Formulation for finite deformation FE analysis of one-dimensional consolidation of saturated soils

by

Masayoshi SHIMIZU

Department of Civil Engineering

(Received September 1, 1994)

Abstract

A theoretical formulation of governing equations for one-dimensional consolidation of saturated soils is developed based on the continuum mechanics. Small deformation is not assumed but large deformation is considered and inviscid but non-linear constitutive properties of soils are taken into account. A formulation of solving the governing equations by the FEM is also developed with the weighted residual method and Galerkin method.

Key words : Clay, Large deformation, One-dimensional consolidation, FEM

1. Introduction

Major factors that make it difficult to mechanically treat the consolidation phenomenon of soils are that:
1. soils are two-phase materials,
2. constitutive relations of soils are non-linear;
3. constitutive properties are essentially viscid or time-depending; and
4. deformation that occurs in the consolidation is not small but large.

Thus, we have to treat problems in which finite deformation occurs in time-depending and non-linear materials.

The theoretical constitution of the system of the governning equations for the consolidation of soils are similar to that for common problems in solid mechanics. Indeed, the differential equations that governs consolidation of soils are
1.equations for the equilibrium of stresses,
2.equations for the mass balance of solid soil particles and liquid pore water, and
3.constitutive laws for the soil of interest.

However, particular attentions have to be paid because of some difficulties cited in the above. In particular, we note that the balance of mass, both for soilid soil particles and for liquid pore water, has to be implemented.

Some finite deformation theories for one-dimensional consolidation of soils and solutions to them have been proposed in which field and constitutive equations are combined to derive one differential equation containning one unknown function (e.g., Mikasa(1967), Gibson et al. (1967)). In such an approach, some kinds of assumptions have to be made for the developement of those theories: the first kind of assumptions are concerned with the simplification or linearization of the equation; the other concerned with the simplification of the constitutive laws of soils. Some of these assumptions restrict the applicability of the solutions, e.g., the deformation is small; constitutive properties are constant in the process of the consolidation, including the assumption of constant coeffiecient of consolidation. Therefore we can not expect full understanding of the consolidation phenomenon, even in one-dimensional consolidation, on the basis of such analytical solutions.

In some cases such as in the determination of consolidation properties from conventional or other types of consolidation tests, an analytical solution is required. However in some cases such as the prediction of ground settlement due to the consolidation of soil layers, analytical solutions will not always provide correct prediction because of the assumptions as mentioned above. Further, as the Authors have pointed out, theories for determining consolidation properties from CRS consolidation tests can be examined only by using the test results on soils of which constitutive relations are known. In the last example, a numerical procedure such as the Finite Element Method (FEM) is required in which constitutive relations should have been given.

The author analyzed the ground settlement caused by the consolidation of soft clay layers with the assumption of small strain (Shimizu, 1991). As for the large deformation analyses, they obtained numerical solutions for the ground settlement by instantaneous loading and the constant-rate-of-strain consolidation tests (Shimizu et al., 1994). However in those analyses the equation for the mass balance of soils were simplified.

The objective of this study is to develope a method by which we can obtain numerical solutions, by the FEM, to the governning equations, in which small deformation is not assumed but finite deformation is considered without any linearization assumption.

In this paper, firstly, the governing equations are derived based on the continuum mechanics; large deformation is considered with a non-linear constitutive law. Secondly, the method of solving the system of the governing equations by the FEM is described. Rigorous analysis of the derived system has not yet been made.

2. Governing equations

Governing equations for the one-dimensional consolidation of soils are formulated. They are
1. the equation for the equilibrium of stresses,
2. equations for the mass balance,
3. constitutive laws for the soil of interest, and
4. the flow law of pore water.

The first two are field equations.

One-dimensional consolidation of soils is described by the vertical movement of a horizontal plane with time; the plane is idealized to one that has no thickness but consists of soil grains and pore water. We define the movement of the plane by the movement of the soil grains that the plane contains; in other words pore water particles that the plane contains can be different at different instances.

Any physical or mechanical quantity, ϕ, associated with material particles is a function of spatial and time variables when the particles move and the body deforms. We distinguish, to avoid confusion, the spatial change of ϕ at a particular time, denoted by $\Delta\phi$, and the change during a time duration δt for a particular material particle, $\delta\phi$.

The saturated soil is a two-phase material that consists of solid soil particles and liquid pore water, whose movement is different each other. In what follows the suffix 's' denotes quantities associated with soil particles and 'w' those with pore water.

2.1 Deformation

(1) Motion

Using a coordinate axis, denoted by ξ, that is fixed in the space, we specify the position of soil particles and pore water particles in the soil by the value of ξ (see Fig.1 and also Appendix 1).

Motions of soil particles and pore water are different: for the motion of soil particles

$$z = \chi_s(Z,t) \tag{2.1}$$

and for that of pore water

$$z = \chi_w(Z,t). \tag{2.2}$$

These equations state that soil particles and pore water particles locating at Z at a reference time t_0 move to z at time t.

(2) Strain of the element (average strain)

Consider a soil element that occupies the spatial domain $Z \le \xi \le Z + \Delta Z$ at a reference time t_0. We describe the deformation of the element by the motion of the soil particles contained in the element. The soil particles move to another domain $z \le \xi \le z + \Delta z$ at a time t, and the thickness of the element changes from ΔZ, at the reference time, to Δz, at the time t, and to $\Delta z'$, at a subsequent time $t+\delta t$, where

$$\Delta z = \chi_s(Z + \Delta Z, t) - \chi_s(Z,t) \tag{2.3}$$

$$\Delta z' = \chi_s(Z + \Delta Z, t + \delta t) - \chi_s(Z, t + \delta t) \tag{2.4}$$

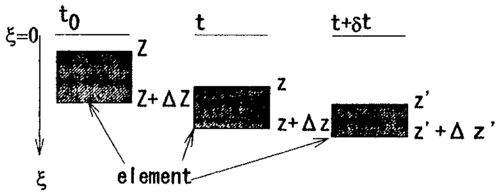

Fig.1: Definition of the spatial coordinate: Time t_0 is the reference time; and the element drawn in the figure consists of the same soil particles.

