Realization of Robust Performance for Interval Systems via Model Feedback

Yoshifumi OKUYAMA and Fumiaki TAKEMORI

Department of Information and Knowledge Engineering
Faculty of Engineering, Tottori University
Tottori, 680-8552 Japan

E-mail: oku@ike.tottori-u.ac.jp

Abstract: The physical parameters of controlled systems are uncertain and are accompanied with nonlinearity. The transfer function of the controlled system should, therefore, be expressed by interval polynomials. This paper describes the realization of robust performance for that type of control systems (interval systems) via model reference feedback. First, we will analyze an invariance problem of dynamic characteristics such that the dominant roots do not break away from a specified circular area, and will present a discrimination algorithm (i.e., a division algorithm) for the extreme points of the uncertain coefficients. Then, we will present a design method of control systems which have a robust performance such that the location of the dominant roots does not vary excessively.

Key words: Robust performance; interval system; uncertainty; model reference feedback; Sturm theorem; Rouché theorem.

1. INTRODUCTION

Since the physical parameters of controlled systems are uncertain and are accompanied with nonlinearity, the transfer function should be expressed by interval polynomials (Barmish [1], Ackermann [2]). This paper describes the existing area of characteristic roots for control systems which are expressed by that type of transfer function. A discrimination method of the number of characteristic roots in a specified area on an s-plane was developed in our paper (Okuyama et al. [3]), when a characteristic equation was expressed as an interval polynomial. The criterion is based on the classic Sturm's theorem. The discrimination algorithm was expressed so that it can be easily programmed on a computer.

This paper examines an invariance problem of dynamic characteristics such that the dominant roots do not break away from a specified circular area (a disc), and presents a discrimination algorithm (i.e., a division algorithm) for the extreme points of the uncertain coefficients. Designing examples of a control system with robust performance via model feedback are presented.

2. INTERVAL POLYNOMIALS

The transfer function of a control system with uncertainty (and nonlinearity) is expressed by interval polynomials. Therefore, the characteristic polynomial of a control system with uncertainties can be written by an interval polynomial as follows:

$$\tilde{F}(s) = \tilde{a}_0 s^n + \tilde{a}_1 s^{n-1} + \cdots + \tilde{a}_{n-1} s + \tilde{a}_n, \quad (1)$$
$$\tilde{a}_i \in [a_i^-; a_i^+], \quad (i = 0, 1, 2, \cdots, n).$$

When expressing interval polynomials in terms of nominal coefficient a_i and maximum error $\Delta\overline{a}_i$, the

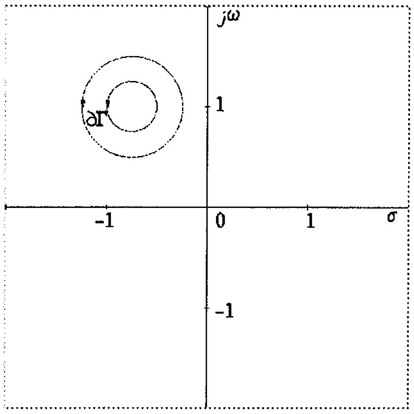

Fig. 1 Circular area ($\sigma_0 = -0.75$ and $\omega_0 = 1.0$, $\rho = 0.3$ or $\rho = 0.6$).

following can be obtained:

$$a_i = \frac{a_i^+ + a_i^-}{2}, \qquad \Delta\overline{a}_i = \frac{a_i^+ - a_i^-}{2},$$
$$|\tilde{a}_i - a_i| = |\Delta a_i| \leq \Delta\overline{a}_i. \tag{2}$$

By using free parameter γ_i interval coefficient $\tilde{a}_i$ can be expressed as follows:

$$\tilde{a}_i = a_i + \Delta a_i = a_i + \gamma_i \cdot \Delta\overline{a}_i, \quad \gamma_i \in [-1;\ 1]. \tag{3}$$

In Eq. (3), non-negative free parameter $\overline{\gamma}$ which is written by

$$|\gamma_i| \leq \overline{\gamma} \in [0;\ 1], \quad \forall i \tag{4}$$

can be found.

