Characteristic Root Areas and Loci Band of Control Systems with Frequency-Dependent Uncertainty

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(Received September 1, 1995)

This paper presents a method to calculate the characteristic root areas and loci band of control systems with frequency-dependent uncertainty. When the upper boundary of the frequency response is also frequency-dependent, the frequency-dependent terms are included in the characteristic equation of the nominal system. This lead to the boundary equations of the root areas for control systems with frequency-dependent uncertainty. Numerical examples of the control systems with multiplicative perturbations including frequency-dependent terms are presented to verify this calculation method and to clarify the effectiveness for analysis and design of actual control systems.

Key words : Characteristic root area, root loci band, multiplicative perturbation, frequency-dependent uncertainty, robust stability

1 Introduction

In this paper, we present a method to calculate the characteristic root area and its array, i.e., root loci band of control systems with uncertainty. The method of calculating the characteristic root areas for additive and multiplicative perturbations is discussed. A little adjustment is performed between the nominal system and the uncertain part particularly in the case of multiplicative perturbation, so that the characteristic root area is not too large, or in other word, boundary curves are not too conservative.

For the control systems with uncertainty, we previously presented the method to directly calculate the existence area and boundary of the characteristic root, assuming that an upper bound was given to the absolute value of the frequency characteristic of an uncertain part[1],[2]. This is based on the idea that the upper bound of the frequency characteristic of uncertain part does not depend on the frequency. When it depends on the frequency, a sufficient condition is provided so that the characteristic root will exist based on two inequalities[3].

This report presents a new method to calculate the characteristic root areas when the upper bound of the frequency characteristic of an uncertain part depends on the frequency. When the upper bound of the absolute values of frequency responses for the uncertain part is also frequency-dependent, the frequency-dependent terms are included in the characteristic equation of the nominal system. In either case, the upper bound of the absolute values for the uncertain part is given, the equation of the boundary curves can be derived, and an algorithm for the numerical calculation of the array of closed curves can be presented.

In this paper we present some numerical examples for it and clarify its effectiveness for analysis and design of actual control systems.

2 Control Systems with Uncertainty

The dynamic characteristics of controlled objects and control elements are often given in experimental data of frequency response. In addition, the frequency response characteristic should be considered in a certain band because of nonlinearity and uncertainty. Also, the band is considered to be dependant on the amplitude by the gain change of amplifier and the saturation (dead zone) of sensor and actuator. Though it is a well-known method to express such a frequency response characteristic band by Nyquist or Bode plot, we will catch it as a root loci band on the s-plane to make it easy to observe the influences on stability with the gain change.

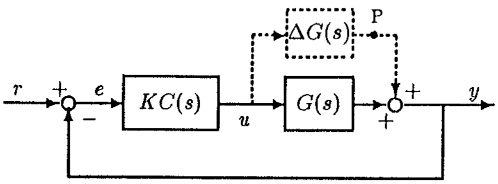

Fig. 1 Control system with uncertainty.

2.1 Additive Perturbation

We consider a control system, as shown in Fig. 1, where the controlled objects have the additive perturbation (uncertainty by interference with other loops). The open loop characteristic viewed from the point P in the figure is represented as

$$L(s) = \frac{\Delta G(s)KC(s)}{1+KC(s)G(s)}, \tag{1}$$

where $G(s)$, $C(s)$, and K is plant, compensator and gain parameter respectively. The robust stability condition concerning the H_∞ norm[4] is obviously given by

$$\|L(s)\|_\infty < 1. \tag{2}$$

If we use

$$F_a(s) = \frac{1+KC(s)G(s)}{KC(s)},$$

then Eq. (2) is equivalently written as

$$|F_a(j\omega)| > |\Delta G(j\omega)| \tag{3}$$

for the frequency response characteristic.

It is necessary to provide an uncertain part $\Delta G(s)$ as an upper bound of the amplitude in experimental data of frequecy response, i.e.,

$$|\Delta G(j\omega)| \le |\rho_a(j\omega)|, \tag{4}$$

where $\rho_a(j\omega)$ is a parameterized frequency transfer function. If the upper bound of the absolute value of frequency responses for the uncertain part is given by a frequncy-dependent radius in Eq. (4), then Eq. (3) is rewritten as follows:

$$|F_a(j\omega)| > |\rho_a(j\omega)|. \tag{5}$$

2.2 Multiplicative Perturbation

When the uncertain part in Fig. 1 is represented by $\Delta G(s) = \Delta(s)G(s)$, it is possible to discuss control systems with multiplicative perturbation in the same way as those with additve perturbation. In that case, the open loop transfer function viewed from the point of P is expressed as

