A Game-Theoretic Approach to the Analysis of Areawide, Multi-Modal Water Utlization System

by

Norio OKADA

(Received May 31, 1977.)

This paper deals with a coalition problem involved in the water development and conservation by different municipalities. Local interests conflicting among different municipalities concern (1) the attainment of economic efficiency, (2) provision of needed amount of water, and (3) regulation of water pollution. Possible coalition patterns formed by municipalities are examined by the use of a game theoretic approach. The paper includes a case study on the Kakogawa River Basin in Hyogo Prefecture, whereby the practicability of the approach is validated.

1 Introduction

With the impetus of increasing complexties of water-related activities and growing deterioration of water resources environment, there is a growing concern and need for the development and conservation of water resources systems. A conspicuous problem which demands a practical solution then is an effective coordination of local interests which conflict among different municipalities. The interests of each municipality concern ① the attainment of economic efficiency, ② provision of needed amount of water, and ③ regulation of water pollution.

The coordination of the conflicting interests can be approached by two kinds of systems. One concerns a coordination by the third sector (any existing agency of a higher level or some provisional authority) which works out an arbitration proposal. The other is a solution-finding by the party concerned, which forms a coalition where it is found to be preferable to the party. The coordination problem of the former type can be modeled by goal programming approach as has already been studied by the author[1),2),3)].

This paper approaches the latter problem by use of game theory and presents a mathematical formulation of the problem[4)]. The mathematical model thus obtained will be applied to the study area of the Kakogawa River Basin in Hyogo Prefecture, whereby the practicability and potential of the methodology will be vindicated.

2 Problem Identification

(1) We deal with a coalition problem involved in the development and utilization of urban water systems by three municipalities which are assumed to be located on the same stream ; one on the upstream, one on the middlestream, and the other on the downstream.

(2) The explicit goal of each municipality concerns the minimization of the associated costs subject to an established set of restrictions (1) to the provision of needed amount of water and (2) to the regulation of water pollution.

(3) The feasible patterns of coalition are (1) separate implementation of projects by each municipality (no coalition), (2) joint implementation of projects between the upstream and middlestream municipalities, (3) between the middlestream and downstream municipalities, and (4) between all the municipalities (entire coalition).

(4) The above coalitions are considered as a three-person (or two-person) cooperative game whose players are municipalities.

(5) For each type of coalition the party concerned attempts to find the optimal strategy to minimize the costs associated with the development and conservation of combined (or separate) urban water systems.

(6) This problem can be formulated as a nonlinear programming problem as will be seen later.

If the total costs are found to be reducible with the coalition, the party's concern is whether it pays to each player in the party. It depends on how to allocate the total costs to each player in the party. This kind of cost allocation problem can be studied by use of the concepts of *core* developed by Bondareva,[5] Shapley[6] and Scarf,[7] and *kernel* due to Peleg,[8] Davis and Maschler.[9] This will be discussed later.

3 Model Formulation of Coalition Strategy

3.1 Constraints

Let $i(j,k) \in S$ represent three (two or one) players (municipalities) in a given coalition S.

For municipality i ($i = 1, 2, 3$; 1=upstream, 2=middlestream, 3=downstream), the optimal strategy is formulated as follows. (The nonnegativity conditions for all the variables are omitted here.)

Available amount of streamflow to be collected is constrained as

$$x_i \leqq q_i^u - F_i \tag{1}$$

x_i : amount of collection = amount of water supply (variable)

q_i^u : amount of streamflow at the farthest upstream point (variable)

F_i : amount of minimum discharge reserved (constant)

Water supply condition is formulated as :

$$D_i - D_i^C \leqq x_i + \sum_t (y_{ti} - y_{it}) + w_i^N \quad (t = j,\ k \in S) \tag{2}$$

y_{ti} (y_{it}) : amount of purified water distributed to (from) municipality i from (to) t (variable)

w_i^N : amount of renovated wastewater reused for water use in municipality i (variable)

D_i : water demand in municipality i

D_i^O : existing water demand (constants)