Based on the expression of Eq. (3) for interval coefficients $\tilde{a}_i$, Eq. (1) can be rewritten as follows:

$$\tilde{F}(s) = F(s) + \Delta F(s), \tag{5}$$

where $F(s)$ and $\Delta F(s)$ are the nominal and the uncertain parts of interval polynomial $\tilde{F}(s)$, respectively, and written as

$$F(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n,$$
$$\Delta F(s) = \Delta a_0 s^n + \Delta a_1 s^{n-1} + \cdots + \Delta a_{n-1} s + \Delta a_n,$$
$$= \gamma_0 \Delta\overline{a}_0 s^n + \gamma_1 \Delta\overline{a}_1 s^{n-1} + \cdots + \gamma_{n-1} \Delta\overline{a}_{n-1} s + \gamma_n \Delta\overline{a}_n.$$

Moreover, the following inequality can be given:

$$|\Delta F(s)| \leq \overline{\gamma} \cdot \max_{\eta_j} |\Delta\overline{F}(s, \eta_j)|, \tag{6}$$

where

$$\Delta\overline{F}(s, \eta_j) = \nu_0 \Delta\overline{a}_0 s^n + \nu_1 \Delta\overline{a}_1 s^{n-1} + \cdots + \nu_{n-1} \Delta\overline{a}_{n-1} s + \nu_n \Delta\overline{a}_n,$$
$$\eta_j = \{\nu_0, \nu_1, \cdots, \nu_n\}_j, \quad \nu_i \in \{-1;\ 1\},$$
$$(j = 1, 2, 3, \cdots, 2^{n+1}).$$

When analyzing a given transformation (i.e., the mapping of circular contour $\partial\Gamma$ as is shown in Fig. 1) in an s-plane, i.e.,

$$s = \rho e^{j\theta} + \sigma_0 + j\omega_0, \tag{7}$$

the following chracteristic polynomial with complex coefficients can be obtained :

$$\Phi(j\alpha) = P(\alpha) + jQ(\alpha), \tag{8}$$

$$P(\alpha) = \tilde{a}_{0,0}\alpha^n + \tilde{a}_{0,1}\alpha^{n-1} + \cdots + \tilde{a}_{0,n-1}\alpha + \tilde{a}_{0,n},$$
$$Q(\alpha) = \tilde{b}_{0,0}\alpha^n + \tilde{b}_{0,1}\alpha^{n-1} + \cdots + \tilde{b}_{0,n-1}\alpha + \tilde{b}_{0,n},$$
$$\tilde{a}_{0,i} \in [a_{0,i}^-;\ a_{0,i}^+],\ \ \tilde{b}_{0,i} \in [b_{0,i}^-;\ b_{0,i}^+],\ \ (i = 0, 1, 2, \cdots, n).$$

Here, α corresponds to the following variable:

$$\alpha = \tan(\theta/2). \tag{9}$$

3. DISCRIMINATION OF THE NUMBER OF ROOTS

If the μ pieces of the characteristic roots are present in specified circle $\partial\Gamma$, the argument change in the mapping of characteristic polynomial (1) should be $2\mu\pi$, regardless of its interval coefficients.

Assume that $f_0(\alpha) = P(\alpha)$, $f_1(\alpha) = Q(\alpha)$. As for $P(\alpha)/Q(\alpha)$ (or $-Q(\alpha)/P(\alpha)$), the following division algorithm can be used (Takagi [4]):

$$f_0(\alpha) = f_1(\alpha)q_1(\alpha) - f_2(\alpha),$$
$$f_1(\alpha) = f_2(\alpha)q_2(\alpha) - f_3(\alpha), \tag{10}$$
$$\cdots$$
$$f_{2n-2} = f_{2n-1}(\alpha)q_{2n-1}(\alpha) - f_{2n}(\alpha).$$

