Upper and Lower Bounds of the Discrepancy of the Sequence $(\alpha n+\beta n\log n)$

GOTO, Kazuo*

1 Abstract

We give upper and lower bounds of the discrepancy of the sequence $(\alpha n+\beta n\log n)$.

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2 Lemmas

The discrepancy D_N of the given sequence $x_1, \dots, x_N$ is defined by

$$D_N = \sup_{0\le\alpha<\beta\le 1}\left|\frac{1}{N}A([\alpha,\beta):N)-(\beta-\alpha)\right|,$$

where $A([\alpha,\beta):N)$ is the number of terms whose fractional part is included in $[\alpha,\beta)\subset[0,1]$.

Lemma 1 (Erdös and Turán inequality [2, Chap.2, 2.5]). *For any positive integer m, we have*

$$D_N \le \frac{6}{m+1}+\frac{4}{\pi}\sum_{h=1}^{m}\frac{1}{h}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h x_n}\right|.$$

Lemma 2 (Goto[1], Theorem 3.1). (i) *Let $0\le\alpha<1$ and $\beta>0$. For any $\epsilon>0$, we have*

$$\left|\sum_{n=2}^{N}e^{2\pi i(\alpha n+\beta n\log n)}\right| \le \frac{1}{\sqrt{\beta}}\frac{e^{\frac{\epsilon+1-\{\alpha+\beta(\log N+1)+\epsilon\}}{2\beta}}}{e^{\frac{1}{2\beta}}-1}\sqrt{N}+O\left(\frac{e^{(1+\epsilon)/2\beta}}{\sqrt{\beta}(e^{1/2\beta}-1)}\right)+O(\log N),$$

where the constants implied by the O's are absolute and $\{x\}$ is the fractional part of x.

(ii) *Let $0<\beta\le 1/(2\log 2)$. For any $0<\epsilon<1$, we have*

$$\left|\sum_{n=2}^{N}e^{2\pi i(\alpha n+\beta n\log n)}\right|$$

* Department of Regional Planning, Faculty of Education and Regional Sciences, Tottori University
(鳥取大学教育地域科学部　地域設計学講座)

$$\geq \frac{1}{\sqrt{\beta}} e^{-\frac{1+\{\alpha+\beta(\log N+1)+\epsilon\}}{2\beta}} \sqrt{N} \cdot \left\{ e^{\frac{1}{2\beta}} - 1 - \frac{e^{\frac{1}{2\beta}}}{e^{\frac{1}{2\beta}}-1} \right\} + O\left(\frac{1}{\epsilon^{3/2}\sqrt{\beta}} \right) + O(\log N)$$

$$\geq \frac{1}{\sqrt{\beta}} e^{-\frac{1}{\beta}} \sqrt{N} \cdot \left\{ e^{\frac{1}{2\beta}} - 1 - \frac{e^{\frac{1}{2\beta}}}{e^{\frac{1}{2\beta}}-1} \right\} + O\left(\frac{1}{\sqrt{\beta}} \right) + O(\log N),$$

where the constants implied by the O's are absolute.

3 Theorem

Throughtout this section, the constants implied by the $\ll$ are absolute.

Theorem 1. *For the discrepancy D_N of the sequence $(\alpha n + \beta n \log n)$, we have*

$$D_N \geq A\frac{1}{\sqrt{N}}, \qquad D_N \leq B\frac{\log N}{\sqrt{N}} g(N),$$

for some positive constans A and B, where $g(N)$ is any function which tends monotonically to infinity.

Proof of Theorem 1. We set $\epsilon = 1$ in Lemma 2. Then

$$\left| \sum_{n=2}^{N} e^{2\pi i h(\alpha n+\beta n\log n)} \right| \leq \frac{1}{\sqrt{\beta h}} \frac{e^{\frac{1}{h\beta}}}{e^{\frac{1}{2h\beta}}-1} \sqrt{N} + O\left(\frac{e^{1/h\beta}}{\sqrt{h\beta}(e^{1/2h\beta}-1)} \right) + O(\log N)$$

$$= \frac{1}{\sqrt{\beta h}} \frac{1+\frac{1}{h\beta}+O\left(\left(\frac{1}{h\beta}\right)^2\right)}{\frac{1}{2h\beta}+O\left(\left(\frac{1}{h\beta}\right)^2\right)} \sqrt{N} + O\left(\frac{1}{\sqrt{\beta h}} \frac{1+\frac{1}{h\beta}+O\left(\left(\frac{1}{h\beta}\right)^2\right)}{\frac{1}{2h\beta}+O\left(\left(\frac{1}{h\beta}\right)^2\right)} \right) + O(\log N)$$

$$\ll \sqrt{h\beta}\sqrt{N}.$$

Therefore

$$\left| \sum_{n=1}^{N} e^{2\pi i h(\alpha n+\beta n\log n)} \right| = |I_N| \ll \sqrt{h\beta}\sqrt{N}.$$

For any function $g(n)$ which tends monotonically to infinity, we have

$$\lim_{N\to\infty} \frac{I_N}{\sqrt{N}g(N)} = 0,$$

which implies that there exists an $N_0(h,\beta)$ such that

$$|I_N| < \sqrt{N}g(N) \quad \text{for all} \quad N \geq N_0(h,\beta).$$

By Lemma 1, we have as $N \to \infty$

$$D_N \ll \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{hN}\sqrt{N}g(N) \ll \frac{1}{m} + \frac{1}{\sqrt{N}}g(N)\log m.$$

We choose $m = [N^{1/2}]$. Then, by the definition of $g(n)$,

$$D_N \ll N^{-1/2} + \frac{1}{\sqrt{N}}g(N)\log N \ll \frac{\log N}{\sqrt{N}}g(N),$$

for all $N \geq \max(N_0, (h+1)^2)$. By the [**KN**,Chap.2, Cor.5.1] and Lemma 2, we have

$$\begin{aligned} 4ND_N &\geq \left|\sum_{n=2}^{N} e^{2\pi i(\alpha n + \beta n \log n)}\right| \\ &\geq \frac{1}{\sqrt{\beta}}\exp(-\frac{1}{\beta})\sqrt{N}\cdot\left\{\exp(\frac{1}{2\beta}) - 1 - \frac{\exp(\frac{1}{2\beta})}{\exp(\frac{1}{2\beta}) - 1}\right\} + O\left(\frac{e^{(1+\epsilon)/\beta}}{e^{1/\beta} - 1}\right) \\ &\gg \sqrt{N}, \end{aligned}$$

where the constant implied by the $\gg$ depends on only β.

Therefore we have $4ND_N \gg \sqrt{N}$ if $0 < \beta < 1/(2\log 2)$. □

References

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Faculty of Education and Regional Sciences,
Tottori University,
Tottori-shi, 680-0945, Japan
e-mail: goto@fed.tottori-u.ac.jp