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$(\mathfrak{g}, K)$-module of $\mathrm{O}(p, q)$ associated with the finite-dimensional representation of $\mathfrak{sl}_2$

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The main aim of this paper is to show that one can construct $(\mathfrak{g}, K)$-modules of $\mathrm{O}(p, q)$ associated with the finite-dimensional representation of $\mathfrak{sl}_2$ by quantizing the moment map on the symplectic vector space $(\mathbb{C}^{p+q})_{\mathbb{R}}$ and using the fact that $(\mathrm{O}(p, q), \mathrm{SL}_2(\mathbb{R}))$ is a dual pair. Then one obtains the K-type formula, the Gelfand-Kirillov dimension and the Bernstein degree of them for all non-negative integers m satisfying $m + 3 \leqslant (p + q)/2$ when $p, q \geqslant 2$ and $p + q$ is even. In fact, one finds that the Gelfand-Kirillov dimension is equal to $p + q - 3$ and the Bernstein degree is equal to $4(m + 1)(p + q - 4)!/((p - 2)!\,(q - 2)!)$.

Keywords: indefinite orthogonal group, moment map on symplectic vector space, canonical quantization, irreducible $(\mathfrak{g}, K)$-module, K-type formula

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1. Introduction

Let G be a Lie group with $\mathfrak{g}_0$ its Lie algebra and $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. An action of G on a symplectic manifold (M, ω) is called symplectic if $g^*\omega = \omega$ for all $g \in G$, and a symplectic action is called Hamiltonian if there exists a smooth G-equivariant map $\mu : M \longrightarrow \mathfrak{g}_0^*$ satisfying the condition (2.3) below, which is called a moment map, where $\mathfrak{g}_0^*$ is the dual vector space of $\mathfrak{g}_0$. We are concerned with the case where the symplectic manifold is a real symplectic vector space (W, ω). It was shown in [3] that when $G = \mathrm{Sp}(n, \mathbb{R}), \mathrm{U}(p, q)$ and $\mathrm{O}^*(2n)$, the canonical quantization of the moment map on $W = \mathbb{R}^{2n}, (\mathbb{C}^{p+q})_{\mathbb{R}}$ and $(\mathbb{C}^{2n})_{\mathbb{R}}$, with a choice of a Lagrangian subspace in each case, yields a representation of $\mathfrak{g}$ that is the differentiation of the oscillator (or Segal-Shale-Weil) representation of $\mathrm{Mp}(n, \mathbb{R}), \mathrm{U}(p, q)$ and $\mathrm{O}^*(2n)$ respectively, where $\mathrm{Mp}(n, \mathbb{R})$ is the metaplectic group, i.e., the double cover of $\mathrm{Sp}(n, \mathbb{R})$.

In this paper, we consider the case where $W = (\mathbb{C}^{p+q})_{\mathbb{R}}$, the real vector space underlying $\mathbb{C}^{p+q}$:

$$W = \left\{ z = x + \mathrm{i}\, y \,\middle|\, x, y \in \mathbb{R}^{p+q} \right\},$$

which we regard as a symplectic vector space equipped with a symplectic form ω given by

$$\omega(z, w) = \mathrm{Im}(z^* I_{p,q} w) \qquad (z, w \in W),$$

and $G = \mathrm{O}(p,q)$, the indefinite orthogonal group defined by

$$\mathrm{O}(p,q) = \left\{ g \in \mathrm{GL}_{p+q}(\mathbb{R}) \,\middle|\, {}^t g I_{p,q} g = I_{p,q} \right\}$$

with $I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$. The action of $G = \mathrm{O}(p,q)$ on W defined by matrix multiplication is symplectic and Hamiltonian. The $\mathrm{O}(p,q)$-case we consider here is closely related to the $\mathrm{U}(p,q)$-case mentioned above. In fact, the symplectic vector space (W,ω) for $\mathrm{O}(p,q)$ is identical to the one for $\mathrm{U}(p,q)$, and the action of $\mathrm{O}(p,q)$ on W is the restriction of the action of $\mathrm{U}(p,q)$ induced from the canonical embedding of $\mathrm{O}(p,q)$ into $\mathrm{U}(p,q)$. Furthermore, the moment map for the $\mathrm{O}(p,q)$-case is the real part of the one for the $\mathrm{U}(p,q)$-case.

The canonical quantization of the moment map $\mu : W = (\mathbb{C}^{p+q})_{\mathbb{R}} \to \mathfrak{g}_0^*$ for $G = \mathrm{O}(p,q)$, with a choice of a Lagrangian subspace V of W, provides a representation π of $\mathfrak{g}$ as in the cases mentioned above, which is shown to be a partial Fourier transformation of the representation $\pi^\sharp$ of $\mathfrak{g}$ obtained by differentiating the left regular representation of G on $C^\infty(V)$. Note that if we restrict the operator $\pi^\sharp(X)$, $X \in \mathfrak{g}$, to a subspace consisting of homogeneous functions on V with respect to the multiplicative group $\mathbb{R}_{>0}$, then the restricted representation is the degenerate principal series of G obtained by inducing up a one-dimensional representation of a parabolic subgroup of G (see [6]).

In the influential paper [5], Howe showed that one can treat the classical invariant theory from a unified viewpoint — the dual pair. In this paper, we focus our attention on the dual pair $(\mathrm{O}(p,q), \mathrm{SL}_2(\mathbb{R}))$, both components of which are noncompact, and apply the representation theory of $\mathfrak{sl}_2$ to cut out $(\mathfrak{g},K)$-modules, which we denote by $M^+(m)$ and $M^-(m)$, $m = 0,1,2,\ldots$, where $M^+(m)$ (resp. $M^-(m)$) consists of all highest (resp. lowest) weight vectors with respect to the $\mathfrak{sl}_2$-action (see Definition 4.1 below for details). Note that $M^+(0) = M^-(0)$ by definition. We will see that such weight vectors are given in terms of harmonic polynomials and the Bessel functions of the first kind. Both $M^\pm(m)$ should correspond to the $(m+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$ under the Howe duality, and in fact are isomorphic to each other. They were originally considered in [12] without the condition of finite-dimensionality.

In the cases of the oscillator representations mentioned above, i.e., when $G = \mathrm{Sp}(n,\mathbb{R})$, $\mathrm{U}(p,q)$ and $\mathrm{O}^*(2n)$, we note that the counterpart G' of G for the dual pair (G,G') is compact, hence, all its irreducible representations are finite-dimensional. Furthermore, the oscillator representations give examples of the minimal representations (we refer to [7] and the references therein for the definition of the minimal representation). When $G = \mathrm{O}(p,q)$, its minimal representation is discussed e.g. in [1, 7–11, 16].