Wastewater treatments are so regulated :

$$w_i^R = D_i - \sum_t (u_{it} - u_{ti}) \quad (t = j(k) \in S) \tag{3}$$

$$u_{it} \leqq D_i \tag{4}$$

$$u_{ti} \leqq D_t \quad (t = j\ (k) \in S) \tag{5}$$

u_{ti} (u_{it}) : amount of wastewater transported from municipality $i(t)$ to $t(i)$ where it undergoes secondary treatment

$$w_i^R = w_i^W - w_i^E \tag{6}$$

$$w_i^N = w_i^R - w_i^O \tag{7}$$

w_i^W : amount of wastewater treated in the secondary process (variable)

w_i^R : treated wastewater which further undergoes tertiary treatment = amount of renovated wastewater (variable)

w_i^E : treated wastewater to be discharged into the receiving water body (variable)

w_i^N : renovated wastewater to be reused for water use (variable)

w_i^O : renovated wastewater to be discharged into the receiving water body (variable)

Streamflow condition is expressed as :

$$q_i^U = \sum_{i'} (w_{i'}^E + w_{i'}^O) - \sum_{i'} x_{i'} \ (i' = j \text{ or } k \in U_i \subset S_i) \tag{8}$$

$$q_i^U = Q \tag{9}$$

$$q_i^U = q_{i*}^L \tag{10}$$

q_i^L : amount of streamflow at the farthest downstream in municipality $i*$ located nearest above municipality i (variable)

U_i : set of those municipalities located farther upstream from municipality i (given *a priori*)

Q : amount of streamflow in the headwaters (constant given *a priori*)

Streamflow quality is regulated as :

$$b_i^L \leqq B_i \tag{11}$$

$$b_i^L = \left\{ b_i^U (q_i^U - x_i) + B^E w_i^E + B^O w_i^O \right\} \tag{12}$$

$$b_i^U = b_{i*}^L \tag{13}$$

b_i^L : streamflow quality (BOD) at the farthest downstream in muniicipality i (variable)

b_i^U : at the farthest upstream (variable)

B_i : streamflow quality standard (BOD) prescribed *a priori* (constant)

B^E : water qulity (BOD) of w_i^E (constant)

B^O : that of w_i^O (constant)

3.2 Objective Function

The objective function is identified as the minimization of the total costs, which is formulated as :

$$\text{Minimize} \quad z = \sum_{i \in S} \left\{ \alpha_i (x_i) + \sum_{t=j,k \in S} (\beta_{it} (y_{it}) + \beta_{ti} (y_{ti}) + \gamma_{it} (u_{it}) + \gamma_{ti}(u_{ti})) + \sum_{J=W,E,R,O,N} \delta_i^J (w_i^J) \right\}$$

$\alpha_i (x_i)$, $\beta_{it} (y_{it})$, $\beta_{ti} (y_{ti})$, $\gamma_{it} (u_{it})$, $\gamma_{ti} (u_{ti})$, $\delta_i^J (w_i^J)$: cost functions with respect to x_i y_{it}, y_{ti}, u_{it}, u_{ti}, w_i^J $(J=W,E,R,O,N)$, respectively.

(14)

3.3 Solution Algorithm

The above formulated model is a class of nonlinear programming problem with a specific property that a small number of equations are nonlinear (objective function (12), and Equation (10)) while the others are linear.

This observation allows us to combine the method of feasible directions due to Zoutendijk[10] and the penalty function method developed by Fiacco.[11] For details of this algorithm developed by the author, see references (1), (2) and (3).

4 Cost Allocation by Game Theory

We now arrive at the discussion of cost allocation based on the notions developed in the field of n-person cooperative games.