If $f_0(\alpha)$ and $f_1(\alpha)$ are of the n-th order in regard to α and coprime,

$$f_2(\alpha), f_3(\alpha), \cdots, f_{2n}$$

are all present and

$$f_{2h}(\alpha),\ f_{2h+1}(\alpha),\ \ (h = 0, 1, \cdots, n-1)$$

become the $(n-h)$-th order polynomial, and f_{2n} becomes a non-zero constant. That is, $f_2(\alpha), f_3(\alpha), \cdots, f_{2n}$ can be expressed as follows:

$$f_2(\alpha) = a_{1,1}\alpha^{n-1} + \cdots + a_{1,n}$$
$$f_3(\alpha) = b_{1,1}\alpha^{n-1} + \cdots + b_{1,n}$$
$$\cdots \tag{11}$$
$$f_{2n-2}(\alpha) = a_{n-1,n-1}\alpha + a_{n-1,n}$$
$$f_{2n-1}(\alpha) = b_{n-1,n-1}\alpha + b_{n-1,n}$$
$$f_{2n} = a_{n,n}.$$

In order to simplify the notation, interval coefficients $\tilde{a}_{i,j}$ (which are obtained by the division algorithm) simply denote $a_{i,j}$.

Here, each coefficient can be given by the following sequential operations:

$$\begin{aligned}
a_{1,p} &= b_{0,p}\left(\frac{a_{0,0}}{b_{0,0}}\right) - a_{0,p}, \\
b_{1,p} &= a_{1,p+1}\left(\frac{b_{0,0}}{a_{1,1}}\right) - b_{0,p}, \\
&\quad (p = 1, 2, \cdots, n) \qquad (12) \\
&\quad \cdots \\
a_{q,p} &= b_{q-1,p}\left(\frac{a_{q-1,q-1}}{b_{q-1,q-1}}\right) - a_{q-1,p}, \\
b_{q,p} &= a_{q,p+1}\left(\frac{b_{q-1,q-1}}{a_{q,q}}\right) - b_{q-1,p}, \\
&\quad (p = q, \cdots, n) \\
&\quad \cdots \\
a_{n,n} &= b_{n-1,n}\left(\frac{a_{n-1,n-1}}{b_{n-1,n-1}}\right) - a_{n-1,n}, \\
&\quad (a_{q,n+1} = 0).
\end{aligned}$$

When a characteristic equation is expressed as an interval polynomial, these sequential operations should be based on interval arithmetic. Interval arithmetic, however, can only be used where each variable (coefficient) is independent. Because the calculation in Eq. (12) was advanced sequentially by using the preceding results, each variable (coefficient) is not independent; therefore, the use of interval arithmetic in the above sequential operation may give a more conservative result. The number of characteristic roots, however, can be discriminated by the extreme point in each coefficient of Eq. (12). (Note that the denominator of Eq. (12) must not become zero, although this is only a problem in certain singular points.)

Since the extreme points of the interval sets of series (array) $a_{q,p}$ and $b_{q,p}$ $(q = 1, 2, \cdots, n, p = q, \cdots, n)$ are given in order according to the extreme points of the uncertain coefficients in interval polynomial (1) and (8), these extreme points can be determined by using all the combinations of the extreme points of the coefficients in the interval polynomial. Therefore, the following theorem in regard to the robustness of control systems can be obtained (Okuyama [3]):

[Theorem] A necessary condition for the μ pieces of the roots of characteristic equation $\tilde{F}(s) = 0$ (which are present in specified circle $\partial\Gamma$) is shown below.

The following coefficient ratios should be calculated (they were present in division algorithm (12)):

$$\frac{b_{0,0}}{a_{1,1}}, \ \frac{b_{1,1}}{a_{2,2}}, \ \cdots, \ \frac{b_{n-1,n-1}}{a_{n,n}}. \qquad (13)$$

for all the combinations of the extreme points of the uncertain coefficients in the family of characteristic equations,

$$\boldsymbol{F}(s) = [a_0^-; a_0^+]s^n + [a_1^-; a_1^+]s^{n-1} + \cdots + [a_n^-; a_n^+] = 0. \qquad (14)$$

If the number of ratios that is to be negative is not changed, a control system that is characterized by Eq. (14) has robust performance in regard to the invariance of the number of characteristic roots in specified circle $\partial\Gamma$. Moreover, when the above also holds for $\overline{\gamma} \in [0; 1]$ in Eqs. (2), (3) and (4), it becomes the necessary and sufficient condition.