The main result of this paper is the K-type formula of $M^\pm(m)$ for nonnegative integers m satisfying

$$m + 3 \leqslant \frac{p+q}{2} \tag{1.1}$$

when $p+q$ is even, from which one can show that $M^\pm(m)$ are irreducible $(\mathfrak{g},K)$-modules if $p,q \geqslant 2$ (Theorem 4.1). The fact that the elements of $M^\pm(m)$ are described in terms of the Bessel function plays a role in the proof of our main result. The K-type formula of $M^+(0) = M^-(0)$, which is associated with the one-dimensional trivial representation of $\mathfrak{sl}_2$, shows that it corresponds to the $(\mathfrak{g}, K)$-module of the minimal representation of $\mathrm{O}(p,q)$. We will see that the Gelfand-Kirillov dimension and the Bernstein degree of the irreducible $M^{\pm}(m)$, which we denote by $\mathrm{Dim}\, M^{\pm}(m)$ and $\mathrm{Deg}\, M^{\pm}(m)$ respectively, are given by

$$\mathrm{Dim}\, M^{\pm}(m) = p+q-3,$$
$$\mathrm{Deg}\, M^{\pm}(m) = \frac{4(m+1)(p+q-4)!}{(p-2)!(q-2)!}$$

(Corollary 4.1).

The rest of this paper is organized as follows. In §2, we calculate the moment map μ on W for $G = \mathrm{O}(p,q)$, and construct the representation π of $\mathfrak{g}$ via canonical quantization of μ. Then we show that π is a partial Fourier transform of the differential representation of the left regular representation of G on $C^{\infty}(V)$. In §3, we give an $\mathfrak{sl}_2$-action that commutes with π, and find both highest weight vectors and lowest weight vectors with respect to the $\mathfrak{sl}_2$-action. We remark that such weight vectors are given in terms of the Bessel functions of the first kind. In §4, we introduce $(\mathfrak{g}, K)$-modules $M^{\pm}(m)$ and prove that $M^{+}(m)$ and $M^{-}(m)$ are isomorphic to each other for any nonnegative integer m. When $p+q$ is even, we find the K-type formula of $M^{\pm}(m)$ for m satisfying (1.1) and show that they are irreducible if p, q are $\geqslant 2$. As a corollary, we obtain the Gelfand-Kirillov dimension and the Bernstein degree of $M^{\pm}(m)$.

Notation

Let $\mathbb{N}$ denote the set of nonnegative integers $\{0,1,2,\dots\}$, and $[p]$ the set $\{1,2,\dots,p\}$ for a positive integer p. We write $\bar{\imath} := p+i$ for $i \in [q]$ for the sake of simplicity. Finally, for $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, we denote the rising and the falling factorials by

$$(\alpha)_n := \prod_{i=1}^{n}(\alpha+i-1) \quad \text{and} \quad (\alpha)_n^{-} := \prod_{i=1}^{n}(\alpha-i+1),$$

respectively.

2. Moment Map and its Quantization

Let G be the indefinite orthogonal group $\mathrm{O}(p,q)$, which we realize by

$$\mathrm{O}(p,q) = \left\{ g \in \mathrm{GL}_{p+q}(\mathbb{R}) \,\middle|\, {}^t g I_{p,q} g = I_{p,q} \right\}$$

with $I_{p,q} = \begin{bmatrix} 1_p & \\ & -1_q \end{bmatrix}$. Let K be a maximal compact subgroup of G given by

$$K = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in G \,\middle|\, a \in \mathrm{O}(p), d \in \mathrm{O}(q) \right\} \simeq \mathrm{O}(p) \times \mathrm{O}(q).$$

We denote the Lie algebra of K and its complexification by $\mathfrak{k}_0$ and $\mathfrak{k}$ respectively.

Let $\{X^{\pm}_{i,j}\}$ be a basis for $\mathfrak{g}_0 = \mathfrak{o}(p,q)$ given by

$$\begin{aligned} X^{+}_{i,j} &= E_{i,j} - E_{j,i} && (i,j \in [p]) \\ X^{+}_{\bar{\imath},\bar{\jmath}} &= E_{\bar{\imath},\bar{\jmath}} - E_{\bar{\jmath},\bar{\imath}} && (i,j \in [q]) \\ X^{-}_{i,j} &= E_{i,\bar{\jmath}} + E_{\bar{\jmath},i} && (i \in [p], j \in [q]), \end{aligned} \tag{2.1}$$

which also forms a basis for $\mathfrak{g} = \mathfrak{o}_{p+q}$, the complexification of $\mathfrak{g}_0 = \mathfrak{o}(p,q)$. We often identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the invariant bilinear form B given by

$$B(X,Y) = \frac{1}{2}\operatorname{tr}(XY) \qquad (X,Y \in \mathfrak{g}),$$

where $\mathfrak{g}^*$ denotes the dual space of $\mathfrak{g}$. Finally, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition of $\mathfrak{g}$ with

$$\mathfrak{k} = \sum_{i,j\in[p]} \mathbb{C}X^+_{i,j} \oplus \sum_{i,j\in[q]} \mathbb{C}X^+_{\bar{i},\bar{j}}, \qquad \mathfrak{p} = \sum_{i\in[p],j\in[q]} \mathbb{C}X^-_{i,\bar{j}}.$$

Let W be the real vector space $(\mathbb{C}^{p+q})_{\mathbb{R}}$ underlying the complex vector space $\mathbb{C}^{p+q}$:

$$W = \left\{ z = x + \mathrm{i}\, y \,\middle|\, x = {}^t(x_1,\dots,x_{p+q}), y = {}^t(y_1,\dots,y_{p+q}) \in \mathbb{R}^{p+q} \right\},$$

which is equipped with a symplectic form ω given by

$$\omega(z,w) = \operatorname{Im}(z^* I_{p,q} w) \qquad (z,w \in W) \tag{2.2}$$

Then G acts on (W,ω) symplectically via $z \mapsto gz$ (matrix multiplication) for $z \in W$ and $g \in G$. Furthermore, the action of G on (W,ω) is Hamiltonian, i.e., there exists a moment map $\mu : W \longrightarrow \mathfrak{g}_0^*$, whose definition we briefly recall: if, in general, a Lie group G acts on a symplectic manifold (M,ω) symplectically, a smooth G-equivariant map $\mu : M \longrightarrow \mathfrak{g}_0^*$ that satisfies

$$\mathrm{d}\langle \mu, X\rangle = \iota(X_M)\omega \qquad \text{for all } X \in \mathfrak{g}_0, \tag{2.3}$$

is called a moment map, where ι stands for the contraction and X_M denotes the vector field on M given by

$$X_M(p) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(-tX).p \qquad (p \in M).$$

Under the identification that $e_i := {}^t(0,\dots,\overset{i}{1},\dots,0) \leftrightarrow \partial_{x_i}$ and $\mathrm{i}\, e_i \leftrightarrow \partial_{y_i}$ for $i = 1,2,\dots,p+q$, the symplectic form ω given in (2.2) can be rewritten as

$$\omega = \sum_{i=1}^{p+q} \epsilon_i \,\mathrm{d}x_i \wedge \mathrm{d}y_i$$

with $\epsilon_i = 1$ for $i \in [p]$ and $\epsilon_{p+i} = -1$ for $i \in [q]$.

