An Adaptive Node Relocation Procedure in FEM with Its Application to Deformation Analysis of Ground

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The finite element technique is the standard procedure of analysis for many problems in geotechnical engineering. However, the question of assessinsg the reliability of the admittedly approximate results memains to be solved.

In this paper a node-relocation adaptive procedure is proposed. The nodal forces caused by the stress discontinuity between elements are used for the adaptive relocation of nodes.

A test problem of a footing on the elastic ground is analyzed for three types of meshes : fine, medium and rough meshes. The error included in the solutions was estimated in two ways : one is to assume that the finest mesh used can give the correct solution and another to use the error estimate proposed by Zienkievicz and Zhu. It is shown through the numerical test problem that the adaptive procedure can reduce the error.

Key words : FEM, Adaptive procedure, Node relocation, Strip footing

1. Introduction

The finite element technique is the standard procedure of analysis for many problems in geotechnical engineering. However, the question of assessing the reliability of the admittedly approximate results remains to be solved.

In the usual formulation for the displacement finite element technique, the continuity of stresses between elements are *a priori* assumed, but the resulted stresses become discontinuous between elements. The main error included in the solution is caused by the discontinuity. The degree of inter-element discontinuity of stresses can be reduced by some adaptive finite element techniques: higher order interpolation method (p-method), element refinement method (h-method) and node relocation method (r-method)[1].

In this paper a node-relocation adaptive procedure is proposed. The concept of the nodal forces caused by the stress discontinuity between elements will be introduced and used for the adaptive relocation of nodes.

Zienkievicz and Zhu[2] proposed a simple method for estimating the error. In the method the correct stresses are approximated with the same interpolation functions as those for displacements and they can be evaluated from the finite element solution by requiring that the weighted residual for the estimated error be zero in the element domain.

In this study, a test problem of a footing on the elastic ground was analyzed for three types of meshes: fine, medium and rough meshes. It will be shown from the results of the analysis that the adaptive procedure can reduce the nodal forces due to stress discontinuity.

The error included in the solutions will be estimated in two ways: one is to assume that the finest mesh used can give the correct solution and another to use the error estimator proposed by Zienkievicz and Zhu.[2] It will be shown through the solutions to the test problem that the adaptive procedure can reduce the error.

In this paper, compressive stresses and strains are taken positive; and the vectors and tensors are referred to a Cartesian coordinate, (x1,x2,x3).

2. Theoretical Basis

2.1. Weighted residuals of governing equations

The equation for the equilibrium of stresses:

$$\frac{\partial \sigma_{ij}}{\partial X_j} - \bar{\bar{b}}_i = 0 \quad (i=1,2,3) \text{ in } V \tag{1}$$

Boundary conditions:

$$-\sigma_{ji}n_j = \bar{\bar{t}}_i \quad (i=1,2,3) \text{ on } \partial V_t \tag{2}$$

$$u_i = \bar{\bar{u}}_i \quad (i=1,2,3) \text{ on } \partial V_u \tag{3}$$

In eqs.(1) to (3), V is a domain to be analyzed, and ∂V is the boundary of V; σ_{ij} is the Cauchy's stress tensor, b_i is the body force per unit volume, u_i is the displacement and n_i is the outer normal unit vector of the boundary. The double bar "=" denotes the quantity specified or given.

In the displacement method of the finite element technique, displacements $u_i(i=1,2,3)$, unknown functions, are approximated with functions, of spatial coordinates, including a finite number of parameters. Approximate functions are denoted temporarily by $\tilde{u}_i(x_1,x_2,x_3)$ (i=1,2,3), i.e.

$$u_i \cong \tilde{u}_i(x_1,x_2,x_3) \quad (i=1,2,3) \tag{4}$$

Stresses are also approximate because they are related, through the constitutive relations, to the strain field compatible with the assumed displacements:

$$\sigma_{ij} \cong \tilde{\sigma}_{ij}(x) \quad (i,j=1,2,3) \tag{5}$$

where

$$\tilde{\sigma}_{ij} = D_{ijpq}\tilde{\varepsilon}_{pq} \quad (i,j=1,2,3) \tag{6}$$

$$\tilde{\varepsilon}_{pq} = -\frac{1}{2}\left(\frac{\partial \tilde{u}_p}{\partial x_q} + \frac{\partial \tilde{u}_q}{\partial x_p}\right) \quad (p,q=1,2,3) \tag{7}$$

Stresses resulted from the assumed displacements do not satisfy the equilibrium of stresses and boundary condition on ∂V_t; however the boundary condition on ∂V_u can easily be satisfied by usual nodal-approximation technique for the approximation of displacements.