The process is initiated by the completion of an inventory of the associated costs for a set of possible coalitions with an aid of the above formulated mathematical model. Let these costs be denoted by $C\ (S)$ which is a function of coalition S. For convenience of calculation this function is normalized for each type ef coalition. That is,

$$v\ (S) = \sum_{i \in S} C\ (S) - C(S) \tag{13}$$

This normalized function which is called a characteristic function of coalition S represents the reduced cost derived from coalition S. This set of characteristic functions define the formulation of n-person cooperative game (N, v). We shall discuss the assignment of the reduced costs to each player of the party. In terms of n-person cooperative game, this kind of assignment problem is identified as the determination of pay-off vector (X_j) which represents a set of assigned portions of the reduced cost to player i. We shall discuss the applicability of the notion of core or kernel.

4.1 Assignment based on the Notion of Core

A vector of pay-off levels is suggested which is feasible for all of the players acting collectively, and an arbitrary coalition is examined to see whether it can provide higher pay-off levels for all of its members. If this is possible, the pay-off vector which was originally suggested is said to be blocked by the coalition. The core of the n-person game consists of those pay-off vectors which are feasible for the entire group of players and which can be blocked by no coalition. The core of the n-person game is defined by the following two constraints.

$$\sum_{i \in S} X_i \geqq v\ (S) \tag{14}$$

$$\sum_{i \in N} X_i = v\ (N) \tag{15}$$

Equation (15) is the expression of the "individual rationality" which means the condition that any coalition $S \subset N$ should guarantee such an imputed pay-off (reduced cost) to each member of the party that would never come under the lowest level of available pay-off produced from coalition S. Equation (15) formulates the "group rationality" which means that the entire pay-offs derived from coalition N should completely be imputed to all the members.

The feasible solution to the above problem which is called an "imputation" constitutes a convex set as shown in **Fig. 1**. This means that the concept of core connot specify a unique assignment (imputation) alternative, although it reduces the number of feasible alternatives. For this purpose we proceed in the following manner by use of the concept of kernel.

4.2 Assignment based on the Notion of Kernel

Let S be an arbitrary coalition. The excess of S with respect to the outcome of game β is

$$e\ (S) = v\ (S) - \sum_{i \in S} X_i^{\beta} \tag{16}$$

The excess of S therefore represents the total amount that the members of S gain (or lose, if $e\ (S)<0$), if they withdraw from the outcome of game β and form the coalition S.

The notion of kernel is defined as the minimization of the maximal excess of S. For a three-person game, the point of kernel, X_1^* is identified as shown in **Fig. 1**. This can be extended to an n-person game as given below.

$$X_1^* = d_i + \frac{1}{n}(v(n) - \sum_{i \in N} d_i), \tag{17}$$

where d_i denotes a degree of contribution to the entire coalition.

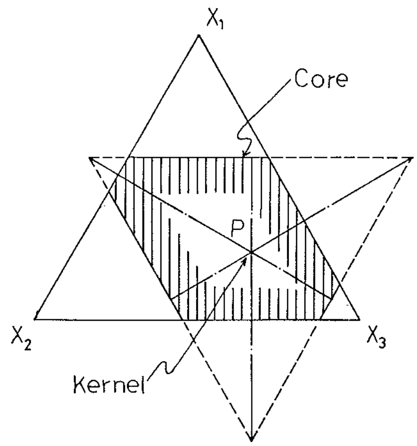

Fig. 1. Illustrated cores and kernel for the three-person game

5 Case Study

The study area is identified as the Kakogawa River Basin in Hyogo Prefecture. The inputs are estimated through a close scrutiny of practical conditions related to the development and conservation of the water resources systems. Some of them are shown in **Figs. 2** and **3**.

The calculated associated costs for each coalition are listed in **Table I**. On this basis the characteristic functions are obtained as shown in **Table II**. The results of cost allocations derived from the combined application of the concepts of core and kernel are listed in **Table III**. The study of this table shows :

(1) The reduced costs are almost the same for both the upstream and middle-stream municipalities, whereas those for the downstream municipality amount to three

