-value (Okuyama and Takemori [12]).

Nonetheless, inequality (27) holds in regard to $q = q_0$ by which $\xi(q,\omega_0)$ is minimized. Thus the following is satisfied:

$$\xi(q_0,\omega) \leq \xi(q_0,\omega_0) = M_0, \qquad \forall\omega. \tag{28}$$

It can be shown that inequality (26) in Theorem-2 is equivalent to (20). □

5. NUMERICAL EXAMPLES

[Example-1] Consider the following controller and controlled system:

$$G(s) = \frac{1}{s(1+s)^2}, \qquad K = 1.2. \tag{29}$$

Figure 4 shows calculation results for $\xi(q,\omega_0)$. When nominal linear gain $K = 1.2$,

$$M_0 = \min_q \xi(q,\omega_0) = \xi(q_0,\omega_0) = 1.5$$

is obtained. Figure 5 shows calculation results of circle array as to $q \geq 0$ for $M = 1.5$, i.e.,$\alpha = 0.667$ and vector locus G_1 for the controlled system.

In this example, q by which $\xi(q,\omega_0)$ is minimized is $q_0 = 2.0$ and the vector locus contacts with an off-axis circle **C** on the real axis. Here, the gain margin is $g_1 = 4.44$ dB and equals

$$-20\log_{10}\frac{M_0}{M_0+1} = 4.44 \text{ dB},$$

which is determined by the point where the argument of contour **C** is -180 degrees. It corresponds to the size of a sector in which nonlinear characteristics are permitted. That is, it is an example that shows Aizerman's conjecture to be valid.

On the other hand, G_2 is a vector locus for $K = 0.57$ by which the peak value becomes $M = 1.5$. It corresponds to a limit of the robust stability condition which can be applied to a time-varying nonlinear

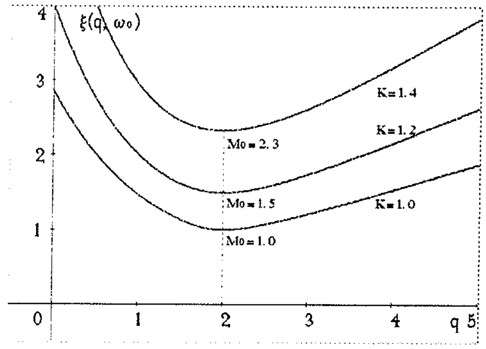

Fig. 4 $\xi(q, \omega_0)$ curves for Example-1.

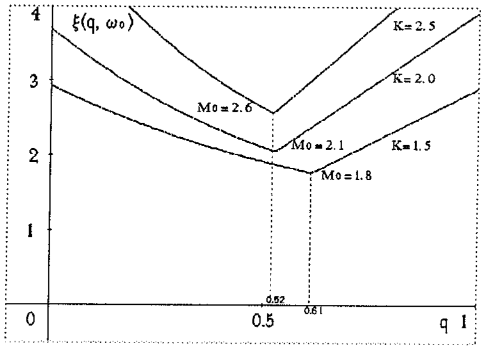

Fig. 6 $\xi(q, \omega_0)$ curves for Example-2.

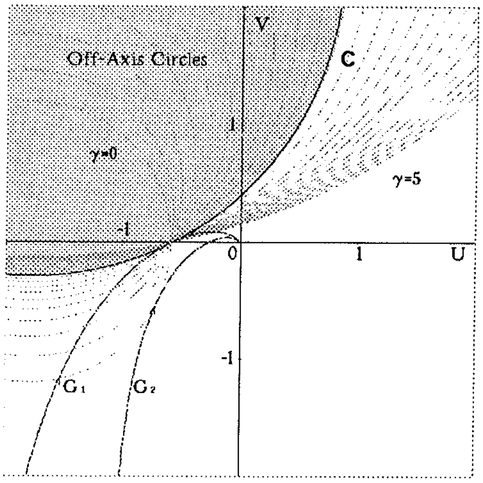

Fig. 5 Off-axis circles and vector loci for Example-1 ($M = 1.5,\ 0 \leq \omega \leq 10$).