$$L(s) = \frac{\Delta(s)KG(s)C(s)}{1+KC(s)G(s)}. \tag{6}$$

If the upper bound of the absolute value of frequency responses for the uncertain part $\Delta(s)$ is given by

$$|\Delta(j\omega)| \leq |\rho_m(j\omega)|, \tag{7}$$

then the robust stability condition Eq. (2) is arranged by

$$|F_m(j\omega)| > |\rho_m(j\omega)|. \tag{8}$$

Here, concerning $j\omega \to s$ the characteristic function $F_m(s)$ can be written as

$$F_m(s) = \frac{1 + KC(s)G(s)}{KC(s)} = \frac{N_m(s)}{D_m(s)}, \tag{9}$$

and a frequency transfer function $\rho_m(j\omega)$ is replaced by a parameterized radius function $\rho_m(s)$.

2.3 Modification

In order to avoid conservativeness, a little adjustment is performed between the nominal system $G(s)$ and the uncertain part $\Delta(s)$ particularly when dealing with multiplicative perturbation[5]. Here, we consider that the parameterized radius function $\rho_m(s)$ is expressed as a radius function $\rho_m(s, \tau)$ with an uncertain parameter τ. In addition, we assume that the phase characteristic of the uncertain part $\Delta(s)$ is similar to that of the radius function $\rho_m(s, \tau)$. Therefore, we can identify the uncertain part $\Delta(s)$ with the radius function $\rho_m(s, \tau)$.

If the uncertain parameter τ of the radius function $\rho_m(s, \tau)$ varies from 0 to $\bar{\tau}$, the modifying term $\hat{g}_m(s, \bar{\tau})$ by which to multiply the nominal system is as follows:

$$\hat{g}_m(s, \bar{\tau}) = 1 + \frac{\rho_m(s, 0) + \rho_m(s, \bar{\tau})}{2}. \tag{10}$$

Therefore, the modified term of uncertainty $\rho_m(s, \tau)$ can be written as

$$\hat{\rho}_m(s, \bar{\tau}) = \frac{\rho_m(s, 0) - \rho_m(s, \bar{\tau})}{2 + \rho_m(s, 0) + \rho_m(s, \bar{\tau})}. \tag{11}$$

2.4 Stability Invariance

The foregoing concept can also be expanded to the case where an nominal closed loop system is unstable. In such a case, we consider the complex variable s on the contour Γ which encloses the left (or right) half plane containing the imaginary axis.

When considering the multiplicative perturbation, the robust stability condition is rewritten as

$$|F_m(s)| > |\rho_m(s)|, \quad s \in \Gamma. \tag{12}$$

This inequality is an invariant condition of the dynamic characteristic[6], that is, a condition so that stability and instability will not change regardless of the uncertainty.

3 Root Loci Band

3.1 Array of Boundary Curves

The invariant condition of the dynamic characteristic in Eq. (12) contains all the properties in the complex frequency s domain. From the aspect of the control system design, it might also be important to know how the area, that is, the "boundary curve" based on the radius of the uncertainty moves according to the change of the gain constant K.

A method to calculate the array of the curves is mentioned below. In the case of the multiplicative perturbation, the locus of a zero s_i $(i = 1, 2, \cdots, n)$ of

$$|F_m(s, K)| = 0 \quad \text{and} \quad |N_m(s, K)| = 0 \tag{13}$$

for $K : 0 \to \infty$ is obviously a root locus, whereas for some positive constant $\bar{\rho}_m$ the array of the closed curves satisfying

$$|F_m(s, K_l)| = \bar{\rho}_m > |\rho_m(s)|, \quad l = 1, 2, \cdots, \infty \tag{14}$$

shows the boundary curves of the area containing characteristic roots of control systems. When we consider Eq. (14) for some positive constant $\bar{\rho}_m$, the close curve $\Gamma_{i,l}$ on the s-plane for $K = K_l$ shows the invariance of the number of zeros in the area. That is, the inequality Eq. (12) corresponds to Rouché's theorem[2]. Consequently, we can consider that the array of the boundary curves for $K = K_1, K_2, \cdots, K_\infty$ is a band of root loci. In this paper, the array of the boundary curves is refered to as the *root loci band.*

However, the boundary curves should satisfy the sign of inequality at the right side of Eq. (14). In the previous paper, we presented the idea to evaluate the area satisfying it with the sign of inequality. Here presented is the method to calculate the root locus band by including a frequency-dependent term of the uncertainty in the left side of inequality.