Proposition 2.1. *The action of $G = \mathrm{O}(p,q)$ on (W,ω) is Hamiltonian, and the moment map $\mu : W \longrightarrow \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ is given by*

$$\begin{aligned}\mu(z) &= -\frac{\mathrm{i}}{2}\left(zz^* - {}^t(zz^*)\right) I_{p,q} \\ &= (-x\,{}^t y + y\,{}^t x) I_{p,q} \\ &= \begin{bmatrix} -x'\,{}^t y' + y'\,{}^t x' & x'\,{}^t y'' - y'\,{}^t x'' \\ -x''\,{}^t y' + y''\,{}^t x' & x''\,{}^t y'' - y''\,{}^t x'' \end{bmatrix}\end{aligned} \tag{2.4}$$

for $z = x + \mathrm{i}\, y \in W$ *with* $x = {}^t(x', x'')$, $y = {}^t(y', y'') \in \mathbb{R}^{p+q}$ *and* $x', y' \in \mathbb{R}^p, x'', y'' \in \mathbb{R}^q$ [a].

Proof. Using the formula

$$\langle \mu(z), X\rangle = \frac{1}{2}\omega(z, Xz)$$

(see e.g. [2, Proposition 1.4.6]), which, in our case, can be written as

$$B(\mu(z), X) = \frac{1}{2} B\big((zz^* - \bar{z}\,{}^t z) I_{p,q}, X\big)$$

for all $X \in \mathfrak{g}_0$, we obtain (2.4) immediately. □

Remark 2.1. Recall that the moment map $\mu_{\mathrm{U}} : W \to \mathfrak{u}(p,q)^* \simeq \mathfrak{u}(p,q)$ for the action of $\mathrm{U}(p,q)$ on our symplectic vector space (W, ω) is given by

$$\mu_{\mathrm{U}}(z) = -\mathrm{i}\, zz^* I_{p,q} \qquad (z = x + \mathrm{i}\, y \in W)$$

where we identify $\mathfrak{u}(p,q)^*$ with $\mathfrak{u}(p,q)$ via the invariant bilinear form B given by $B(X, Y) = (1/2)\,\mathrm{tr}\,(XY)$. Therefore, the moment map μ in the proposition is related to μ_{U} by

$$\mu(z) = \frac{\mu_{\mathrm{U}}(z) + \overline{\mu_{\mathrm{U}}(z)}}{2}.$$

Namely, one has $\mu = \mathrm{Re}\, \mu_{\mathrm{U}}$.

We define a Poisson bracket by

$$\{f, g\} = \omega(\xi_g, \xi_f),$$

where ξ_f denotes the Hamiltonian vector field on W corresponding to $f \in C^\infty(W)$, i.e. the vector field that satisfies $\iota(\xi_f)\omega = \mathrm{d}f$. Then the Poisson bracket among the coordinate functions are given by

$$\{x_i, y_j\} = -\delta_{i,j}\epsilon_i, \qquad \{x_i, x_j\} = \{y_i, y_j\} = 0$$

for $i, j = 1, 2, \ldots, p + q$. Dirac's quantization conditions requires that

$$\{f_1, f_2\} = f_3 \quad \text{implies} \quad [\hat{f}_1, \hat{f}_2] = -\mathrm{i}\,\hbar \hat{f}_3$$

for $f_i \in C^\infty(W)$ (see e.g. [15]). Thus, we quantize the coordinate functions as follows:

$$\begin{aligned} \hat{x}_i &= x_i, & \hat{y}_i &= -\mathrm{i}\,\hbar\partial_{x_i}, & &(i = 1, \ldots, p), \\ \hat{x}_{\bar{j}} &= -\mathrm{i}\,\hbar\partial_{y_{\bar{j}}}, & \hat{y}_{\bar{j}} &= y_{\bar{j}}, & &(j = 1, \ldots, q), \end{aligned} \tag{2.5}$$

where ∂_{x_i} and $\partial_{y_{\bar{j}}}$ denote $\partial/\partial x_i$ and $\partial/\partial y_{\bar{j}}$ respectively. In what follows, we set $\hbar = 1$ for brevity.

The quantization (2.5) corresponds to a Lagrangian subspace V of W given by

$$V = \langle e_1, \ldots, e_p, \mathrm{i}\, e_{\bar{1}}, \ldots, \mathrm{i}\, e_{\bar{q}}\rangle_{\mathbb{R}} \tag{2.6}$$

[a] More precisely, one should denote $x = {}^t({}^t x', {}^t x'')$ etc.; we will use this abbreviated notation in what follows.

in the sense that the quantized operators are realized in $\mathcal{PD}(V)$, the ring of polynomial coefficient differential operators on V. Therefore, the quantized moment map $\hat{\mu}$ is given by

$$\hat{\mu} = (-\hat{x}\,{}^t\hat{y} + \hat{y}\,{}^t\hat{x})I_{p,q} = \begin{bmatrix} \mathrm{i}\,(x'\,{}^t\partial_{x'} - \partial_{x'}\,{}^tx') & x'\,{}^ty'' + \partial_{x'}\,{}^t\partial_{y''} \\ \partial_{y''}\,{}^t\partial_{x'} + y''\,{}^tx' & \mathrm{i}\,(y''\,{}^t\partial_{y''} - \partial_{y''}\,{}^ty'') \end{bmatrix},$$

where

$$\hat{x} = {}^t(\hat{x}_1, \ldots, \hat{x}_{p+q}) = {}^t(x', -\mathrm{i}\,\partial_{y''}),$$
$$\hat{y} = {}^t(\hat{y}_1, \ldots, \hat{y}_{p+q}) = {}^t(-\mathrm{i}\,\partial_{x'}, y''),$$

and

$$x' = {}^t(x_1, \ldots, x_p), \qquad \partial_{x'} = {}^t(\partial_{x_1}, \ldots, \partial_{x_p}),$$
$$y'' = {}^t(y_{\bar{1}}, \ldots, y_{\bar{q}}), \qquad \partial_{y''} = {}^t(\partial_{y_{\bar{1}}}, \ldots, \partial_{y_{\bar{q}}}).$$

Note that $x_1, \ldots, x_p, y_{\bar{1}}, \ldots, y_{\bar{q}}$ are considered to be the coordinate functions on V with respect to the basis $e_1, \ldots, e_p, \mathrm{i}\,e_{\bar{1}}, \ldots, \mathrm{i}\,e_{\bar{q}}$.