Referring to Fig.1, we define the average strains of the element at the time t and the subsequent time $t+\delta t$ as:

$$\bar{\varepsilon}(t) = -\frac{\Delta z - \Delta Z}{\Delta Z} \tag{2.5}$$

and

$$\bar{\varepsilon}(t + \delta t) = -\frac{\Delta z' - \Delta Z}{\Delta Z} \tag{2.6}$$

respectively. The strain was defined to be zero at the reference configuration. The increment of the average strain during the time increment δt becomes

$$\delta\bar{\varepsilon} = \bar{\varepsilon}(t + \delta t) - \bar{\varepsilon}(t) = -\frac{\Delta z' - \Delta z}{\Delta Z} \tag{2.7}$$

Following relations will be used later:

$$\Delta z = \left(1 - \bar{\varepsilon}(t)\right)\Delta Z, \tag{2.8}$$

and

$$\Delta z' = \left(1 - \bar{\varepsilon}(t + \delta t)\right)\Delta Z. \tag{2.9}$$

(3) Strain field

At the limits of $\Delta Z \to 0$ and therefore $\Delta z \to 0$ and $\Delta z' \to 0$, the average strain and average strain increment of the element lead to the strain and strain increment, respectively, at the upper end of the element as:

$$\varepsilon(z,t) = \lim_{\Delta Z \to 0} \bar{\varepsilon}(t) = -\left(\frac{\partial z}{\partial Z} - 1\right) \tag{2.10}$$

$$\varepsilon(z', t + \delta t) = \lim_{\Delta Z \to 0} \bar{\varepsilon}(t + \delta t) = -\left(\frac{\partial z'}{\partial Z} - 1\right) \tag{2.11}$$

$$\delta\varepsilon(z,t;\delta t)=\varepsilon(z',t+\delta t)-\varepsilon(z,t)=-\left(\frac{\partial z'}{\partial Z}-\frac{\partial z}{\partial Z}\right) \tag{2.12}$$

The following relation will be used later:

$$\frac{\partial z'}{\partial z}=\frac{\partial z'}{\partial Z}\frac{\partial Z}{\partial z}=\frac{1-\varepsilon(z',t+\delta t)}{1-\varepsilon(z,t)} \tag{2.13}$$

The strain and strain increment at the upper end of the element give their current fields because the element having so far been considered is quite arbitrary; Z and therefore z or z' are arbitrary.

The strain rate can be defined as

$$\dot{\varepsilon}=\lim_{\delta t\to 0}\frac{\delta\varepsilon(z,t;\delta t)}{\delta t} \tag{2.14}$$

(4) Velocity gradient

Current velocities of soil particles at $\xi=z$ and $z+\Delta z$ are defined, according to eqs.(2.1) and (2.2), as

$$v_s(z,t)=\lim_{\delta t\to 0}\frac{1}{\delta t}\{\chi_s(Z,t+\delta t)-\chi_s(Z,t)\} \tag{2.15}$$

and

$$v_s(z+\Delta z,t)=\lim_{\delta t\to 0}\frac{1}{\delta t}\{\chi_s(Z+\Delta Z,t+\delta t)-\chi_s(Z+\Delta t,t)\}. \tag{2.16}$$

The velocity gradient at the current configuration is defines as

$$\frac{\partial v_s(z,t)}{\partial z}=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\{v_s(z+\Delta z,t)-v_s(z,t)\} \tag{2.17}$$

Using the definitions for the velocity , eqs.(2.15) and (2.16), we can express the current gradient of velocity as:

$$\begin{aligned}\frac{\partial v_s(z,t)}{\partial z}&=\lim_{\Delta z\to 0}\frac{1}{\Delta z}\left\{\lim_{\delta t\to 0}\frac{1}{\delta t}(\Delta z'-\Delta z)\right\}\\&=\lim_{\delta t\to 0}\frac{1}{\delta t}\left\{\lim_{\Delta z\to 0}\frac{1}{\Delta z}(\Delta z'-\Delta z)\right\}\\&=\lim_{\delta t\to 0}\frac{1}{\delta t}\left\{-\frac{\delta\varepsilon(z,t;\delta t)}{1-\varepsilon(z,t)}\right\}\\&=-\frac{\dot{\varepsilon}(z,t)}{1-\varepsilon(z,t)}\end{aligned} \tag{2.18}$$

or

$$\dot{\varepsilon}(t)=-\{1-\varepsilon(z,t)\}\frac{\partial v_s}{\partial z}. \tag{2.18'}$$

Eqs.(2.3) to (2.14) were used.

2.2 Equilibrium of stresses

It is preferable to express the equilibrium of stresses in an incremental form because the soil has non-linear constitutive properties. To derive the incremental form of equation for the equilibrium of stresses, we consider two configurations at a time t and a subsequent time $t+\delta t$.

In finite element analyses, the configuration at the time $t+\delta t$ is determined based on the configuration at the preceding time t; we calculate the changes of physical or mechanical quantities during δt. The changes have to satisfy the equilibrium of stresses at time $t+\delta t$.