3.2 Frequency-Dependent Uncertainty

When the upper bound of the absolute values of frequency responses for the uncertain part is frequency-dependent, the frequency-dependent terms such as the right side of Eq. (12) can be included on the left side of the inequality. If the frequency-dependent radius function is given by

$$\rho_m(s) = \epsilon \cdot \frac{N_\rho(s)}{D_\rho(s)}, \qquad \epsilon:\ \text{const.}, \tag{15}$$

then the characteristic functions Eqs. (5) and (11) are written as follows:

$$F_m^*(s) = \frac{N_m(s)}{D_m(s)} \cdot \frac{D_\rho(s)}{N_\rho(s)}. \tag{16}$$

Therefore, the inequality corresponding to Eq. (14) is

$$|F^*_m(s, K_l)| = \bar{\rho}^*_m > |\epsilon| , \quad l = 1, 2, \cdots, \infty , \qquad (17)$$

and the right side of Eq. (14) is modified as the following equation:

$$f(\sigma, \omega, K_l) = |F^*(s, K_l)| - \bar{\rho}^* = 0, \; l = 1, 2, \cdots, \infty. \qquad (18)$$

In later examples, we will use Eq. (18) as the equation of the root loci.

4 Algorithm

4.1 Equation of Boundary Curve

The equation of the boundary curves corresponding to Eq. (18), in other words, *root contours* $\Gamma_{i,l}$ is generally represented by

$$f(\sigma, \omega, K_l) = |F(s, K_l)| - \bar{\rho}, \;\; l = 1, 2, \cdots, \infty. \quad (19)$$

Roots $s = (\sigma, \omega)$ of this equation are obtained sequentially by using Newton's algorithm together with the gradient method in the complex s-plane. Note that any boundary curve of Eq. (19) is a simple closed curve, that is, a Jordan curve.

The algorithm obtaining the closed boundary curves represented by Eq. (19) will be shown in several calculation steps. A course of computation is illustrated in Fig. 2.

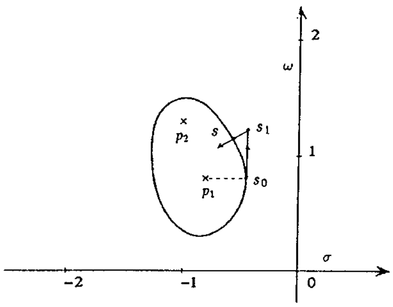

Fig. 2 A root contour and its calculation.

4.2 Algorithm

The algorthm for the calculation of Eq. (19) is as follows[2]:

(1) Let $i := 1$, $l := 1$ be the initial setting.

(2) Calculate root locus according to Eq. (13) from $K = K_{l-1}$ $(l = 1, 2, \cdots)$ to the appropriate gain $K = K_l$, by using the two-dimensional Newton method.

When calculating the root locus, it is necessary to pay close attention to the vicinity of the points where roots are breakaway from real quantities to complex ones, or break-in from complex quantities to real ones[3], i.e., multiple roots, a node or a saddle point in Eq. (19). However, the method to calculate such points is not discussed here in detail.

(3) Proceed from the root $p_{i,l}$ of the nominal system for $K = K_l$ to a direction of zero argument and then calculate root $s_0 = (\sigma_0, \omega_0)$ of Eq. (19) by using the Newton method.

(4) Proceed from the point s_0 to the tangential direction (the orthogonal direction to ∇F) and at this time proceed from the point s_1 to the direction of $-\nabla F$. And then calculate root $s = (\sigma, \omega)$ of Eq. (19) by using the Newton method.

(5) Repeat **step(3)** until returning the vicinity of point s_0.

(6) Let $l := l + 1$ and repeat from **step(1)** until $K = K_{\max}$ is satisfied.

(7) Let $i := i + 1$ and repeat from **step(1)** until $i = n$ is satisfied.

5 Numerical Examples

[Example 1]

Consider a control system as shown in Fig. 1, where plant (controlled system) is given by

$$G(s) = \frac{1}{s(1+s)}. \qquad (20)$$

Suppose the parameterized radius function $\rho_m(s, \tau)$ for the plant uncertainty of multiplicative perturbation $\Delta(s)$ can be written as a first-order lag characteristic as follws:

$$\rho_m(s, \tau) = \frac{-\tau s}{1 + \tau s}, \qquad 0 \le \tau \le \bar{\tau}. \qquad (21)$$