Theorem 2.1. *For $X \in \mathfrak{g}$, set $\pi(X) := \mathrm{i}\,\langle\hat{\mu}, X\rangle$. Then $\pi : \mathfrak{g} \to \mathcal{PD}(V)$ is a Lie algebra homomorphism. In terms of the basis* (2.1)*, it is given by*

$$\pi(X) = \begin{cases} -x_j\partial_{x_i} + x_i\partial_{x_j} & \text{if} \quad X = X^+_{i,j}; \\ -y_{\bar{j}}\partial_{y_{\bar{i}}} + y_{\bar{i}}\partial_{y_{\bar{j}}} & \text{if} \quad X = X^+_{\bar{i},\bar{j}}; \\ \mathrm{i}\,(x_i y_{\bar{j}} + \partial_{x_i}\partial_{y_{\bar{j}}}) & \text{if} \quad X = X^-_{i,j}. \end{cases} \tag{2.7}$$

Proof. This is proved in the same manner as [3, Theorem 2.3] (or, one can verify the commutation relations by direct calculation). □

There is another canonical quantization that corresponds to the same Lagrangian subspace V of W as given in (2.6). Namely, if we quantize the coordinate functions as

$$\begin{aligned} \hat{x}_i &= x_i, & \hat{y}_i &= -\mathrm{i}\,\partial_{x_i}, & &(i = 1, \ldots, p), \\ \hat{x}_{\bar{j}} &= y_{\bar{j}}, & \hat{y}_{\bar{j}} &= \mathrm{i}\,\partial_{y_{\bar{j}}}, & &(j = 1, \ldots, q), \end{aligned} \tag{2.8}$$

then the quantized moment map, which we denote by $\hat{\mu}^\sharp$, is given by

$$\hat{\mu}^\sharp = (-\hat{x}\,{}^t\hat{y} + \hat{y}\,{}^t\hat{x})I_{p,q} = \mathrm{i}\begin{bmatrix} x'\,{}^t\partial_{x'} - \partial_{x'}\,{}^tx' & x'\,{}^t\partial_{y''} + \partial_{x'}\,{}^ty'' \\ y''\,{}^t\partial_{x'} + \partial_{y''}\,{}^tx' & y''\,{}^t\partial_{y''} - \partial_{y''}\,{}^ty'' \end{bmatrix},$$

where

$$\hat{x} = {}^t(\hat{x}_1, \ldots, \hat{x}_{p+q}) = {}^t(x', y''),$$
$$\hat{y} = {}^t(\hat{y}_1, \ldots, \hat{y}_{p+q}) = {}^t(-\mathrm{i}\,\partial_{x'}, \mathrm{i}\,\partial_{y''}).$$

Hence one obtains a representation $\pi^\sharp : \mathfrak{g} \to \mathcal{PD}(V)$ if one sets $\pi^\sharp(X) := \mathrm{i}\,\langle\hat{\mu}^\sharp, X\rangle$ for $X \in \mathfrak{g}$. It is given in terms of the basis (2.1) by

$$\pi^\sharp(X) = \begin{cases} -x_j\partial_{x_i} + x_i\partial_{x_j} & \text{if} \quad X = X^+_{i,j}; \\ -y_{\bar{j}}\partial_{y_{\bar{i}}} + y_{\bar{i}}\partial_{y_{\bar{j}}} & \text{if} \quad X = X^+_{\bar{i},\bar{j}}; \\ -(x_i\partial_{y_{\bar{j}}} + y_{\bar{j}}\partial_{x_i}) & \text{if} \quad X = X^-_{i,j}. \end{cases} \tag{2.9}$$

Remark 2.2. (i) Comparing (2.8) with (2.5), one sees that $\pi^\sharp$ is related to π through the partial Fourier transform on $\mathbb{R}^{p+q}$ with respect to the variables $y_{\bar{1}}, \ldots, y_{\bar{q}}$. In fact, if we denote the dual variable of $y_{\bar{j}}$ by $\eta_{\bar{j}}$, $j = 1, 2, \ldots, q$, then π and $\pi^\sharp$ interchange with each other under the correspondence

$$-\mathrm{i}\,\partial_{y_{\bar{j}}} \longleftrightarrow \eta_{\bar{j}}, \quad y_{\bar{j}} \longleftrightarrow \mathrm{i}\,\partial_{\eta_{\bar{j}}} \qquad (j = 1, \ldots, q);$$

the former operators $-\mathrm{i}\,\partial_{y_{\bar{j}}}$ and $\eta_{\bar{j}}$ are the realizations of $\hat{x}_{\bar{j}}$, while the latter operators $y_{\bar{j}}$ and $\mathrm{i}\,\partial_{\eta_{\bar{j}}}$ are the realizations of $\hat{y}_{\bar{j}}$.

(ii) Recall that one can obtain $\pi^\sharp$ by differentiating the left regular representation of $G = \mathrm{O}(p,q)$ on $C^\infty(V)$, the space of complex-valued smooth functions on V, where G acts on V by matrix multiplication under the identification of V with $\mathbb{R}^{p+q}$ given by ${}^t(x', \mathrm{i}\,y'') \leftrightarrow {}^t(x', y'')$ (see e.g. [6,12]). As one can see from (2.7) and (2.9), $\pi^\sharp(X)$ coincides with $\pi(X)$ for all $X \in \mathfrak{k}$. Thus, the action π restricted to $\mathfrak{k}_0$ lifts to the action of K on $C^\infty(V)$.

3. Dual Pair $(\mathrm{O}(p,q), \mathfrak{sl}_2(\mathbb{R}))$

Henceforth, let us denote $x' = {}^t(x_1, \ldots, x_p)$ and $y'' = {}^t(y_{\bar{1}}, \ldots, y_{\bar{q}})$ by

$$x = {}^t(x_1, \ldots, x_p) \quad \text{and} \quad y = {}^t(y_1, \ldots, y_q)$$

respectively for the sake of simplicity if there exists no risk of confusion. Namely, we regard $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_q)$ as the canonical coordinate functions on $\mathbb{R}^p$ and on $\mathbb{R}^q$ respectively.