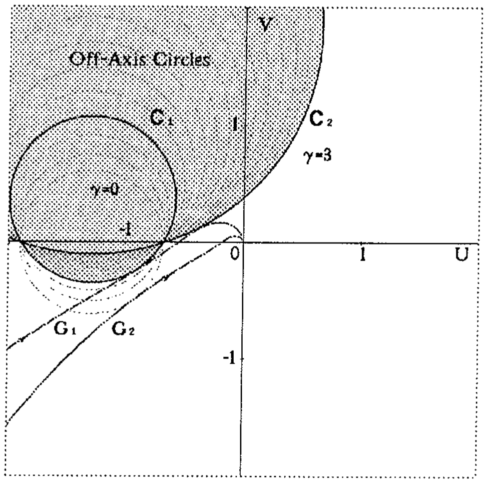

Fig. 7 Off-axis circles and vector loci for Example-2 ($M = 2.1,\ 0 \leq \omega \leq 30$).

control system. Although the robust control system is usually designed based on this concept, the robust stability condition is more conservative. Here, the gain margin becomes $g_2 = 10.8$ dB.

[Example-2] Consider the following controller and controlled system:

$$G(s) = \frac{2(1+s)(1-0.5s)}{s(1+5s)(1+0.2s)}, \quad K = 2.0. \tag{30}$$

Figure 6 shows calculation results for $\xi(q, \omega_0)$. When nominal linear gain $K = 2.0$,

$$M_0 = \min_q \xi(q, \omega_0) = \xi(q_0, \omega_0) = 2.1$$

is obtained. Figure 7 shows calculation results of circle array as to $q \geq 0$ for $M = 2.1$, i.e., $\alpha = 0.476$ and vector locus G_1 for the controlled system.

In this example, q by which $\xi(q, \omega_0)$ is minimized is $q_0 = 0.5$. Obviously, the vector locus contacts with off-axis circles $\mathbf{C}_1$ and $\mathbf{C}_2$ except on the real axis. The gain margin is $g_1 = 5.15$ dB, which is different from

$$-20 \log_{10} \frac{M_0}{M_0 + 1} = 3.36 \text{ dB},$$

which is determined by the point where the arguments of contours $\mathbf{C}_1$ and $\mathbf{C}_2$ become -180 degrees. On the other hand, G_2 is a vector locus for $K = 0.67$ by which the peak value becomes $M_0 = 2.1$. It is a limit for a robust stability condition which can be applied to a nonlinear time-varying system. The gain margin in this case is $g_2 = 14.6$ dB.

Figure 8 is an example of broken (polygonal) line nonlinear characteristic $N(e)$. Figure 9 shows the time response for the control system. Because the stability region for a linear characteristic is $0 < K < 3.63$, the response in Fig. 9 is a counter example to Aizerman's conjecture.

6. CONCLUSIONS

In this paper, by applying a small gain theorem about the L_2 gain to the nonlinear subsystem with a free parameter, the stability criterion in the frequency

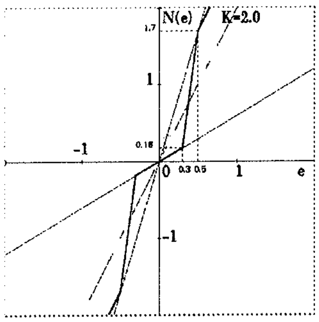

Fig. 8 Broken line nonlinearity.

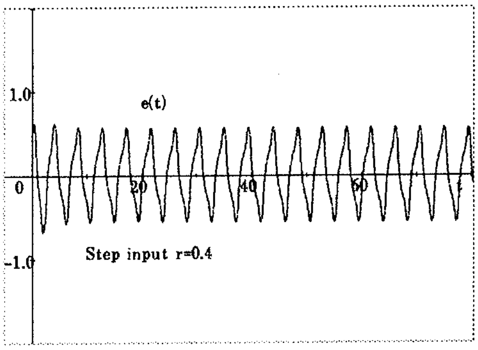

Fig. 9 Time response for Example-2.

domain of the control system with a time-invariant nonlinearity was given. By drawing an off-axis circle array on the Nyquist diagram a robust stability condition in relation to the vector locus of the open loop frequency response was presented. The evaluation of robust performance concerning the off-axis circle diagram will be used in the design of robust control systems.

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