(1) Equilibrium of stresses at $t+\delta t$

The equilibrium of forces acting on/in an element of thickness $\Delta z'$ at time $t+\delta t$ yields the following equation (see Fig.2):

$$\sigma(z',t+\delta t)-\sigma(z'+\Delta z',t+\delta t)+\bar{b}(t+\delta t)\Delta z'=0, \tag{2.19}$$

with

$$\bar{b}(t+\delta t)=\frac{1}{\Delta z'}\int_{z'}^{z'+\Delta z'}\rho(\xi,t+\delta t)g\,d\xi \tag{2.20}$$

where $\rho(\xi,t)$ is the total density of soil, and g is the acceleration of gravity. At the limit of $\Delta z'\to 0$,

$$b(z',t+\delta t)=\lim_{\Delta z'\to 0}\bar{b}(t+\delta t)=\rho(z',t+\delta t)g. \tag{2.21}$$

Introducing the stress increment defined as

$$\delta\sigma(z,t;\delta t)=\sigma(z',t+\delta t)-\sigma(z,t) \tag{2.22}$$

and

$$\delta\sigma(z+\Delta z,t;\delta t)=\sigma(z'+\Delta z',t+\delta t)-\sigma(z+\Delta z,t), \tag{2.23}$$

eq.(2.19) is expressed in terms of the increments:

$$\{\delta\sigma(z+\Delta z,t;\delta t)-\delta\sigma(z,t;\delta t)\}+ \{\sigma(z+\Delta z,t)-\sigma(z,t)\}-\bar{b}(t+\delta t)\Delta z'=0 \quad (2.24)$$

In the above we consider the stress equilibrium at time t, which is

$$\sigma(z,t)-\sigma(z+\Delta z,t)+\bar{b}(t)\Delta z=0, \quad (2.25)$$

with

$$\bar{b}(t)=\frac{1}{\Delta z}\int_z^{z+\Delta z}\rho(\xi,t)g\,d\xi. \quad (2.26)$$

Indeed, eq.(2.24) is modified to

$$\{\delta\sigma(z+\Delta z,t;\delta t)-\delta\sigma(z,t;\delta t)\}-\bar{b}(t+\delta t)\Delta z'+\bar{b}(t)\Delta z=0 \quad (2.27)$$

Dividing by Δz and taking the limit Δz→0,

$$\frac{\partial\delta\sigma(z,t;\delta t)}{\partial z}-\delta b(z,t;\delta t)+\frac{b(z,t)+\delta b(z,t;\delta t)}{1-\varepsilon(z,t)}\delta\varepsilon(z,t;\delta t)=0, \quad (2.28)$$

where

$$\delta b(z,t;\delta t)=b(z',t+\delta t)-b(z,t) \quad (2.29)$$

In the above, the term δbδε is a small quantity of higher order than other terms. If we neglect it, eq.(2.28) leads to

$$\frac{\partial\delta\sigma(z,t;\delta t)}{\partial z}-\delta b(z,t;\delta t)+\frac{b(z,t)}{1-\varepsilon(z,t)}\delta\varepsilon(z,t;\delta t)=0 \quad (2.30)$$

Eq.(2.28) or (2.30) is the stress equilibrium that the increment δσ, δε and δb have to satisfy. The term including δε expresses the effect of finite strain or geometrical non-linearity. The term δb reflects the strain and is not explicitly given. An iterative scheme is needed in the finite element analysis because of the evaluation of δb. Further we note here that the gradient is evaluated at the current (time t) configuration.

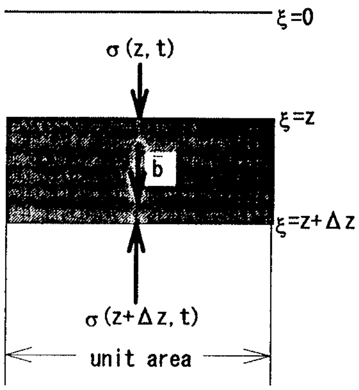

Fig.2: Forces acting on/in the element at time t.

If we assume that the strain increment is negligibly small, then eq.(2.28) and (2.30) reduce to:

$$\frac{\partial\delta\sigma(z,t;\delta t)}{\partial z}-\delta b(z,t;\delta t)=0 \quad (2.31)$$

The last equation can be used for the problems in which the strain is assumed to be small but the constitutive relations are non-linear and the body force changes (Shimizu,1991).

2.3 Balance of mass for saturated soils

(1) General

We derive the equation for the balance of mass for saturated soils. Consider a soil element that occupy the space $z\le\xi\le z+\Delta z$ at time t, as shown in Fig.2. As explained earlier, our element always consists of the same soil particles and the velocities of the boundaries are equal to the velocity of soil particles at the boundaries.

The balance of mass for a soil element during finite time increment δt requires the following conditions if no interchange of mass between soil particles and pore water is assumed: for soil grains

$$M_S(t+\delta t)-M_S(t)=0 \quad (2.32)$$

and for the pore water

$$M_W(t+\delta t)-M_W(t)=\int_t^{t+\delta t}Q_W(\tau)d\tau, \quad (2.33)$$

where M_S and M_W are the mass of soil grains and that of pore water contained in the element, respectively;