If we denote the Casimir element of $\mathfrak{g}$ by $\Omega_{\mathfrak{g}}$, then the corresponding Casimir operator is given by

$$\begin{aligned}\pi(\Omega_{\mathfrak{g}}) &= (E_x - E_y)^2 + (p-q)(E_x - E_y) - 2(E_x + E_y) \\ &\quad - \Big(r_x^2 r_y^2 + r_x^2 \Delta_x + r_y^2 \Delta_y + \Delta_x \Delta_y\Big) - pq,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned} E_x &= \sum_{i \in [p]} x_i \partial_{x_i}, & r_x^2 &= \sum_{i \in [p]} x_i^2, & \Delta_x &= \sum_{i \in [p]} \partial_{x_i}^2, \\ E_y &= \sum_{j \in [q]} y_j \partial_{y_j}, & r_y^2 &= \sum_{j \in [q]} y_j^2, & \Delta_y &= \sum_{j \in [q]} \partial_{y_j}^2. \end{aligned} \tag{3.2}$$

Now, taking account of the fact that our realization of the representation operators of $\mathfrak{g}$ given in (2.7) is a partial Fourier transform of the ones given in [6,12] as we mentioned in Remark 2.2 (i) above, we define elements H, X^+, X^- of $\mathcal{PD}(V)$ by

$$H = -E_x - \frac{p}{2} + E_y + \frac{q}{2}, \quad X^+ = -\frac{1}{2}(\Delta_x + r_y^2), \quad X^- = \frac{1}{2}(r_x^2 + \Delta_y). \tag{3.3}$$

Then, it is immediate to see that the commutation relations among them are given by

$$[H, X^+] = 2X^+, \quad [H, X^-] = -2X^-, \quad [X^+, X^-] = H.$$

Proposition 3.1. *Let* $\mathfrak{g}' := \mathbb{C}\text{-span}\,\{H, X^+, X^-\}$. *Then* $\mathfrak{g}'$ *is a Lie subalgebra of* $\mathcal{PD}(V)^{\mathfrak{g}}$ *isomorphic to* $\mathfrak{sl}_2$ *(=* $\mathfrak{sl}_2(\mathbb{C})$*), where* $\mathcal{PD}(V)^{\mathfrak{g}}$ *denotes the commutant of* $\mathfrak{g}$ *in* $\mathcal{PD}(V)$.

Proof. Note that $\pi(X^+_{i,j})$, $i, j \in [p]$, span the Lie subalgebra isomorphic to $\mathfrak{o}_p$ commuting with E_x, Δ_x and r_x^2, and that $\pi(X^+_{\bar{i},\bar{j}})$, $i, j \in [q]$, span the Lie subalgebra isomorphic to $\mathfrak{o}_q$ commuting with E_y, Δ_y and r_y^2. Hence, it remains to show that each $\pi(X^-_{i,j})$ commutes with H, X^+ and X^- given in (3.3)

We will only show here that $[\pi(X^-_{i,j}), X^+] = 0$. The other cases can be shown similarly. Now, one sees

$$\begin{aligned}
-2\mathrm{i}\,[\pi(X^-_{i,j}), X^+] &= \left[x_i y_j + \partial_{x_i}\partial_{y_j}, -\Delta_x - r_y^2\right] \\
&= \sum_{k=1}^{p} \left[\partial_{x_k}^2, x_i\right] y_j - \sum_{l=1}^{q} \partial_{x_i} \left[\partial_{y_j}, y_l^2\right] \\
&= \sum_{k=1}^{p} 2\delta_{k,i}\partial_{x_k} y_j - \sum_{l=1}^{q} 2\partial_{x_i}\delta_{j,l} y_l \\
&= 2\partial_{x_i} y_j - 2\partial_{x_i} y_j = 0.
\end{aligned}$$

This completes the proof. □

If one denotes the Casimir element of $\mathfrak{g}'$ by $\Omega_{\mathfrak{g}'}$, then the corresponding Casimir operator that is defined by

$$\begin{aligned}
\pi(\Omega_{\mathfrak{g}'}) &= H^2 + 2(X^+X^- + X^-X^+) \\
&= H^2 - 2H + 4X^+X^- \\
&= H^2 + 2H + 4X^-X^+
\end{aligned}$$

is concretely written in terms of the operators given by (3.2) as follows:

$$\begin{aligned}
\pi(\Omega_{\mathfrak{g}'}) &= (E_x - E_y)^2 + (p-q)(E_x - E_y) - 2(E_x + E_y) \\
&\quad - \left(r_x^2\, r_y^2 + r_x^2\,\Delta_x + r_y^2\,\Delta_y + \Delta_x\,\Delta_y\right) + \frac{1}{4}(p-q)^2 - (p+q).
\end{aligned} \tag{3.4}$$

It follows from (3.1) and (3.4) that

$$\pi(\Omega_{\mathfrak{g}}) = \pi(\Omega_{\mathfrak{g}'}) - \frac{1}{4}\,(p+q)^2 + (p+q)$$

(see [4, 12]).

In what follows, we denote by $\mathcal{H}^k(\mathbb{R}^n)$ the space of homogeneous harmonic polynomials on $\mathbb{R}^n$ of degree k and set $\mathcal{H}(\mathbb{R}^n) := \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{R}^n)$. It is well known that $\mathcal{H}^k(\mathbb{R}^n)$ is an irreducible $\mathrm{O}(n)$-module and its dimension is given by

$$\begin{aligned}
\dim \mathcal{H}^k(\mathbb{R}^n) &= \binom{k+n-1}{n-1} - \binom{k+n-3}{n-1} \\
&= \frac{(k+n-3)!}{k!\,(n-2)!}\,(2k+n-2)
\end{aligned}$$

if $n \geqslant 2$ and $k \in \mathbb{N}$, where $\binom{\nu}{i}$ denotes the binomial coefficient. Note that it can be further rewritten as

$$\dim \mathcal{H}^k(\mathbb{R}^n) = \frac{2(k+n/2-1)}{(n-2)!}\,(k+1)(k+2)\cdots(k+n-3). \tag{3.5}$$

Now, we will find a highest weight vector with respect to the $\mathfrak{g}'$-action (3.3), i.e. a function f on V which satisfies

$$Hf = \lambda f \quad \text{and} \quad X^+ f = 0 \tag{3.6}$$

for some $\lambda \in \mathbb{C}$. Taking account of the fact that the algebra of polynomial functions on V, which we denote by $\mathcal{P}(V)$, can be written as

$$\begin{aligned}\mathcal{P}(V) &\simeq \mathbb{C}[x_1, \ldots, x_p] \otimes \mathbb{C}[y_1, \ldots, y_q] \\ &\simeq \bigoplus_{k=0}^{\infty} \left(\mathbb{C}[r_x^2] \otimes \mathcal{H}^k(\mathbb{R}^p)\right) \otimes \bigoplus_{l=0}^{\infty} \left(\mathbb{C}[r_y^2] \otimes \mathcal{H}^l(\mathbb{R}^q)\right) \\ &\simeq \bigoplus_{k,l=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \otimes \mathbb{C}[r_x^2, r_y^2],\end{aligned}$$

we will seek for a function that satisfies (3.6) of the form

$$f(x,y) = h_1(x) h_2(y) \phi(r_x^2, r_y^2), \tag{3.7}$$

where $h_1 \in \mathcal{H}^k(\mathbb{R}^p), h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ are harmonic polynomials, and $\phi(s,t) \in \mathbb{C}[[s,t]]$ is a formal power series (**Caution**: we do *not* assume that ϕ is a *polynomial*). Namely, our function f on V lives in the space $\tilde{\mathcal{E}}$ defined by

$$\tilde{\mathcal{E}} := \bigoplus_{k,l=0}^{\infty} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \otimes \mathbb{C}[[r_x^2, r_y^2]] \qquad \text{(algebraic direct sum).} \tag{3.8}$$

Recall that the action π of $\mathfrak{k}_0$ lifts to the action of K on $\tilde{\mathcal{E}}$ as we mentioned in Remark 2.2 (ii), which we denote by the same letter π.