We define residuals for eqs.(1), (2) and (3) as:

$$r_{\sigma i} \equiv \frac{\partial \tilde{\sigma}_{ji}}{\partial x_j} - \bar{\bar{b}}_i \quad (i=1,2,3) \tag{8}$$

$$r_{ti} \equiv -\tilde{\sigma}_{ji}n_j - \bar{\bar{t}}_i \quad (i=1,2,3) \tag{9}$$

$$r_{ui} \equiv \tilde{u}_i - \bar{\bar{u}}_i = 0 \quad (i=1,2,3) \tag{10}$$

Weighted residual in a domain V can be given as

$$R = \int_V w_i r_{\sigma i} dV + \int_{\partial V_t} w_i r_{ti} d(\partial V_t) \tag{11}$$

where the condition (10) was taken into account. In the above equation, w_i (i=1,2,3) are arbitrary weighting functions. In the weighted residual method, the problem is to find a finite number of parameters included in the approximate functions u_i by imposing that

$$R = 0 \quad \text{for any weighting functions} \tag{12}$$

2.2. Residual due to stress discontinuity between elements

In the context of the finite element method, the whole domain V is divided into finite elements V_e (e=1,2,.,N_e; N_e is the total number of elements). The residual, R, in V is considered to be the sum of residuals R_e, (e=1,2,.N_e) in V_e and their boundary ∂V_{et} (e=1,2...N_e):

$$R = \sum_{e=1}^{N_e} R_e \tag{13}$$

where

$$R_e = \int_{V_e} w_i r_{\sigma i} dV + \int_{\partial V_{et}} w_i r_{ti} d(\partial V_{et}) \tag{14}$$

The second term appears only for such elements that a part of their boundaries is also a part of ∂V_t.

By using eqs.(8) and (9) in eq.(14), we can obtain the expression for R_e as

$$R_e = \int_{\partial V_e} w_i \tilde{\sigma}_{ji} n_j d(\partial V_e)$$

$$- \int_{V_e} \frac{\partial w_i}{\partial x_j} \tilde{\sigma}_{ji} dV - \int_{V_e} w_i \bar{\bar{b}}_i dV + \int_{\partial V_{et}} w_i(-\tilde{\sigma}_{ji} n_j - \bar{\bar{t}}_i) d(\partial V_{et}) \tag{15}$$

We divide the boundary ∂V_e of an element e into ∂V_{et} and $\overline{\partial V_{et}}$, where

$$\partial V_{et} \cap \overline{\partial V_{et}} = \phi \tag{16}$$

The boundary $\overline{\partial V_{et}}$ is usually an inter-element boundary. The residual R_e can be rewritten as:

$$R_e = R_{e\sigma} + R_{eu} \tag{17}$$

where

$$R_{e\sigma} = \int_{\overline{\partial V_{et}}} w_i \tilde{\sigma}_{ji} n_j d(\overline{\partial V_{et}}) \tag{18}$$

$$R_{eu} = -\int_{V_e} \frac{\partial w_i}{\partial x_j}\sigma_{ji}dV - \int_{V_e} w_i\bar{\bar{b}}_i dV - \int_{\partial V_{et}} w_i\bar{\bar{t}}_i d(\partial V_{et}) \tag{19}$$

In a usual formulation for the displacement finite element method, the condition of inter-element continuity of stresses is *a priori* assumed, i.e.,

$$R_\sigma \equiv \sum_{e=1}^{N_e} R_{e\sigma} = 0 \tag{20}$$

In facts, for two adjacent elements, weighting functions take the same value on the inter-element boundary ; the unit outer normal vectors on the inter-element boundary have opposite directions for these two elements; and therefore, if the continuity of stresses between elements is assumed, the sum of $R_{e\sigma}$ will be reduced to zero in the whole domain V.

Thus the displacement FEM requires the following condition:

$$R_u \equiv \sum_{e=1}^{N_e} R_{eu} = 0 \quad \text{for any weighting functions} \tag{21}$$

This condition leads to the so-called stiffness equation, which is expressed in a discretized form.

However, stresses computed from the displacement finite element solution are not continuous between elements and the condition (20) can not be guaranteed. From this point of view, in this study, the residual R_σ is considered to be related to a measure for the error included in results from the displacement method.