Q_W is the mass of the water that inflows into the element through the boundaries per unit time. They are defined as:

$$M_s(t) = \int_z^{z+\Delta z} m_s(\xi,t)d\xi \tag{2.34}$$

with

$$m_s(\xi,t) = \rho_s(\xi,t)\{1-n(\xi,t)\}, \tag{2.35}$$

$$M_w(t) = \int_z^{z+\Delta z} m_w(\xi,t)d\xi \tag{2.36}$$

with

$$m_w(\xi,t) = \rho_w(\xi,t)n(\xi,t), \tag{2.37}$$

and

$$Q_w(t) = -[q_w(z+\Delta z,t) - q_w(z,t)] \tag{2.38}$$

with

$$q_w(z+\Delta z,t) = v_r(z+\Delta z,t)n(z+\Delta z,t)\rho_w(z+\Delta z,t) \tag{2.39}$$

$$q_w(z,t) = v_r(z,t)n(z,t)\rho_w(z,t), \tag{2.40}$$

in which v_r is the relative velocity of pore water to soil grains defined as

$$v_r = v_w - v_s \tag{2.41}$$

At the limit of $\delta t \to 0$, the conditions, eq.(2.32) and (2.33) become, for soil particles,

$$\frac{d}{dt}M_s(t) = 0 \tag{2.32'}$$

and for the pore water

$$\frac{d}{dt}M_w(t) = Q_w(t), \tag{2.33'}$$

To implement the conditions, eqs.(2.32') and (2.33'), in the finite element analyses, the time increment δt has to be sufficiently small because they were derived by taking the limit of $\delta t \to 0$.

(2) Mass balance for soil particles

The condition, eq.(2.32'), leads to the following differential form for the mass balance for soil grains (see Appendix 2):

$$(1-n)\frac{\dot{\rho}_s}{\rho_s} - \dot{n} + (1-n)\frac{\partial v_s}{\partial z} = 0, \tag{2.42}$$

where

$$\dot{\rho}_s = \frac{\partial \rho_s}{\partial t} + \frac{\partial \rho_s}{\partial z}v_s \tag{2.43}$$

and

$$\dot{n} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial z}v_s. \tag{2.44}$$

All the quantities are defined at a time t. The quantity $\dot{\rho}_s$ is the material derivative of the density of soil grains; on the other hand $\dot{n}$ means only apparently the material derivative of n because n is not associated with soil grains.

(3) Mass balance for pore water
By the way described in Appendix 2, the condition, eq.(2.33'), leads to the following differential form:

$$n\frac{\dot{\rho}_w}{\rho_w} + \dot{n} + \frac{\partial}{\partial z}(nv_r) + n\frac{\partial v_s}{\partial z} = 0 \tag{2.45}$$

(4) Mass balance for saturated soils
Eliminating $\dot{n}$ from eqs.(2.42) and (2.45), we obtain the general equation for the mass balance of saturated soils as below:

$$n\frac{\dot{\rho}_w}{\rho_w} + (1-n)\frac{\dot{\rho}_s}{\rho_s} + \frac{\partial}{\partial z}(nv_r) + \frac{\partial v_s}{\partial z} = 0 \tag{2.46}$$

If the incompressibility of soil grains is assumed, then

$$n\frac{\dot{\rho}_w}{\rho_w} + \frac{\partial}{\partial z}(nv_r) + \frac{\partial v_s}{\partial z} = 0. \tag{2.47}$$

Moreover if the incompressibility of pore water is assumed, then

$$\frac{\partial}{\partial z}(nv_r) + \frac{\partial v_s}{\partial z} = 0, \tag{2.48}$$

which is a usual equation for the incompressible pore water flow in a deformable soil skeleton.

In this study we use eq.(2.46) bacause we do not assume the incompressibility of pore water to hold the generality. One of advantages of taking into account the compressibility of pore fluid was demonstrated in the problem of wave-induced generation of pore water pressure in sand seabeds (Shimizu, 1994). One another reason is that, if we assume it, some difficulty will arise when instantaneous loading on the ground is analyzed by the FEM.

2.4 Constitutive equations for soils

We can express non-linear constitutive equations of soils by the relations between strain rate and effective stress rate as:

$$\dot{\varepsilon} = m_v \dot{\sigma}' \tag{2.49}$$

or

$$\dot{\sigma}' = \frac{1}{m_v}\dot{\varepsilon}, \tag{2.50}$$

where $\dot{\sigma}'$ is the effective stress defined as $\dot{\sigma}' = \dot{\sigma} - \dot{p}$. The non-linearity is taken into account by the non-constant coefficient of volume compressibility m_v. The coefficient is a function of σ' or ε. The function can be evaluated from the *e*-log*p* relation (Shimizu,1991).

2.5 Compressibility of pore water

The density of pore water is assumed to be a unique function of pore water pressure p as:

$$\frac{\dot{\rho}_w}{\rho_w} = \beta \dot{p}, \tag{2.51}$$

where β is the coefficient of volume compressibility of water; it can be assumed to be constant.

2.6 Darcy's law

We assume that the flow of pore water in saturated soils obeys Darcy's law, which is

$$nv_r = -k\frac{\partial h}{\partial z}, \tag{2.52}$$

where k is the coefficient of permeability and h is the total head. The total head is the sum of the elevation head and the pressure head as

$$h = -(z - z_0) + \frac{p}{\rho_w g}, \tag{2.53}$$

In the above z_0 is the datum for the head, which is a constant arbitrarily chosen, and p is the pore water pressure. We note that the law does not control the velocity of pore water itself but its relative velocity to the velocity of soil particles; Gibson et al.(1967) gave a physical interpretation of Darcy's law. Imai(1987) also gave an interpretation of the law.

Considering that ρ_w is a function of spatial and time variables as well as p is, we have

$$\frac{\partial h}{\partial z} = -1 + \frac{1+\beta p}{\rho_w g}\frac{\partial p}{\partial z} \tag{2.54}$$

where the relation that

$$\frac{1}{\rho_w}\frac{\partial \rho_w}{\partial z} = \beta\frac{\partial p}{\partial z} \tag{2.55}$$

was used (see eq.(2.51)).