(Proof) The necessity is obvious from Sturm's theorem (i.e., the results of Eqs. (11) and (12)). The sufficiency can be proven by using Rouché's theorem.

If the number of roots of characteristic equation

$$F(s) + \gamma \cdot \Delta\overline{F}(s, \eta_j) = 0, \quad \forall\gamma \in [0; \overline{\gamma}], \quad \forall\eta_j \qquad (15)$$

is invariant inside the specified contour, the following must holds as to $s \in \partial\Gamma$:

$$F(s) \neq -\gamma \cdot \Delta\overline{F}(s, \eta_j), \quad \forall\gamma \in [0; \overline{\gamma}], \quad \forall\eta_j.$$

Thus, the following inequality can be given:

$$|F(s)| > \overline{\gamma} \cdot \max_{\eta_j} |\Delta\overline{F}(s, \eta_j)|, \quad \forall s \in \partial\Gamma. \qquad (16)$$

As a result, we can obtain

$$|F(s)| > |\Delta F(s)|, \quad \forall s \in \partial\Gamma \qquad (17)$$

from Eq. (6).

By using Rouché's theorem we may conclude that the number of roots of characteristic equation

$$\tilde{F}(s) = F(s) + \Delta F(s) = 0 \qquad (18)$$

is invariant inside the specified contour, regardless of uncertainty $\Delta F(s)$, i.e., free parameters $\gamma_i \in [-1; 1]$ in Eq. (3). □

[Example 4.1] Consider the family of characteristic equations with low frequency uncertainty (e.g., sector nonlinearities (Okuyama & Takemori [5, 6]) as follows:

$$0.005s^4 + 0.25s^3 + [0.9; 1.1]s^2 + [1.1; 1.4]s + [0.8; 1.2] = 0. \qquad (19)$$

When a circle with center $(-0.75, j)$ and radius $r = 0.6$ is specified, the number of the characteristic

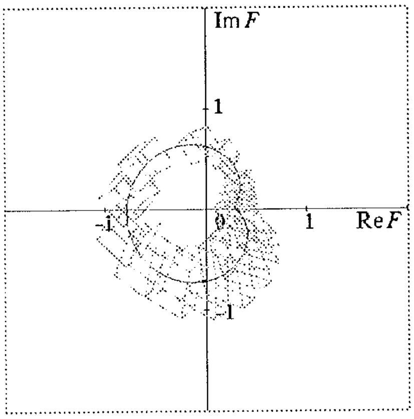

Fig. 2 Polyhedral mapping of $\partial\Gamma$ (ρ = 0.6) for Example 4.1.

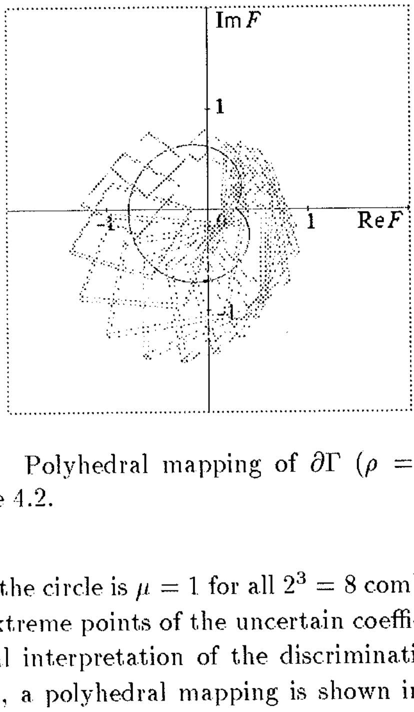

Fig. 3 Polyhedral mapping of $\partial\Gamma$ (ρ = 0.6) for Example 4.2.

roots in the circle is $\mu = 1$ for all $2^3 = 8$ combinations in the extreme points of the uncertain coefficients. A graphical interpretation of the discrimination theorem, i.e., a polyhedral mapping is shown in Fig. 2. In this figure, 2^3 vertices of the polyhedra indicate extreme points of the interval sets(Bhattacharyya [7]).