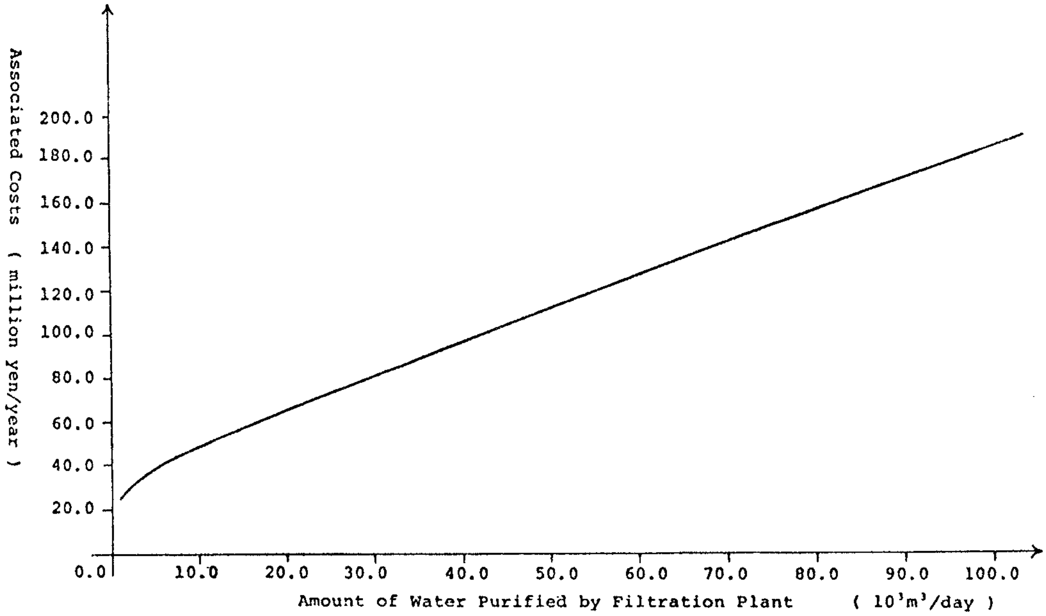

Fig. 2 Cost-scale relation for filtration plant

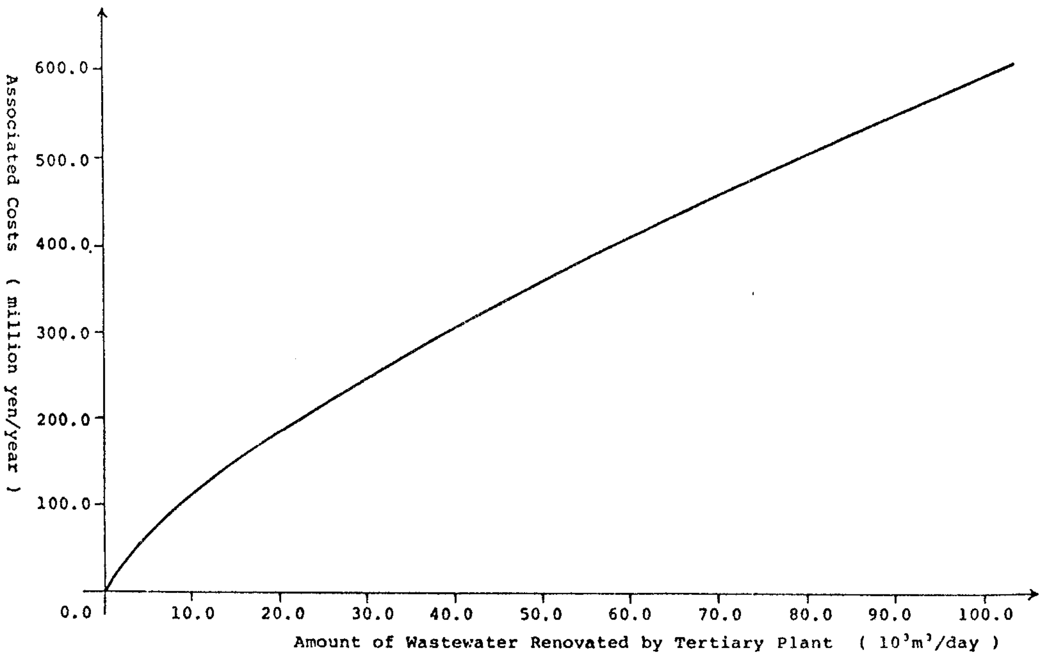

Fig. 3 Cost-scale relation for tertiary treatment plant

Table I. Associated Costs for each coalition (10^3 Yen/year)