We can justify the approximation of $1+\beta p \cong 1$ as follows: the order of magnitude of β is $5\times10^{-10}\mathrm{m}^2/\mathrm{N}$, that of p encountered in practical consolidation problems including laboratory tests is at most $2\times10^{6}\mathrm{N}/\mathrm{m}^2$ and

$$\beta p < 10^{-3}. \tag{2.56}$$

With this approximation, eq.(2.54) yields

$$nv_r = -k\left(-1 + \frac{1}{\rho_w g}\frac{\partial p}{\partial z}\right) \tag{2.57}$$

as Darcy's law. This is the equation that we should use in FE analyses.

Gibson et al. (1967) used, in Darcy's law, the excess pore water pressure, which is defined as the deviation of pore water pressure from the hydrostatic component. However, here, such a separation of the pore water pressure is not done because confusion is sometimes seen in the definition of the excess pore water pressure (Gibson et al., 1989).

3. Finite element formulation

In this section we transform the differential equations derived in the preceding section to a discrete system of equations for the FEM. We follow the procedures as:
1. To derive weak forms or integral forms of the differential equations by using the weighted residual method.
2. To introduce the finite element approximation of unknown functions to discretize the unknowns.
3. To discretize the weak form governing equations by using the Galerkin method.

In the finite element analyses, we calculate increments of unknowns in the time increment δt from a time t to a subsequent time $t+\delta t$.

3.1. Weighted residual

(1) Equilibrium of stresses
The equilibrium of stresses is given by:

$$\frac{\partial \delta\sigma(z,t;\delta t)}{\partial z} - \delta b(z,t;\delta t) + \frac{b(z,t)+\delta b(z,t;\delta t)}{1-\varepsilon(z,t)}\delta\varepsilon(z,t;\delta t) = 0, \tag{2.28bis}$$

or

$$\frac{\partial \delta\sigma(z,t;\delta t)}{\partial z} - \delta b(z,t;\delta t) + \frac{b(z,t)}{1-\varepsilon(z,t)}\delta\varepsilon(z,t;\delta t) = 0 \tag{2.30bis}$$

In what follows we base the formulation for FEM on eq.(2.28bis).

The weighted residual of the above equation in an arbitrary domain $z_1 \le z \le z_2$ is defined as

$$r_e \equiv \int_{z_1}^{z_2} w_e\left(\frac{\partial\delta\sigma}{\partial z} - \delta b + \frac{b+\delta b}{1-\varepsilon}\delta\varepsilon\right) = 0, \tag{3.1}$$

where w_e is a weighting function. The weighting function can be chosen arbitrarily. With the integration by parts, eq.(3.1) is modified to

$$r_e \equiv \{w_e(z_2)\delta\sigma(z_2) - w_e(z_1)\delta\sigma(z_1)\} + \int_{z_1}^{z_2}\left\{-\frac{\partial w_e}{\partial z}\delta\sigma - w_e\delta b + w_e\frac{b+\delta b}{1-\varepsilon}\delta\varepsilon\right\}dz \tag{3.2}$$

In this equation we consider the relationships between the increments of total stresses and surface traction at the boundaries such as

$$\delta\sigma(z_1) = \delta q(z_1) \tag{3.3}$$

$$\delta\sigma(z_2) = -\delta q(z_2), \tag{3.4}$$

where q is the surface traction (see Fig.3), and the definition of effective stress increments such as

$$\delta\sigma = \delta\sigma' + \delta p. \tag{3.5}$$

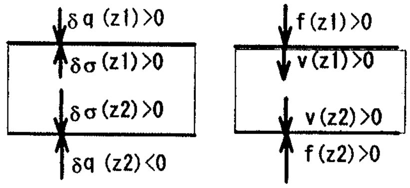

Fig.3: Definition of flux and surface traction.

Further, considering the constitutive relations, eq.(2.49), we obtain the weighted residual for the equilibrium of stresses as

$$r_e \equiv -\{w_e(z_1)\delta q(z_1) + w_e(z_2)\delta q(z_2)\} + \int_{z_1}^{z_2}\left(-\frac{\partial w_e}{\partial z}\frac{\delta\varepsilon}{m_v} - w_e\delta b + w_e\frac{b+\delta b}{1-\varepsilon}\delta\varepsilon - \frac{\partial w_e}{\partial z}\delta p\right)dz \tag{3.6}$$

(2) Mass balance
The equation that describes the balance of mass, at t+dt for saturated soils was already derived as eq.(2.47). The equation is valid at any time, and we have the equation for the mass balance at $t'(=t+\delta t)$ as

$$n(t')\frac{\dot{\rho}_w(t')}{\rho_w(t')} + \frac{\partial}{\partial z'}v_a(t') + \frac{\partial v_s(t')}{\partial z'} = 0, \tag{3.7}$$

where

$$v_a(t') = n(t')v_r(t') \tag{3.8}$$

In the above, and in what follows, the spatial variable z' at time $t'(=t+\delta t)$ is not written for simplicity of description.

Introducing, in eq.(3.7), the compressibility of pore water, eq(2.51), and velocity gradient of soil particles, eq.(2.18), we obtain the following:

$$n(t')\beta\dot{p}(t')+\frac{\partial v_a(t')}{\partial z'}-\frac{1}{1-\varepsilon(t')}\dot{\varepsilon}(t')=0 \tag{3.9}$$

The gradient at t' has to be transformed to the gradient at t, because in the FE analyses, the configuration at the preceding time t is known but that at $t+\delta t$ is not known. In facts

$$\frac{\partial v_a(t')}{\partial z'}=\frac{\partial v_a(t')}{\partial z}\frac{\partial z}{\partial z'}=\frac{\partial v_a(t')}{\partial z}\frac{1-\varepsilon(t)}{1-\varepsilon(t')} \tag{3.10}$$

Eq.(3.9) is rewritten to:

$$n(t')\beta\frac{1-\varepsilon(t')}{1-\varepsilon(t)}\dot{p}(t')+\frac{\partial v_a(t')}{\partial z}-\frac{1}{1-\varepsilon(t)}\dot{\varepsilon}(t')=0 \tag{3.11}$$

where eq.(2.13) was used.