The calculated results show that the number of roots in the specified circle did not change. Moreover, the number of the dominant roots was maintained for all $\overline{\gamma} \in [0;\ 1]$.

On the other hand, when a circle with radius r = 0.3 is specified, the calculated results show that there are some cases where the dominant root does not exist in the specified area.

[Example 4.2] Consider the family of characteristic equations with high frequency uncertainties as:

$$[0.001; 0.009]s^4 + [0.1; 0.4]s^3 + [0.8; 1.2]s^2 + 1.25s + 1.0 = 0. \quad (20)$$

When a circle with center $(-0.75, j)$ and radius r = 0.6 is specified, the number of the characteristic roots did not change. A polyhedral mapping is as shown in Fig. 3.

On the other hand, when a circle with radius r = 0.3 is specified, the calculated results show that there are some cases where the dominant root does not exist in the specified area.

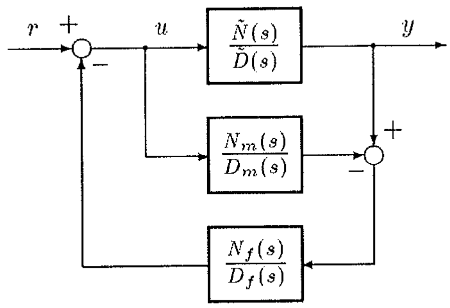

Fig. 4 Realization of robust performace via model feedback.

4. REALIZATION OF ROBUST PERFORMANCE

Consider a model feedback system, as is shown in Fig. 4. Here, $\tilde{G} = \dfrac{\tilde{N}(s)}{\tilde{D}(s)}$, $G_m = \dfrac{N_m(s)}{D_m(s)}$, $G_f = \dfrac{N_f(s)}{D_f(s)}$ are uncertain(interval) plants, plant model and feedback compensator, respectively. Here, we will assume that $\tilde{N}(s)$ and $\tilde{D}(s)$ are expressed by interval polynomials as shown in Eq. (1).

By using this type of feedback structure, a robust performance for the uncertain control system can be realized (Okuyama [8]). If feedback compensator G_f is chosen as

$$N_f(s) = D_m(s), \qquad D_f(s) = N_0(s,\tau)N_m(s), \quad (21)$$

modified control signal u is given by

$$\hat{u}(s) = \frac{N_0(s,\tau)}{N_0(s,\tau)-1}\left(\hat{r}(s) - \frac{D_m(s)}{N_0(s,\tau)N_m(s)}\hat{y}(s)\right), \quad (22)$$

where $\hat{u}(s)$, $\hat{r}(s)$ and $\hat{y}(s)$ denote the Laplace transform of u, r and y, respectively. Here, $N_0(s,\tau)$ is an $(n-m)$-th order polynomial which should be designed in the feedback system as is shown in Fig. 4. In this paper, we will choose

$$N_0(s,\tau) = (\tau s + 1)^{n-m}, \quad (23)$$

where τ is an appropriate (small) positive number.

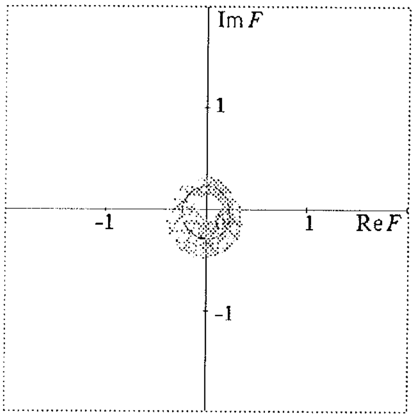

Fig. 5 Polyhedral mapping of $\partial\Gamma$ ($\rho = 0.3$) for Example 5.1.

From Eq. (22) the following can be obtained:

$$(N_0(s,\tau)-1)\hat{u}(s) = (N_0(s,\tau)\hat{r}(s) - \frac{D_m(s)}{N_m(s)}\hat{y}(s). \quad (24)$$

As is obvious from Eq. (24), when $\tau s \to 0$, $\hat{y}(s) \to \frac{N_m(s)}{D_m(s)}\hat{r}(s)$. When operating in lower frequencies, the transfer characteristics from r to y becomes approximately $\frac{N_m(s)}{D_m(s)}$ and it is invariant regardless of uncertainties (i.e., interval set parameters) in plant $\tilde{G}$. In other words, it can be shown that the model reference feedback system has a robust performance.