We can not adopt R_σ itself as a measure for estimating the error because the residual R_σ is defined in terms of arbitrary weighting functions and its value can not be determined uniquely. The weighted residual R_σ may be divided, as will be described in the next paragraph, into two parts: one is the vector of nodal values of weighting functions and another the vector of nodal forces caused by stress discontinuity. We will find that the vector of nodal forces depend on the choice of weighting functions but not on the values of weighting functions and it can be determined uniquely from a finite element solution.

2.3. Error indicator

We use the matrix notation hereafter for the convenience of expression. A vector quantity is denoted by {.} in which components are arranged in a column and its transpose by <.>. A matrix is denoted by [.] and its transpose by the superscript "T". The bar in a vector, for instance, $\{\bar{a}\}$, means that components of the vector {a} are nodal values.

The approximate functions for displacement and strain are given as:

$$\{\tilde{u}\} = [N_u]\{\bar{u}\} \tag{22}$$

$$\{\tilde{\varepsilon}\} = [B]\{\tilde{u}\} \tag{23}$$

The constitutive relation is given by:

$$\{\tilde{\sigma}\} = [D]\{\tilde{\varepsilon}\} \tag{24}$$

We use the same type for weighting function as that for displacement functions (Galerkin method), i.e.,

$$\{w\} = [N_u]\{\bar{w}\} \tag{25}$$

By using expressions (22) to (25) in eq.(18), we can rewrite the expression for $R_{e\sigma}$ as:

$$R_{e\sigma} = \langle\bar{w}\rangle\{f_\sigma\} \tag{26}$$

where

$$\{f_\sigma\} = [k_\sigma]\{\bar{u}\} \tag{27}$$

$$[k_\sigma] = \int_{\overline{\partial V_{et}}} [N]^T[n][D][B][N_u]d(\overline{\partial V_{et}}) \tag{28}$$

The matrix $[n]$ is composed of components of outer normal unit vector on $\overline{\partial V_{et}}$; for example, in two dimensional problems

$$[n] = \begin{vmatrix} n_1 & 0 & n_2 \\ 0 & n_2 & n_1 \end{vmatrix} \tag{29}$$

A component of the vector $\{f_\sigma\}$ can be considered a component of a force vector applying at a node in an element e.

By the usual assembly procedure, in which the continuity condition of weighting functions and displacement are taken into account, the weighted residual R_σ is given as:

$$R_\sigma = \langle\bar{W}\rangle\{F_\sigma\} \tag{30}$$

with

$$\{F_\sigma\} = [K_\sigma]\{\bar{U}\} \tag{31}$$

where $\langle\bar{W}\rangle$, $\{\bar{U}\}$ and $[K_\sigma]$ are expanded forms of $\langle\bar{w}\rangle$, $\{\bar{u}\}$ and $[k_\sigma]$, respectively.

The vector $\{F_\sigma\}$ is also the expanded form of $\{f_\sigma\}$.

The nodal force vector $\{F_\sigma\}$ is considered to be a measure for estimating errors included in the solution because it is expected that the larger the degree of inter-element discontinuity of stresses is, the larger the values of components of $\{F_\sigma\}$ would be. It will be examined later if $\{F_\sigma\}$ can really be a measure for error estimate.

We adopt the L2 norm of $\{F_\sigma\}$, denoted by $||F_\sigma||$, as the magnitude of $\{F_\sigma\}$ in the whole domain:

$$||F_\sigma|| = [\langle F_\sigma\rangle\{F_\sigma\}]^{1/2} \qquad (32)$$

2.4. Adaptive procedure for relocating nodes

(i) Solve the following stiffness equation, ordinarily derived from the condition (21).

$$[K]\{\bar{U}\}=\{\bar{\bar{F}}\} \qquad (33)$$

in which [K] is the stiffness matrix, and $\{\bar{\bar{F}}\}$ is the external force vector.
(ii) Calculate the nodal force vector $\{F_\sigma\}$ according to eq.(31) from the solution obtained in the step (i).
(iii) Solve the stiffness equation by applying $\{F_\sigma\}$ as

$$[K]\{\bar{U}^*\} = \{F_\sigma\} \qquad (34)$$

and obtain displacement $\{\bar{U}^*\}$.
(iv)Relocate nodes by $\{\bar{U}^*\}$ according to:

$$\{X^*\} = \{X_0\} + \{\bar{U}^*\} \qquad (35)$$

in which $\{X_0\}$ is the vector of coordinates of nodes used in the steps (i) and (iii), and $\{X^*\}$ is coordinates of nodes for subsequent computation.