Lemma 3.1. *Let* $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ *and* $r^2 = \sum_{i=1}^n x_i^2$. *For a homogeneous harmonic polynomial* $h = h(x_1, \ldots, x_n)$ *on* $\mathbb{R}^n$ *of degree* d *and for a smooth function* $\varphi(u)$ *in a single variable* u*, we have*

$$\Delta(h\varphi(r^2)) = (4d + 2n) h\varphi'(r^2) + 4r^2 h\varphi''(r^2). \tag{3.9}$$

Proof. Since $\partial_{x_i}\varphi(r^2) = 2x_i\varphi'(r^2)$ and $\partial_{x_i}^2\varphi(r^2) = 2\varphi'(r^2) + 4x_i^2\varphi''(r^2)$, one obtains

$$\Delta\varphi(r^2) = 2n\varphi'(r^2) + 4r^2\varphi''(r^2).$$

Thus,

$$\begin{aligned}\Delta(h\varphi(r^2)) &= \sum_{i=1}^n \left(\partial_{x_i}^2 h \cdot \varphi(r^2) + 2\partial_{x_i} h \cdot \partial_{x_i}\varphi(r^2) + h \cdot \partial_{x_i}^2 \varphi(r^2)\right) \\ &= 4dh\varphi'(r^2) + h\Delta\varphi(r^2) \\ &= 4dh\varphi'(r^2) + h\left(2n\varphi'(r^2) + 4r^2\varphi''(r^2)\right) \\ &= (4d + 2n)h\varphi'(r^2) + 4r^2 h\varphi''(r^2).\end{aligned}$$

This completes the proof. □

For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ (resp. $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$), we define its shifted degree by $\kappa_+(h_1) := k + p/2$ (resp. $\kappa_-(h_2) := l + q/2$), which we denote just by κ_+ (resp. κ_-) if there is no risk of confusion.

It follows from Lemma 3.1 that if f is of the form in (3.7) then

$$\begin{aligned} X^+ f &= -\frac{1}{2}(\Delta_x(h_1h_2\phi) + r_y^2 h_1h_2\phi) \\ &= -2h_1h_2\left(r_x^2\,(\partial_s^2\phi)(r_x^2, r_y^2) + \kappa_+(\partial_s\phi)(r_x^2, r_y^2) + \frac{r_y^2}{4}\,\phi(r_x^2, r_y^2)\right), \end{aligned}$$

which shows that $f = h_1(x)h_2(y)\phi(r_x^2, r_y^2)$ satisfies $X^+ f = 0$ if and only if ϕ is a solution to a differential equation

$$s\partial_s^2\phi + \kappa_+\partial_s\phi + \tfrac{t}{4}\phi = 0 \tag{3.10}$$

with $\kappa_+ = \kappa_+(h_1) = k + p/2$. Solving the differential equation (3.10) by power series, one obtains that

$$\phi(s,t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\kappa_+)_n}\left(\frac{s\,t}{4}\right)^n, \tag{3.11}$$

where a_0 is an arbitrary formal power series in t. Note that if one defines a power series Ψ_α by

$$\Psi_\alpha(u) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(\alpha)_n} u^n = 1 - \frac{u}{\alpha} + \frac{u^2}{2!\,\alpha(\alpha+1)} - \frac{u^3}{3!\,\alpha(\alpha+1)(\alpha+2)} + \cdots \tag{3.12}$$

for $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$, then it converges on the whole $\mathbb{C}$ and is a unique solution to a differential equation

$$u\Psi_\alpha''(u) + \alpha\Psi_\alpha'(u) + \Psi_\alpha(u) = 0 \tag{3.13}$$

that satisfies the initial condition $\Psi_\alpha(0) = 1$.

In the sequel, we set

$$\psi_\alpha^{(n)} := \Psi_\alpha^{(n)}(r_x^2 r_y^2/4) \qquad (n \in \mathbb{N}) \tag{3.14}$$

for brevity, where $\Psi_\alpha^{(n)}(u)$ denotes the n-th derivative of $\Psi_\alpha(u)$ in u.

If, in addition, f satisfies that $Hf = \lambda f$ for some $\lambda \in \mathbb{C}$, then the factor a_0 in (3.11) is equal to t^{μ_-} up to a constant multiple, with $\mu_- = (1/2)(\lambda + \kappa_+ - \kappa_-) \in \mathbb{N}$, $\kappa_+ = \kappa_+(h_1)$ and $\kappa_- = \kappa_-(h_2)$.

Thus, for given $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, if $f = f(x,y)$ of the form (3.7) is a highest weight vector with respect to the $\mathfrak{g}'$-action, i.e. it satisfies (3.6), then $\phi(r_x^2, r_y^2)$ is a constant multiple of $r_y^{2\mu_-}\psi_{\kappa_+}$, and the weight of f is given by

$$\lambda = -\kappa_+ + \kappa_- + 2\mu_- \quad (\mu_- \in \mathbb{N}). \tag{3.15}$$

Similarly, for given $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, one can show that if $f = f(x,y)$ of the form (3.7) is a lowest weight vector with respect to the $\mathfrak{g}'$-action, i.e. it satisfies

$$Hf = \lambda f \quad \text{and} \quad X^- f = 0 \tag{3.16}$$

for some $\lambda \in \mathbb{C}$, then $\phi(r_x^2, r_y^2)$ is a constant multiple of $r_x^{2\mu_+}\psi_{\kappa_-}$, and the weight of f is given by

$$\lambda = -\kappa_+ + \kappa_- - 2\mu_+ \quad (\mu_+ \in \mathbb{N}). \tag{3.17}$$

Let us summarize the above argument in the following.