The characteristic equation of the closed loop system can be expressed as

$$(N_0(s,\tau)-1) + \frac{D_m(s)\tilde{N}(s)}{N_m(s)\tilde{D}(s)} = 0. \quad (25)$$

When uncertainties of plant $\tilde{G}$ exist only in the denominator, that is, $\tilde{N}(s) = N_m(s)$, the characteristic interval polynomial which corresponds to Eq. (1) can be written as follows:

$$\tilde{F}(s) = N_0(s,\tau)D_m(s) + (N_0(s,\tau)-1)\Delta D(s), \quad (26)$$

where $\tilde{D}(s) = D_m(s) + \Delta D(s)$. As for Eq. (26) the invariance of the dominant roots is examined in the following examples.

[Example 5.1] Assume that denominator polynomial $\tilde{D}(s)$ of an uncertain plant is expressed by an interval polynomial as shown in Eq. (19) and numerator polynomial $\tilde{N}(s)$ is written by $\tilde{N}(s) = N_m(s) = 1$. In addition, we choose

$$N_0(s,\tau) = (0.1s+1)^4. \quad (27)$$

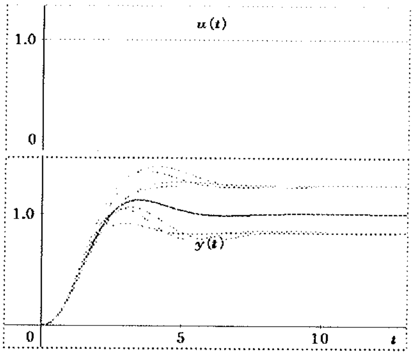

Fig. 6 Step responses for Example 5.1, when model reference feedback was not used.

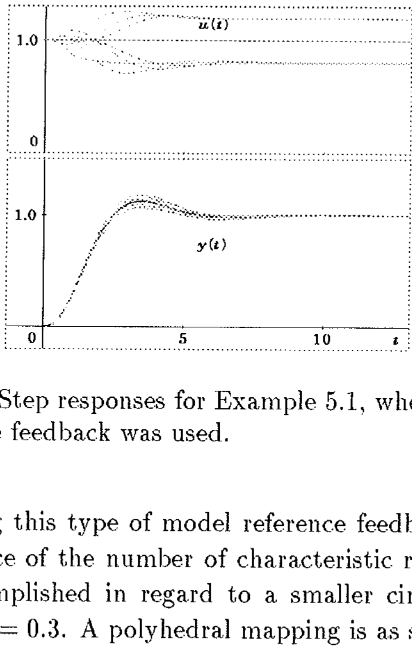

Fig. 7 Step responses for Example 5.1, when model reference feedback was used.

By using this type of model reference feedback, the invariance of the number of characteristic roots can be accomplished in regard to a smaller circle with radius $\rho = 0.3$. A polyhedral mapping is as shown in Fig. 5.

Step responses in regard to the extreme points for uncertain control system $\tilde{G}(s)$ are as shown in Fig. 6 and Fig. 7, when the model reference feedback was not used and was used, respectively. As is clear from the figure, the robust performance according to the above was realized.

[Example 5.2] Assume that denominator polynomial $\tilde{D}(s)$ of an uncertain plant is expressed by an interval polynomial as shown in Eq. (20) and numerator polynomial $\tilde{N}(s)$ is written by $\tilde{N}(s) = N_m(s) = 1$. Here, we choose the same $N_0(s,\tau)$ as in Eq. (27). The invariance of the number of characteristic roots can be accomplished in regard to a smaller circle with radius $\rho = 0.3$. A polyhedral mapping is as shown in Fig. 8.