The weighted residual of eq.(3.11) is given as

$$r_p\equiv\int_{z_1}^{z_2}w_p\left[n(t')\beta\frac{1-\varepsilon(t')}{1-\varepsilon(t)}\dot{p}(t')+\frac{\partial v_a(t')}{\partial z}-\frac{1}{1-\varepsilon(t)}\dot{\varepsilon}(t')\right]=0. \tag{3.12}$$

Taking into account the following relations, eqs.(3.13) to (3.15), the weighted residual r_p is modified to eq.(3.16).

$$w_p\frac{\partial v_a(t')}{\partial z}=\frac{\partial}{\partial z}\left(w_p v_a(t')\right)-\frac{\partial w_p}{\partial z}v_a(t'), \tag{3.13}$$

and

$$v_a(t')=-k(t')\left[-1+\frac{1}{\rho_w(t')g}\frac{\partial p(t')}{\partial z'}\right], \tag{3.14}$$

where

$$\frac{\partial p(t')}{\partial z'}=\left\{\frac{\partial p(t)}{\partial z}+\frac{\partial\delta p}{\delta z}\right\}\frac{1-\varepsilon(t)}{1-\varepsilon(t')}. \tag{3.15}$$

$$\begin{aligned}r_p\equiv&\left\{-w_p(z_1)f(z_1)+w_p(z_2)f(z_2)\right\}\\&+\left\{-w_p(z_1)\delta f(z_1)+w_p(z_2)\delta f(z_2)\right\}\\&+\int_{z_1}^{z_2}w_p\frac{1-\varepsilon(t')}{1-\varepsilon(t)}\beta n(t')\dot{p}(t')dz\\&-\int_{z_1}^{z_2}\frac{\partial w_p}{\partial z}k(t')dz\\&+\int_{z_1}^{z_2}\frac{\partial w_p}{\partial z}\frac{k(t')}{\rho_w(t')g}\frac{1-\varepsilon(t)}{1-\varepsilon(t')}\left(\frac{\partial p(t)}{\partial z}+\frac{\partial\delta p}{\partial z}\right)dz\\&-\int_{z_1}^{z_2}w_p\frac{1}{1-\varepsilon(t)}\dot{\varepsilon}(t')dz,\end{aligned} \tag{3.16}$$

where f denotes the flux at the boundary planes. The flux is defined positive when water inflows the element.(see Fig.3)

(3) Finite difference in the time domain

The weighted residuals include $\dot{\varepsilon}$ and $\dot{p}$ that are the rates at the time $t+dt$. We approximate the rates by such finite differentiation as

$$\dot{\varepsilon}|_{t+\delta t}\cong\frac{\varepsilon(t+\delta t)-\varepsilon(t)}{\delta t}=\frac{\delta\varepsilon}{\delta t}, \tag{3.17}$$

and

$$\dot{p}|_{t+\delta t}\cong\frac{p(t+\delta t)-p(t)}{\delta t}=\frac{\delta p}{\delta t} \tag{3.18}$$

Further the strain increment is replaced by the displacement increment as

$$\delta\varepsilon=-(1-\varepsilon(z,t))\frac{\partial\delta u}{\partial z} \tag{3.19}$$

where

$$\delta u=u(z',t+\delta t)-u(z,t) \tag{3.20}$$

with

$$u(z,t)=z-Z \tag{3.21}$$

and

$$u(z',t+\delta t)=z'-Z \tag{3.22}$$

Finally we obtain the weighted residuals of the equation for the equilibrium of stresses and the equation for the mass balance as

$$\begin{aligned}r_e\equiv&-\left\{w_e(z_1)\delta q(z_1)+w_e(z_2)\delta q(z_2)\right\}\\&+\int_{z_1}^{z_2}\left(\frac{\partial w_e}{\partial z}\frac{1-\varepsilon}{m_v}-w_e(b+\delta b)\right)\frac{\partial\delta u}{\partial z}dz\end{aligned}$$

$$-\int_{z_1}^{z_2}\frac{\partial w_e}{\partial z}\delta p dz-\int_{z_1}^{z_2}w_e\delta b dz=0 \tag{3.23}$$

$$\begin{aligned}r_p \equiv &\{-w_p(z_1)f(z_1)+w_p(z_2)f(z_2)\}\\ &+\{-w_p(z_1)\delta f(z_1)+w_p(z_2)\delta f(z_2)\}\\ &+\frac{1}{\delta t}\int_{z_1}^{z_2}w_p\frac{1-\varepsilon(t')}{1-\varepsilon(t)}\beta n(t')\delta p dz\\ &-\int_{z_1}^{z_2}\frac{\partial w_p}{\partial z}k(t')dz\\ &+\int_{z_1}^{z_2}\frac{\partial w_p}{\partial z}\frac{k(t')}{\rho_w(t')g}\frac{1-\varepsilon(t)}{1-\varepsilon(t')}\left(\frac{\partial p(t)}{\partial z}+\frac{\partial \delta p}{\partial z}\right)dz\\ &+\frac{1}{\delta t}\int_{z_1}^{z_2}w_p\frac{\partial \delta u}{\partial z}dz,\end{aligned} \tag{3.24}$$

3.2 Discretization

Unknown functions are displacement increment, δu and pore pressure increment δp. It is natural that δu is approximated with a nodal function of higher order than δp in the weighted residuals, eqs.(3.18) and (3.19), because the order of derivative for δu is higher than that for δp.