Case	Upstream	Middlestream	Downstream	Sum
1	5580.3	11283.4	32445.0	49308.9
2	15681.8		29878.5	45560.3
3	5839.0	39534.7		45337.7
4	33865.0	10495.8		44360.8
5	40553.4			40553.4

Table II. Characteristic function for each coalition

$V(1) = V(2) = V(3) = 0.0$

$V(12) = 1182.09$

$V(23) = 4193.77$

$V(13) = 4160.54$

$V(123) = 8755.58$

Table III. Costs allocated to each municipality for the entire coalition

(10^6 Yen/year)

	Upstream	Middlestream	Downstream	Sum
Associated costs	5580.5	11283.4	32445.0	49308.9
Reduced costs	−1903.6	−1936.8	−4915.2	−8755.6
Allocated costs	3676.9	9346.6	27529.8	40553.3

times as much as the former.

(2) This seems to be derived from the fact that the attainment of economic efficiency depends largely on the participation of the largest party (downstream municipality) in the coalition.

(3) This is also explained by the analysis of **Fig. 4**, which shows the consistency of the water utilization pattern for the upstream municipality irrespective of the coalition patterns.

(4) This implies that the entire coalition is essentially effective to the downstream

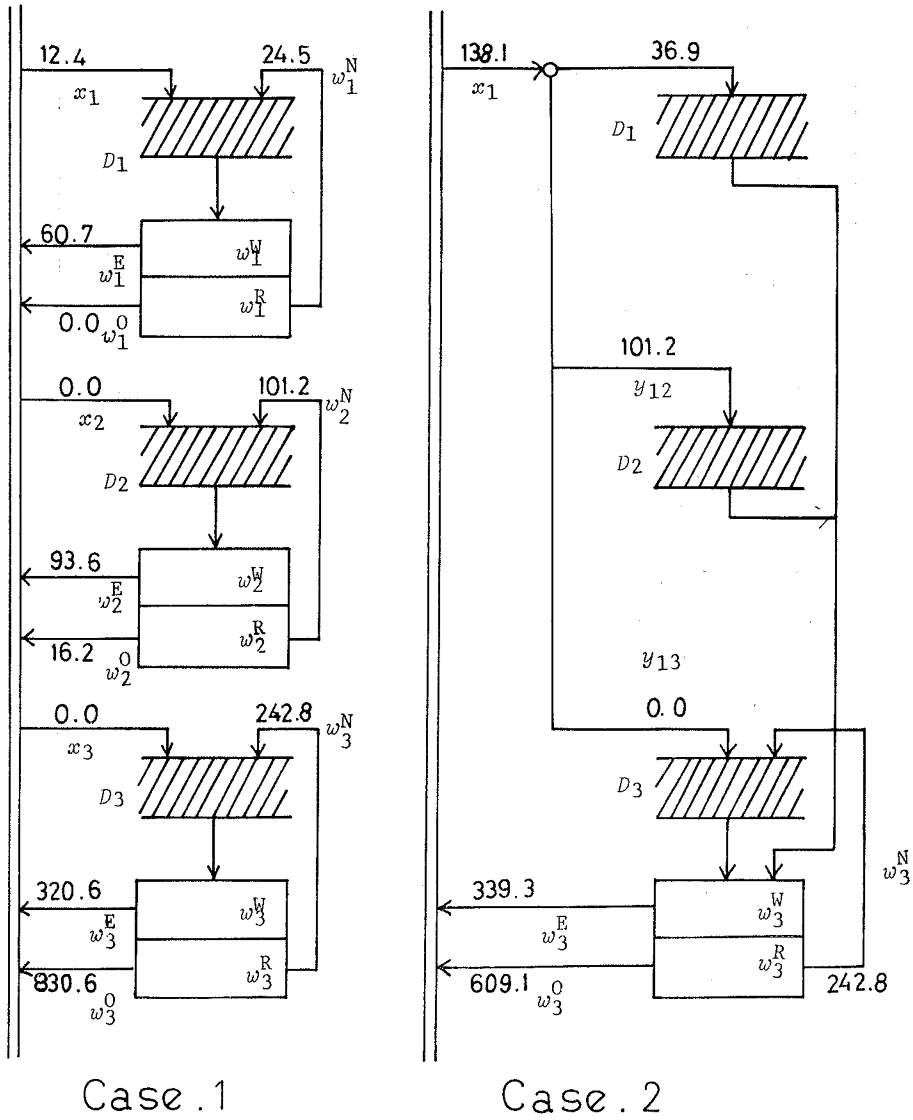

Fig. 4 Calculated water utilizaiion patterns

municipality.

(5) It is preferable also to the upstream and downstream municipalities, because it turns that the streamflow regulations are committed totally to the downstream municipality, thus resulting in an increased availability of the amount of collection in the upstream and middlestream municipalities.

(6) To sum up, the entire coalition with the allocation of the associated costs based on the notions of core and kernel, is preferable to all the municipalities.

6 Conclusion

The central question of this paper has been addressed to a coalition problem involved in the water development and conservation by different municipalities. The paper discussed the potential and applicability of an n-person cooperative game to the analysis of effective coalition patterns available to the party concerned. The case study on the Kakogawa River Basin has revealed the practicability of the methodology in planning areawide multi-modal water resources utilization systems.