Proposition 3.2. *Given $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, let $f = f(x,y)$ be a function of the form*

$$f(x,y) = h_1(x)h_2(y)\phi(r_x^2, r_y^2) \qquad (\phi(r_x^2, r_y^2) \in \mathbb{C}[[r_x^2, r_y^2]]). \tag{3.18}$$

If f given in (3.18) *is a highest (resp. lowest) weight vector with respect to the $\mathfrak{g}'$-action, i.e. it satisfies* (3.6) *(resp.* (3.16)*), then $\phi(r_x^2, r_y^2)$ is a constant multiple of $r_y^{2\mu_-}\psi_{\kappa_+}$ (resp. $r_x^{2\mu_+}\psi_{\kappa_-}$) and the weight λ of f is equal to $-\kappa_+ + \kappa_- + 2\mu_-$ (resp. $-\kappa_+ + \kappa_- - 2\mu_+$).*

Here $\kappa_+ = \kappa_+(h_1) = k + p/2$, $\kappa_- = \kappa_-(h_2) = l + q/2$, $\mu_+, \mu_- \in \mathbb{N}$, and $\psi_{\kappa_\pm}$ is an element of $\mathbb{C}[[r_x^2, r_y^2]]$ given by (3.14) *with $\alpha = \kappa_\pm$.*

Taking account of the discussion so far, let us introduce the subspace $\mathcal{E}$ of $\tilde{\mathcal{E}}$ by

$$\mathcal{E} := \mathbb{C}\text{-span}\left\{ h_1(x)h_2(y)r_x^{2a}r_y^{2b}\psi_\alpha \in \tilde{\mathcal{E}} \,\middle|\, \begin{array}{l} h_1 \in \mathcal{H}(\mathbb{R}^p),\ h_2 \in \mathcal{H}(\mathbb{R}^q), \\ a, b \in \mathbb{N},\ \alpha \in \mathbb{C} \setminus (-\mathbb{N}) \end{array} \right\}.$$

Then one will find that $\mathcal{E}$ is stable under the action of $(\mathfrak{g}, K)$ as well as that of $\mathfrak{g}'$ (see Propositions 3.3 and 3.4 below).

Remark 3.1. (i) The function Ψ_α given in (3.12) can be written in terms of the Bessel function J_ν of the first kind of order ν

$$J_\nu(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{t}{2}\right)^{\nu+2n}$$

that solves the Bessel's differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \frac{1}{t}\frac{\mathrm{d}w}{\mathrm{d}t} + \left(1 - \frac{\nu^2}{t^2}\right) w = 0 \tag{3.19}$$

(see e.g. [14]). Namely, one has

$$\Psi_\alpha(u) = \Gamma(\alpha)\, u^{-(\alpha-1)/2} J_{\alpha-1}(2\, u^{1/2}). \tag{3.20}$$

Therefore,

$$\psi_\alpha = \Gamma(\alpha) \left(\frac{r_x r_y}{2}\right)^{-(\alpha-1)} J_{\alpha-1}(r_x r_y).$$

Note that (3.13) corresponds to (3.19) under (3.20).

(ii) Recall that our representation π is related to $\pi^\sharp$ via the partial Fourier transform with respect to $y_1, \ldots, y_q$, as we mentioned in Remark 2.2 (i). Namely, one can obtain $\pi^\sharp$ by replacing $-\mathrm{i}\,\partial_{y_j}$ and y_j in π by η_j and $\mathrm{i}\,\partial_{\eta_j}$, $j = 1, \ldots, q$, respectively. Under this correspondence, one finds that $H = -E_x - p/2 + E_y + q/2$ and $X^+ = -\frac{1}{2}(\Delta_x + r_y^2)$ transform, up to constant multiples, into the shifted degree operator $\tilde{E}_{p,q}$ and the d'Alembertian $\Box_{p,q}$ on $\mathbb{R}^{p+q}$ that are given by

$$\tilde{E}_{p,q} = \sum_{i=1}^{p} x_i \partial_{x_i} + \sum_{j=1}^{q} \eta_j \partial_{\eta_j} + \frac{p+q}{2},$$
$$\Box_{p,q} = \sum_{i=1}^{p} \partial_{x_i}^2 - \sum_{j=1}^{q} \partial_{\eta_j}^2,$$

respectively. Therefore, the highest weight vector f satisfying $Hf = \lambda f$ for some $\lambda \in \mathbb{C}$ and $X^+ f = 0$ corresponds to a homogeneous solution $\tilde{f}$ to the equation $\Box_{p,q}\tilde{f} = 0$.

Note that $\Psi_\alpha^{(n)}$ is equal to $\Psi_{\alpha+n}$ up to a constant multiple. In fact, differentiating both sides of (3.13) n times, one obtains

$$u\Psi_\alpha^{(n+2)}(u) + (\alpha+n)\Psi_\alpha^{(n+1)}(u) + \Psi_\alpha^{(n)}(u) = 0. \tag{3.21}$$

Since $\Psi_{\alpha+n}$ is a unique solution to (3.13) with α replaced by $\alpha+n$ that satisfies $\Psi_{\alpha+n}(0) = 1$, it follows that $\Psi_\alpha^{(n)} = (-1)^n/(\alpha)_n \Psi_{\alpha+n}$. Thus, one obtains

$$\psi_\alpha^{(n)} = \frac{(-1)^n}{(\alpha)_n}\psi_{\alpha+n} \qquad (n \in \mathbb{N}). \tag{3.22}$$

In what follows, we set $\rho_x := r_x^2/2$ and $\rho_y := r_y^2/2$ for economy of space. Then, it follows from (3.21) and (3.22) that

$$\rho_x \rho_y \psi_{\alpha+2} = \alpha(\alpha+1)(\psi_{\alpha+1} - \psi_\alpha) \tag{3.23}$$

for $\alpha \in \mathbb{C} \setminus (-\mathbb{N})$. Furthermore, setting $\rho := r^2/2$, one can rewrite (3.9) as

$$\frac{1}{2}\Delta(h\varphi(\rho)) = \left(d + \frac{n}{2}\right) h\varphi'(\rho) + h\rho\varphi''(\rho), \tag{3.24}$$

where h, Δ, r^2 and φ are as in Lemma 3.1.

Proposition 3.3. *For $f = h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha \in \mathcal{E}$, one has*

$$H\big(h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha\big) = (-\kappa_+ + \kappa_- - 2a + 2b) h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha, \tag{3.25}$$
$$X^+\big(h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha\big) = h_1 h_2 \Big(-a(\kappa_+ + a - 1)\rho_x^{a-1}\rho_y^b \psi_\alpha + \tfrac{\kappa_+ + 2a - \alpha}{\alpha}\rho_x^a \rho_y^{b+1}\psi_{\alpha+1}\Big), \tag{3.26}$$
$$X^-\big(h_1 h_2 \rho_x^a \rho_y^b \psi_\alpha\big) = h_1 h_2 \Big(b(\kappa_- + b - 1)\rho_x^a \rho_y^{b-1}\psi_\alpha - \tfrac{\kappa_- + 2b - \alpha}{\alpha}\rho_x^{a+1}\rho_y^b \psi_{\alpha+1}\Big). \tag{3.27}$$

In particular, the $\mathfrak{g}'$-action preserves the K-type of each element of $\mathcal{E}$.

Proof. It is immediate to show (3.25), and we will only show (3.26) here; the other case (3.27) can be shown similarly.