Applying $\{F_\sigma\}$ in the step (iii) means that we preliminarily relocate nodes by the magnitude which could produce the nodal forces due to stress discontinuity. The effects of relocating nodes on the reduction of $||F_\sigma||$ will be shown in the Results section.

The boundary condition in the step (iii) is such that:
(1) the boundary is the same as that in the step (i);
(2) nodes on the boundary can move only along the boundary; and therefore
(3) nodes at corners of the boundary can not move.
An example of the boundary condition will be shown in relation to the test problem which will be described later.

3. Numerical tests

The problem in which a rigid strip footing was loaded on the surface of an elastic ground was treated to test the effectiveness of the procedure described above. The problem and values for elastic parameters are shown in Fig.1.

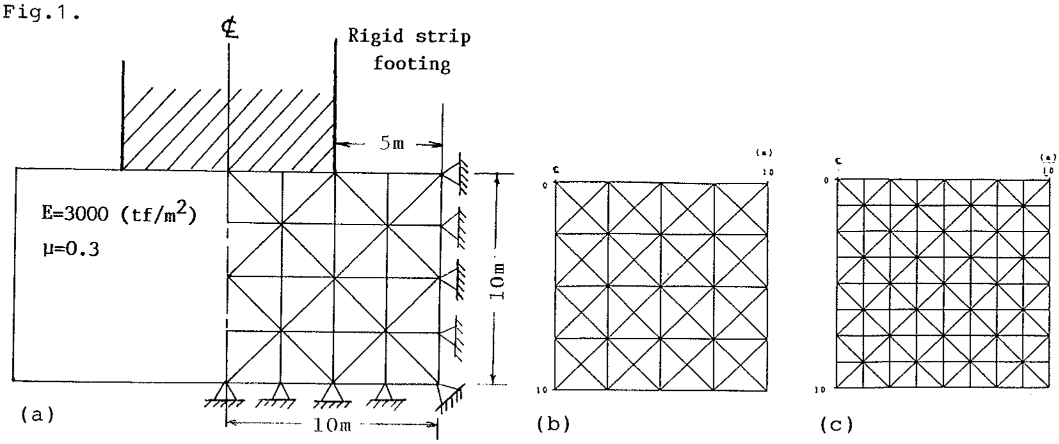

Fig.1: The test problem and elastic constants. (a) N_e=32; (b) N_e=64; and (c) N_e=128.

Constant strain triangular elements were used. Three types of meshes were analyzed: the mesh of N_e=32, 64 and 128. All the types are regular in the beginning of computation, as shown in Fig.1. The footing was vertically lowered by increments of 1mm. The boundary condition for the adaptive relocation of nodes is shown in Fig.2 for the case of N_e=32.

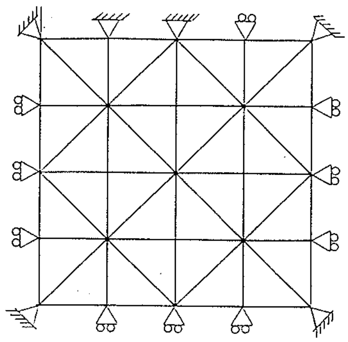

Fig.2: Boundary condition for the adaptive procedure (N_e=32)

4. Results

The relocation of nodes resulted from the adaptive procedure, when the footing was lowered by 1 mm below the original level, is shown in Figs.3(a) to (f). Figures (a) to (d) are results for the mesh of N_e=32, in which we can see how nodes are relocated with the iteration for adaptive procedure. The figure (e) is for N_e=64 and (f) for N_e=128.

We can see in these figures that the magnitude of the relocation of nodes is the largest for the mesh of N_e=32 than others of N_e=64 and 128. We can see