As in usual textbooks for FEM, we introduce vector notation in what follows. The notation <x> denotes a row vector and {x} denotes a column vector. A matrix is denoted by [x].

The increments of displacement and pore pressure are approximated in an element by the functions:

$$\delta u=<N_u>\{\delta\bar{u}\}, \tag{3.25}$$

and

$$\delta p=<N_p>\{\delta\bar{p}\}, \tag{3.26}$$

where $<N_u>$ and $<N_p>$ are nodal approximation functions; $\{\delta\bar{u}\}$ and $\{\delta\bar{p}\}$ are vectors of which components are nodal values of du and dp, respectively. $<N_u>$ is a second order function of a normalized space variable and $<N_p>$ a first order function. The detail of the nodal approximation can be referred to a textbook, e.g., (Dhatt et al., 1984).

According to the Galerkin method, we also discretize weighting functions, w_e and w_p, as

$$w_e=<N_u>\{\bar{w}_e\}, \tag{3.27}$$

$$w_p=<N_p>\{\bar{w}_p\}. \tag{3.28}$$

Weighted residuals for an element are thus

$$\begin{aligned}r_e=<\bar{w}_e>[&([k_1]-[k_2]-[k_3])\{\delta\bar{u}\}\\ &-[l]\{\delta\bar{p}\}-\{\delta b\}-\{\delta s\}]\end{aligned} \tag{3.29}$$

$$\begin{aligned}r_p=<\bar{w}_p>&\left[\frac{1}{\delta t}[l]^{\mathrm{T}}\{\delta\bar{u}\}+\left(\frac{1}{\delta t}[c]+[a]\right)\{\delta\bar{p}\}\right.\\ &-\{e\}+\{f\}+\{\delta f\}+[a]\{\bar{p}\}]\end{aligned} \tag{3.30}$$

where

$$[k_1]=\int_{z_1}^{z_2}\{B_u\}\frac{1-\varepsilon}{m_v}<B_u>dz \tag{3.31}$$

$$[k_2]=\int_{z_1}^{z_2}\{N_u\}\delta b<B_u>dz \tag{3.32}$$

$$[k_3]=\int_{z_1}^{z_2}\{N_u\}b<B_u>dz \tag{3.33}$$

$$[l]=\int_{z_1}^{z_2}\{B_u\}<N_p>dz \tag{3.34}$$

$$\{\delta s\}=\{N_u(z_1)\}\delta q(z_1)+\{N_u(z_2)\}\delta q(z_2) \tag{3.35}$$

$$[\delta b]=\int_{z_1}^{z_2}\{N_u\}\delta b dz \tag{3.36}$$

$$[a]=\int_{z_1}^{z_2}\{B_p\}\frac{k(t')}{\rho_w(t')g}\frac{1-\varepsilon(t)}{1-\varepsilon(t')}<B_p>dz \tag{3.37}$$

$$[c]=\int_{z_1}^{z_2}\{N_p\}\beta n(t')\frac{1-\varepsilon(t')}{1-\varepsilon(t)}<N_p>dz \tag{3.38}$$

$$\{e\}=\int_{z_1}^{z_2}<B_p>k(t')dz \tag{3.39}$$

$$\{f\}=-\{N_p(z_1)\}f(z_1)+\{N_p(z_2)\}f(z_2) \tag{3.40}$$

$$\{\delta f\}=-\{N_p(z_1)\}\delta f(z_1)+\{N_p(z_2)\}\delta f(z_2), \tag{3.41}$$

where

$$<B_u>=\frac{\partial}{\partial z}<N_u> \tag{3.42}$$

and

$$<B_p>=\frac{\partial}{\partial z}<N_p> \tag{3.43}$$

3.4 Assemble

Considering the continuity of unknown and weighting functions, the weighted residuals in the global system can be obtained (assembly). The arbitrariness of the weighting functions leads to the global system of equations to be solved as

$$\begin{bmatrix} [K_1]-[K_2]-[K_3] & -[L] \\ [L]^T & [C]+\delta t[A] \end{bmatrix} \begin{Bmatrix} \{\delta \bar{U}\} \\ \{\delta \bar{P}\} \end{Bmatrix} = \begin{Bmatrix} \{\delta B\}+\{\delta S\} \\ \delta t(\{E\}-\{F\}-\{\delta F\}-[A]\{\bar{P}\}) \end{Bmatrix}, \quad (3.44)$$

where capital letters are used to specify matrices or vectors in the global system.

4. Conclusions

A theoretical formulation was made in which small deformation is not assumed but finite deformation is considered with a non-linear constitutive law. Then the method of solving, by the FEM, the derived system of the governing equations was presented. Rigorous analysis of the system that was developed here has not yet been made.

Appendix 1: Motion of material points

Using a coordinate axis, denoted by x, that is fixed in the space, we specify the position of any material particle in the body by the value of x (see Fig.1). The one-dimensional motion of material particles can be described by the following equation:

$$z=\chi(Z,t), \quad (A1)$$

which indicates that material points locating at the position ξ=Z at a reference time t_0 move to the position ξ=z at some subsequent time t. From this definition,

$$Z=\chi(Z,t_0). \quad (A2)$$

Moreover, the position at time $t+\delta t$ is given by

$$z'=\chi(Z,t+\delta t) \quad (A3)$$

Consider material points of which motion is described by eq.(A1). If a physical or mechanical quantity, ϕ, associated with these material points changes during the time duration δt, then the change df is given by

$$\delta\phi(z,t;\delta t)=\phi(z',t+\delta t)-\phi(z,t), \quad (A4)$$

which gives the time rate of ϕ as

$$\dot{\phi}=\lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}=\frac{\partial\phi}{\partial z}v_\phi+\frac{\partial\phi}{\partial t}, \quad (A5)$$

where

$$v_\phi=\frac{\partial\chi(Z,t)}{\partial t}, \quad (A6)$$

which is the velocity of the material points with which the physical quantity ϕ is associated: e.g., when the material points refer to soil grains, v_ϕ means the velocity of soil grains; when the material points refer to the pore water, it means the velocity of pore water; and so on.