In such a system, the modifying term Eq. (10) is written as

$$\hat{g}_m(s, \bar{\tau}) = \frac{1 + (\bar{\tau}/2)s}{1 + \bar{\tau}s}, \qquad (22)$$

and the modified term of uncertainty Eq. (11) is expressed as

$$\hat{\rho}_m(s, \bar{\tau}) = \frac{-(\bar{\tau}/2)s}{1 + (\bar{\tau}/2)s}. \qquad (23)$$

For the case where $\bar{\tau} = 0.4$ and compensator is not used (C(s)=1), the calculation result of an array of the boundary curves, that is, the root loci band for $K: 0.2 \to 20$ is shown in Fig. 3(a). Fig. 3(b) shows the case where the following phase lead compensator is used:

$$C(s) = \frac{1 + 0.4s}{1 + 0.2s}.$$

As is obvious from the figure, the control system with uncertainty is successfully stabilized up to the considerably higher gain K.

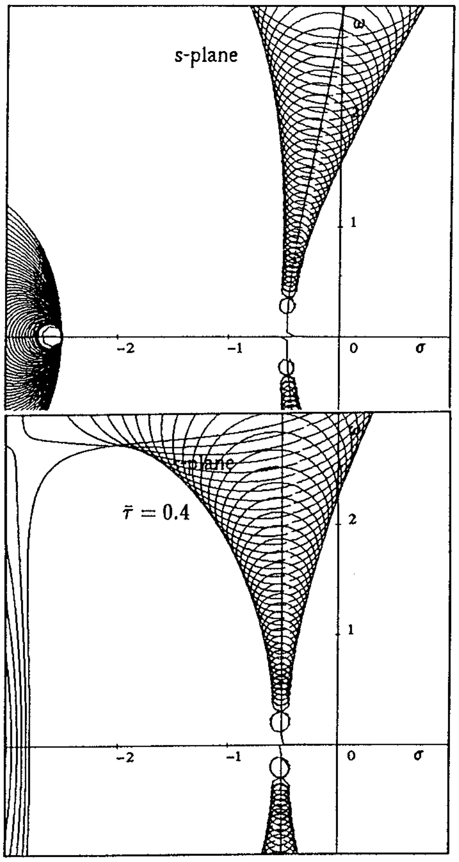

Fig.3 Root loci band of control system with first-order lag uncertainty.

[Example 2]

Consider a control system with a transport lag (dead time) uncertainty for multiplicative perturbation. Obviously, the modifying term Eq. (10) is written as

$$\hat{g}_m(s, \bar{\tau}) = \frac{1 + e^{-\bar{\tau}s}}{2} \qquad (24)$$

and the modified term of uncertainty Eq. (11) is expressed as

$$\hat{\rho}_m(s, \bar{\tau}) = \frac{1 - e^{-\tau s}}{1 + e^{-\tau s}}. \qquad (25)$$

Suppose plant is the same as that in Example 1, that is,

$$G(s) = \frac{1}{s(1+s)}.$$

For $\bar{\tau} = 0.4$, the calculation result of an array of the boundary curves, that is, the root loci band as to "dominant poles"for $K: 0.2 \to 20$ is shown in Fig. 4.

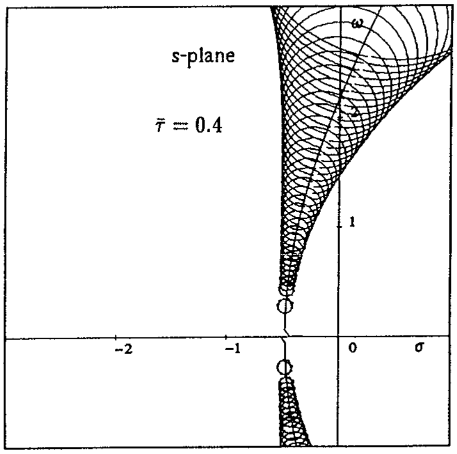

Fig.4 Root loci band of control system with transport lag uncertainty.

[Example 3]

Consider the following third order plant with an oscillating mode:

$$G(s) = \frac{1}{s(2 + 2s + s^2)}. \qquad (26)$$

The calculation result of an array of the boundary curves for $K: 0.2 \to 20$ is as shown in Fig. 5.

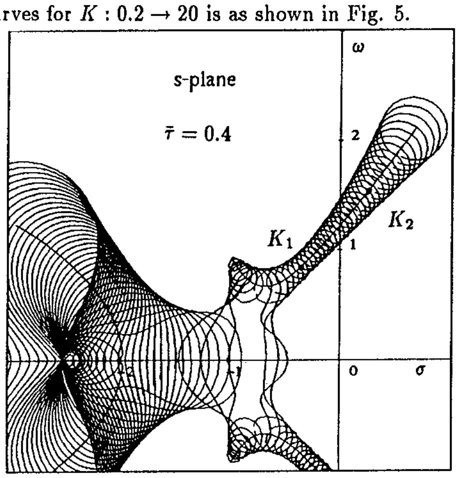

Fig.5 Root loci band for controlled system with oscillating mode.