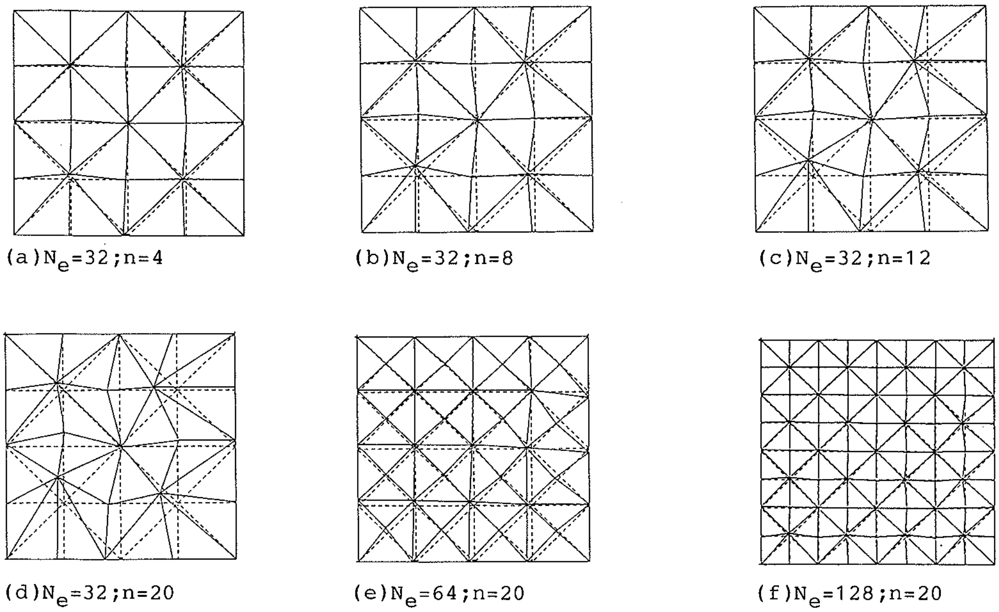

Fig.3: Meshes of relocated nodes. (a) to (d) are for N_e=32; (e)N_e=64; and (f)N_e=128. "n" is the number of iteration for the adaptive procedure.

also that elements near the footing tend to become smaller with the iteration. In such elements, stresses induced by the settlement of the footing are expected to be relatively large.

The adaptive procedure is a method by which we can expect for the nodal forces, defined in eq.(31), to be reduced to be null. The relationships between $||F_\sigma||$ per node, which is defined as $||F_\sigma||$ divided by the number of nodes, and the number of iteration are presented in Fig.4. From this figure, in facts, we can observe that the norm decreases with the adaptive procedure; the effect of the adaptive iteration on the decrease of the norm is appreciable for the case of N_e=32.

In Fig.5, the computed load-settlement relationships are presented. The relationships are resulted after 20th iteration for adaptive procedure in each increment of the settlement. They are compared with those calculated from

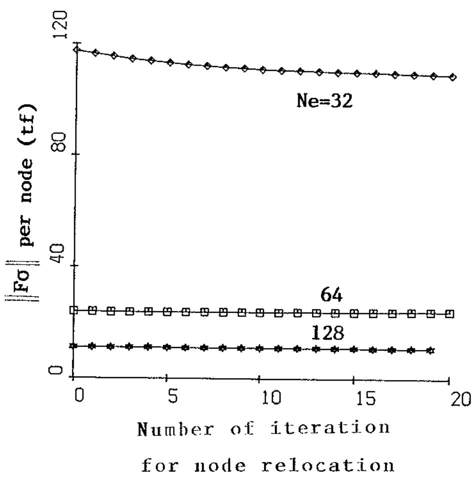

Fig.4: Change in the norm of nodal force vector.

the solution obtained without the adaptive procedure. For the rough mesh of Ne-32, the reduction of the load is appreciable when the adaptive procedure was performed; the magnitude of the reduction due to the adaptive procedure is less for the mesh of Ne=64.

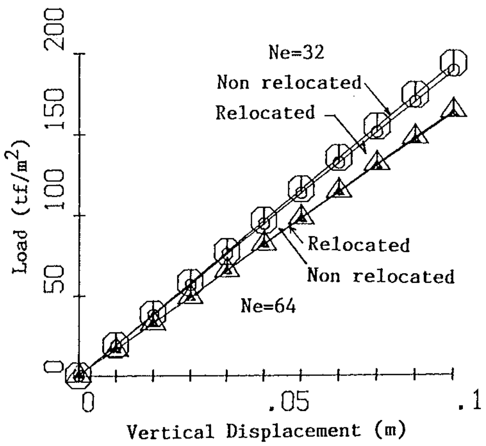

Fig.5: Load-settlement relationships.

5. Discussion

It was assumed that the norm of the nodal force vector, $||F_\sigma||$, would be able to be an error indicator in a finite element solution. To examine the assumption, we have to estimate the error included in the finite element solution. For estimating the error, we adopt two ways: one is to assume that the solution for the finest mesh of N_e=128 may be correct and another to estimate the error by the error estimator proposed by Zienkiewicz and Zhu[2].

Basically we can define the error in computed stresses as:

$$\{e_\sigma\} = \{\tilde{\sigma}_h\} - \{\sigma^*\} \tag{36}$$

where $\{e_\sigma\}$ is the vector of the error, $\{\tilde{\sigma}_h\}$ is the vector of stresses computed for a mesh h and $\{\sigma^*\}$ the vector of correct stresses.