In light of these considerations the presented methodology represents a fruitful research approach, although it has to await further development and revision of the approach such as the quantification of utility levels to be incorporated in the pay-off vector.

Acknowledgment

The author is much indebted to Prof. Kazuhiro Yoshikawa at Kyoto University for his valuable advice and comments on this study. Throughout this study the anthor has enjoyed an unfailing assistance of Haruhiko Watanabe, postgraduate student of Kyoto University.

References

1) N. Okada, Comprehensive Systems Analysis of Areawide, Multi-modal Water Resources Utilization Systems, Doctor Thesis Presented to the Fuculty of Kyoto University, Oct., 1976.
2) N. Okada, and K. Yoshikawa, Nonlinear Programming Approach to the Water Assignment Problem for a Single Basin, Proc. of JSCE, No. 247, March, 1976 (in Japanese).
3) Okada, and K. Yoshikawa, Nonlinear Programming Approach for the Analysis of Intrabasin, Multi-modal Water Utilization System, Preprint of the 1977-Vancouver Panpacific Regional Conferece.
4) H. Watanabe, Norio Okada, and K. Yoshikawa, Application of Game Theory to the Analysis of Intrabasin Water Use System, Preprint of the 1977 JSCE Kinki-Branch Conference, Apr., 1977 (in Japanese).
5) O. Bandareva, The Core of an N Person Game, Vestaik Leningrad University, 17, No. 13, pp. 141–142, 1962.
6) L. S. Shapley, On Balanced Sets and Cores, RAND Corp. Memorandum, RM-601-PR, June, 1965.
7) H. E. Scarf, The Core of an N-Person Game, Econometrica, Vol. 35, No. 1, Jan., 1967.
8) B. Peleg, On the Bargaining Set for m-quota games, Advances in Game Theory, M. Dresher, L. S. Shapley, and W. A. Tucker, eds., Annals of Mathematical Studies, 52, pp. 501–515, Princeton University Press, 1964.
9) M. Davis, and M. Mashler, Existence of Stable Pay-off Configurations for Cooperative Games, Bull. Am. Math. Soc., 69, pp. 106–108, 1963.
10) G. Zoutendijk, Methods of Feasible Directions, Elsevier Publishing Company, 1960.
11) A. V. Fiacco, and G.P. McCormic, Computational Algorithm for Sequential Unconstrained Minimization Tachniques for Non-linear programming, Management Science, Vol. 10, pp. 816–818, 1966.