Appendix 2: Mass balance

$$\frac{d}{dt}M_s=\lim_{\delta t\to 0}\frac{1}{\delta t}\{M_s(t+\delta t)-M_s(t)\} \quad (A7)$$

where

$$M_s(t+\delta t)=\int_{z'}^{z'+\Delta z'}m_s(\xi,t+\delta t)d\xi \quad (A8)$$

and

$$M_s(t)=\int_z^{z+\Delta z}m_s(\xi,t)d\xi \quad (A9)$$

with

$$m_s(\xi,t)=\rho_s(\xi,t)\{1-n(\xi,t)\}, \quad (A10)$$

When δ t is small,

$$z'\cong z+v_s(z,t)\delta t \quad (A11)$$

$$z'+\Delta z'\cong z+\Delta z+v_s(z+\Delta z,t)\delta t \quad (A12)$$

$$M_s(t+\delta t)=\int_{z+v_s(z,t)\delta t}^{z}m(\xi,t+\delta t)d\xi$$

$$+\int_z^{z+\Delta z} m(\xi, t+\delta t) d\xi$$
$$+\int_{z+\Delta z}^{z+\Delta z+v_s(z+\Delta z,t)\delta t} m(\xi, t+\delta t) d\xi \qquad \text{(A13)}$$

Considering that the domains for integral in the first and third terms are small, we can change the above to

$$M_s(t+\delta t) = -m_s(z, t+\delta t) v_s(z,t)\delta t$$
$$+\int_z^{z+\Delta z}\left\{m_s(\xi,t) + \frac{\partial m_s(\xi,t)}{\partial t}\delta t\right\}d\xi$$
$$+m_s(z+\Delta z, t+\delta t)v_s(z+\Delta z, t)\delta t \qquad \text{(A14)}$$

Further,

$$M_s(t+\delta t) = \int_z^{z+\Delta z}\left[\frac{\partial}{\partial \xi}\{m_s(z,t+\delta t)v_s(z,t)\}\delta t + \left\{m_s(\xi,t) + \frac{\partial m_s(\xi,t)}{\partial t}\delta t\right\}\right]d\xi \qquad \text{(A15)}$$

Inserting eqs.(A9) and (A15) into eq.(A7), we obtain

$$\frac{d}{dt}M_s = \int_z^{z+\Delta z}\left[\frac{\partial}{\partial \xi}\{m_s(z,t+\delta t)v_s(z,t)\} + \frac{\partial m_s(\xi,t)}{\partial t}\delta t\right]d\xi \qquad \text{(A16)}$$

Considering that the element is arbitrary or the integration domain is arbitrary, the condition that $\frac{d}{dt}M_s = 0$ leads to

$$\frac{\partial}{\partial \xi}\{m_s(\xi,t+\delta t)v_s(z,t)\} + \frac{\partial m_s(\xi,t)}{\partial t} = 0 \qquad \text{(A17)}$$

After some diferential operation, we get eq.(2.42).

As for the mass balance for the pore water,

$$\frac{dM_w}{dt} = \int_z^{z+\Delta z}\left[\frac{\partial}{\partial \xi}\{m_w(\xi,t)v_s(\xi,t)\} + \frac{\partial m_w(\xi,t)}{\partial t}\right] \qquad \text{(A.18)}$$

$$Q_w = -\int_z^{z+\Delta z}\frac{\partial}{\partial \xi}\{n(\xi,t)v_s(\xi,t)\rho_w(\xi,t)\}d\xi \qquad \text{(A19)}$$

Inserting these relations into eq.(2.33'), we have eq.(2.42).

References

Dhatt G, Touzot G andCantin G (1984): Finite Element Method Displayed, John Wiley & Sons, p.509.

Gibson RE, England GL and Hussey MJL (1967):The theory of one-dimensional consolidation of saturated clays; I Finite non-linear consolidation of thin homogeneous layers, Geotechnique, Vol.17, pp.261-273.

Gibson RE, Schiffman RL and Whitman RV (1989): On two definitions of excess pore water pressure, Geotechnique 39, No.1, pp.169-171.

Imai G (1987): Principle of soil mechanics, Japanese society for Soil Mechanics and Foundation Engineering, pp.45-91. (in Japanese)

Mikasa M (1967): The consolidation of soft clay -A new consolidation theory and its application-, Kajima Publishing Co., p.126

Shimizu M (1991):Finite element analysis of land subsidence due to the variation of groundwater level, Proc. of the 4th Int. Sympo. on Land Subsidence, Int. Asso. of Hydrology Sciences, Publication No.200, pp.645-654.

Shimizu M (1994): A simplified one-dimensional theory for the wave-induced liquefaction of sandy seabeds, Proc. of the 4th Int. Offshore and Polar Engineering Conference, pp.600-606.

Shimizu M, Umetani A and Narahara T (1994): Numerical tests of constant-rate-of-strain consolidation tests by large deformation FEM, Proc of the 29th National Conf. on Soil Mechanics and Foundation Enginering, Vol.1, pp.331-332. (in Japanese)