Step responses in regard to the extreme points for uncertain control system $\tilde{G}(s)$ are as shown in Fig. 9

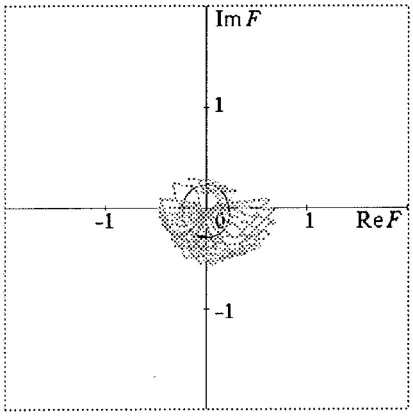

Fig. 8 Polyhedral mapping of $\partial\Gamma$ ($\rho = 0.3$) for Example 5.2.

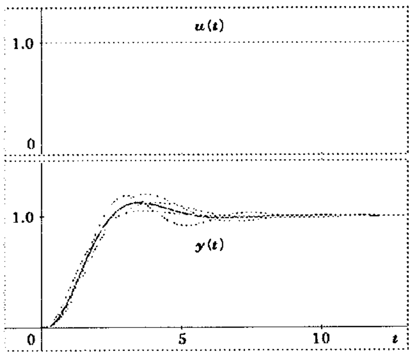

Fig. 9 Step responses for Example 5.2, when model reference feedbak was not used.

and Fig. 10, when the model reference feedback was not used and was used, respectively.

5. CONCLUSION

This study described the existing area of characteristic roots for control systems which are expressed by transfer functions that are composed of interval polynomials. A discrimination method of the number of characteristic roots in a specified circle on an s-plane was presented, when a characteristic equation was expressed as an interval polynomial. A theorem was given in reference to the extreme point results which corresponds to the weak-Kharitonov's theorem for interval polynomials. The theorem can be used as an invariant condition of the number of characteristic roots in the specified circle. In particular, in this paper, the invariance of the dominant roots in the circlar area and the realization of control systems with robust performance using model reference feedback were examined.

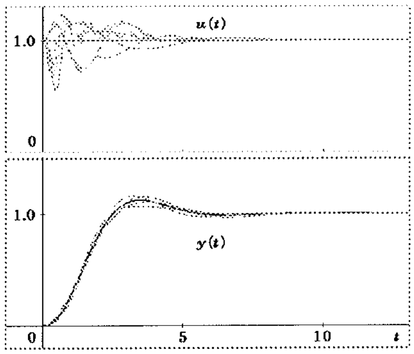

Fig. 10 Step responses for Example 5.2, when model reference feedbak was used.

REFERENCES

[1] Barmish, B. and H. Kang: A Survey of Extreme Point Results of Robust Control Systems, Automatica, vol. 29, pp. 13-35, 1993.

[2] Ackermann, J.: Robust Control Systems with Uncertain Physical Parameters, Springer-Verlag, 1993.

[3] Okuyama, Y., F. Takemori and H. Chen: On Sturm's Theorem for Interval Polynomials, Proc. of American Control Conference, Philadelphia, pp. 1883-1885, 1998.

[4] Takagi, T.: Lecture Notes in Algebra (Daisugaku Kougi), Kyoritsu Shuppan, 1965 (in Japanese).

[5] Okuyama, Y., H. Chen and F. Takemori: Robust Stability Evaluation for Control Systems with Time-Invariant Nonlinearity in Gain-Phase Plane, Computer Aided Control Systems Design (Edited by L. Boullart, et al.), Pergamon, pp. 109-114, 1997.

[6] Okuyama, Y. and F. Takemori: On the Robust Stabilization of Sampled-Data Control Systems with a Sector Nonlinearity, Preprints of 4th IFAC Symposium on Nonlinear Control Systems Design, Enschede, pp. 284-289, 1998.

[7] Bhattacharyya, S. P., et al.: Robust Parameteric Control with Application, Workshop #10, 35th IEEE Conference on Decision and Control, Kobe, 1996.

[8] Okuyama, Y.: Estimation of Parameter Sensitivity of Linear Control Systems and Synthesis of Passive Adaptive Control Systems, J. of SICE, Vol.3, pp. 759-767, 1964 (in Japanese).

(Received August 25, 1998)