In an actual control system, the gain parameter K changes due to the saturation (dead zone) of sensor and actuator[7]. From the figure, at the gain parameter K_1 we can see that the control system is stable regardless of the existence of the uncertainty (robust stable). On the other hand, at the gain parameter K_2 we can see that the control system is unstable regardless of the existence of the uncertainty ('robust unstable'). When the gain parameter change depends on the amplitude of the system; for instance, the gain parameter K incleases as the amplitude parameter a_k decreases, and vice versa, the gain parameter K decleases as the amplitude parameter $a_k$incleases, we will estimate a limit cycle between the gain parameter K_1 and K_2.

[Example 4]

Consider plant is the same as that in Example 3, that is,

$$G(s) = \frac{1}{s(2+2s+s^2)}. \quad (27)$$

When the following second order compensator

$$C(s) = \frac{5+2s+s^2}{5+0.4s+0.008s^2}$$

is used, an array of the boundary curves for the gain parameter changes $K : 0.2 \rightarrow 15$ is calculated as shown in Fig. 6. As is obvious from the root loci band, we can see that the control system is perfectly stabilized for any gain parameter K.

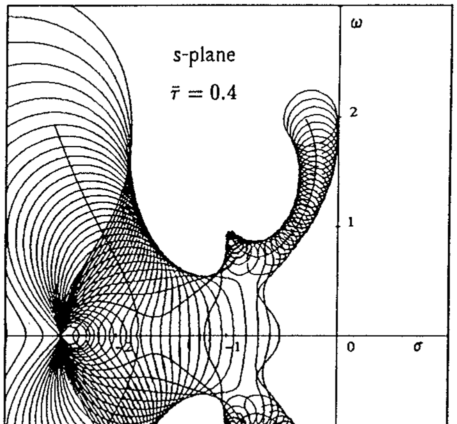

Fig. 6 Root loci band for perfectly stabilized system.

These numerical examples show the applicability of this method to the robust control systems design.

6 Conclusions

In this paper, we have presented a method to calculate the characteristic root areas and loci band of the control system with a frequency-dependent uncertainty. An actual control system has "scattering" in experimental data of frequency response, because it contains uncertainty and nonlinearity. Therefore, it is necessary to deal with the frequency response characteristic in a band. In addition, it is considered that the band depends on the amplitude by the gain changes of amplifier and the saturation (dead zone) of sensor and actuator[8]. Therefore, we suppose that it is effective in analysis and design to graphically express the dynamic characteristic of the control system with the root loci band as described in this paper.

With such a graphical expression of the control system characteristic, we can clarify the existence of the big gain change, amplitude dependency, and limit cycle which are often disregarded in design of the robust control system.

References

[1] Y. Okuyama et al., "Characteristic Root Area of Control Systems with Uncertainties and its Simple Calculation Methd," SICE Symp. on Control Theory, pp.75-78, 1984 (in Japanese).

[2] Y. Okuyama, "Root loci band and its applications," SICE Symp. on Control Theory, pp.79-82, 1985 (in Japanese).

[3] Y. Okuyama and F. Takemori, "Root Loci Band for Control Systems with Uncertainty and its Calculation," Proc. of the 38th ISCIE, pp.233-234, 1994 (in Japanese).

[4] B. A. Francis, A Course in H_∞ Control Theory, Springer-Verlag, 1987.

[5] Y. Okuyama, H. Chen and F. Takemori, "A Calculation Method of Root Loci Band and its Applications to Robust Control System Design," Proc. of the 9th KACC, pp.350-353, 1994.

[6] Y. Okuyama, H. Chen and F. Takemori, "Graphical Analysis and Design of Multi-Loop Control Systems with Model Uncertainty: Considering Frequency Range," SICE Symp. on Control Theory, pp.91-94, 1995.

[7] Y. G.F. Franklin et al., Feedback Control of Dynamic Systems, 2nd ed., Addison-Wesley, 1991.

[8] Y. Okuyama, H. Chen and F. Takemori, "Robust Stability Evaluation of Control Systems Composed by Amplitude Dependent Frequency Response Characteristics," The 38th JACC of Japan, 1995 (to appear).