The magnitude of the error can be evaluated by the L2 norm of $\{e_\sigma\}$, $||e_\sigma||$. Further the relative error can be given by ε_σ defined as

$$\varepsilon_\sigma = \frac{||e_\sigma||}{||\sigma^*||} \tag{37}$$

The relative error ε_σ is different from point to point in the domain. For evaluating the magnitude of error in the global domain, we introduce the global error E_σ defined as follows

$$E_\sigma = \int_V \varepsilon_\sigma dV \tag{38}$$

The global error E_σ was estimated by assuming that the solution obtained for the mesh of N_e=128 is correct. The

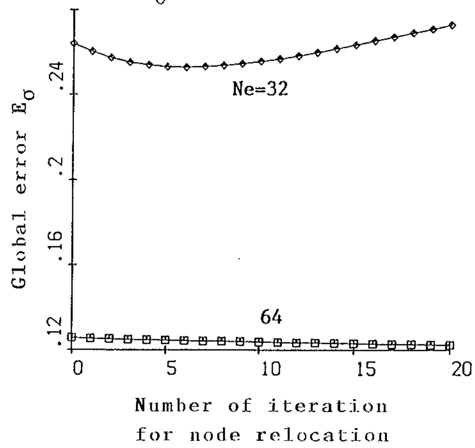

Fig.6: Global error vs. number of iteration

results are presented in Fig.6. In the case when N_e=32, the global error decreases with the number of iteration up to 7 but subsequently it increases. For the finer mesh of N_e=64, the global error decreases with the number of iteration examined.

Zienkievicz and Zhu[2] proposed a method by which the correct stresses $\{\sigma^*\}$ can be estimated. In the method the correct stresses are approximated by the same interpolation functions as those for displacements and the weighted residual for eq.(36) is required to be zero in each element. They introduced the following quantity as the error estimator for an element:

$$||e_\sigma||_e = (\int_{V_e} \langle e_\sigma \rangle \{e_\sigma\} dV_e)^{1/2} \qquad (39)$$

The global error corresponding to eq.(39) can be defined as

$$E_Z = [\sum_{e=1}^{N_e} ||e_\sigma||_e{}^2]^{1/2} \qquad (40)$$

The suffix Z is used to distinguish it from E_σ defined in eq.(38).

In Fig.7, the global error E_Z is shown as a function of the number of iteration; the ratio of E_Z to E_{Z0}, which is the value of E_Z calculated before the adaptive procedure, is used.

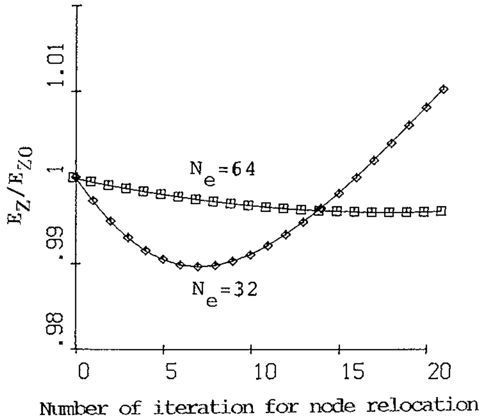

Fig.7: Global error estimated with the error estimator by Zienkievicz and Zhu[2].

We can see in the figure that, for the rough mesh of N_e=32, E_Z decreases with the number of iteration less than 8 and, for the finer mesh of N_e=64, E_Z continues to decrease with the number of iteration. The rate of the reduction of E_Z is appreciable for the rough mesh.

In Fig.6 and Fig.7, we find that the variation of the error, estimated in two ways, is very similar. As a conclusion, the adaptive procedure proposed in this study can reduce the error although the effective number of iteration for the adaptive procedure is limited.

6. Conclusions

An adaptive node-relocation procedure was proposed in which the vector of nodal forces caused by the stress discontinuity between elements is required to be reduced. The error was estimated in two ways: one is to assume that the finest mesh among examined meshes can give the correct solution; and another to use the error estimator proposed by Zienkiewicz and Zhu[2].

It was shown that, with these two methods for the error estimation, the adaptive procedure can reduce the error. However, when the number of iteration for the procedure exceeds a certain value, the error tends to increase. Therefore in applying the proposed procedure to practical problems, the error estimator, for instance, by Zienkievicz and Zhu[2] should be used to judge the end of the iteration.

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