Setting $\varphi(u) := u^a \Psi_\alpha(\rho_y u)$, one sees

$$\varphi'(u) = au^{a-1}\Psi_\alpha(\rho_y u) + u^a \rho_y \Psi_\alpha'(\rho_y u),$$
$$\varphi''(u) = a(a-1)u^{a-2}\Psi_\alpha(\rho_y u) + 2au^{a-1}\rho_y \Psi_\alpha'(\rho_y u) + u^a \rho_y^2 \Psi_\alpha''(\rho_y u).$$

Hence it follows from (3.24) that

$$\begin{aligned}\frac{1}{2}\Delta_x(h_1\rho_x^a\psi_\alpha) &= a(\kappa_+ + a - 1)h_1\rho_x^{a-1}\psi_\alpha + (\kappa_+ + 2a)h_1\rho_x^a\rho_y\psi_\alpha' + h_1\rho_x^{a+1}\rho_y^2\psi_\alpha'' \\ &= h_1\left(a(\kappa_+ + a - 1)\rho_x^{a-1}\psi_\alpha + (\kappa_+ + 2a - \alpha)\rho_x^a\rho_y\psi_\alpha' - \rho_x^a\rho_y\psi_\alpha\right)\end{aligned}$$

since $\rho_x\rho_y\psi_\alpha'' = -\alpha\psi_\alpha' - \psi_\alpha$. Therefore, one obtains that

$$\begin{aligned}X^+(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) &= -\frac{1}{2}(\Delta_x + 2\rho_y)(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) \\ &= -a(\kappa_+ + a - 1)h_1h_2\rho_x^{a-1}\rho_y^b\psi_\alpha - (\kappa_+ + 2a - \alpha)h_1h_2\rho_x^a\rho_y^{b+1}\psi_\alpha',\end{aligned}$$

which, by (3.22), equals the right-hand side of (3.26). This completes the proof. □

We conclude this section by calculating the action of $\mathfrak{p}$ on $\mathcal{E}$, i.e. $\pi(X_{i,j}^-)f$ for $X_{i,j}^- \in \mathfrak{p}$ and $f \in \mathcal{E}$.

For a homogeneous polynomial $P = P(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ of degree d, set

$$P^\dagger := P - \frac{r^2}{4(d + n/2 - 2)}\Delta P,$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ and $r^2 = \sum_{i=1}^n x_i^2$. Note that if $\Delta^2 P = 0$ then $P^\dagger$ is harmonic by Lemma 3.1, and that if $h = h(x_1, \ldots, x_n)$ is harmonic then $\Delta(x_i h) = 2\partial_{x_i} h$ and $\Delta^2(x_i h) = 0$.

Proposition 3.4. *For* $f = h_1h_2\rho_x^a\rho_y^b\psi_\alpha \in \mathcal{E}$, *one has*

$$\begin{aligned}&-\mathrm{i}\,\pi(X_{i,j}^-)\big(h_1h_2\rho_x^a\rho_y^b\psi_\alpha\big) \\ &= (\partial_{x_i}h_1)(\partial_{y_j}h_2)\rho_x^a\rho_y^b\left(\tfrac{(\kappa_++a-\alpha)(\kappa_-+b-\alpha)}{(\kappa_+-1)(\kappa_--1)}\psi_\alpha + \tfrac{(\alpha-1)(\kappa_++\kappa_-+a+b-\alpha-1)}{(\kappa_+-1)(\kappa_--1)}\psi_{\alpha-1}\right) \\ &\quad + (\partial_{x_i}h_1)(y_jh_2)^\dagger\left(-\tfrac{\kappa_++a+b-\alpha}{\alpha(\kappa_+-1)}\rho_x^{a+1}\rho_y^b\psi_{\alpha+1} + \tfrac{b(\kappa_++a-1)}{\kappa_+-1}\rho_x^a\rho_y^{b-1}\psi_\alpha\right) \\ &\quad + (x_ih_1)^\dagger(\partial_{y_j}h_2)\left(-\tfrac{\kappa_-+a+b-\alpha}{\alpha(\kappa_--1)}\rho_x^a\rho_y^{b+1}\psi_{\alpha+1} + \tfrac{a(\kappa_-+b-1)}{\kappa_--1}\rho_x^{a-1}\rho_y^b\psi_\alpha\right) \\ &\quad + (x_ih_1)^\dagger(y_jh_2)^\dagger\left(-\tfrac{a+b+1-\alpha}{\alpha}\rho_x^a\rho_y^b\psi_{\alpha+1} + ab\rho_x^{a-1}\rho_y^{b-1}\psi_\alpha\right),\end{aligned} \tag{3.28}$$

where one regards $\partial_{x_i}h_1/(\kappa_+ - 1)$ *(resp.* $\partial_{y_j}h_2/(\kappa_- - 1)$*) as zero when* $\kappa_+ = 1$ *(resp.* $\kappa_- = 1$*).*

Proof. Since $\partial_{x_i}\psi_\alpha = \rho_y x_i\psi_\alpha'$ and $\partial_{y_j}\psi_\alpha = \rho_x y_j\psi_\alpha'$, one obtains

$$\begin{aligned}&-\mathrm{i}\,\pi(X_{i,j}^-)f = (\partial_{x_i}\partial_{y_j} + x_iy_j)(h_1h_2\rho_x^a\rho_y^b\psi_\alpha) \\ &= (\partial_{x_i}h_1)(\partial_{y_j}h_2)\rho_x^a\rho_y^b\psi_\alpha + (x_ih_1)(\partial_{y_j}h_2)\left(a\rho_x^{a-1}\rho_y^b\psi_\alpha + \rho_x^a\rho_y^{b+1}\psi_\alpha'\right) \\ &+ (\partial_{x_i}h_1)(y_jh_2)\left(b\rho_x^a\rho_y^{b-1}\psi_\alpha + \rho_x^{a+1}\rho_y^b\psi_\alpha'\right) \\ &+ (x_ih_1)(y_jh_2)\left(\rho_x^a\rho_y^b\psi_\alpha + ab\rho_x^{a-1}\rho_y^{b-1}\psi_\alpha + (a+b+1)\rho_x^a\rho_y^b\psi_\alpha' + \rho_x^{a+1}\rho_y^{b+1}\psi_\alpha''\right).\end{aligned} \tag{3.29}$$

Now, by definition, one has

$$x_ih_1 = (x_ih_1)^\dagger + \frac{\rho_x}{\kappa_+ - 1}\partial_{x_i}h_1 \quad \text{and} \quad y_jh_2 = (y_jh_2)^\dagger + \frac{\rho_y}{\kappa_- - 1}\partial_{y_j}h_2. \tag{3.30}$$

Substituting (3.30) into (3.29), and using the relations (3.22) and (3.23), one sees that the coefficient of $(\partial_{x_i} h_1)(\partial_{y_j} h_2)$ in (3.29) equals the one of $(\partial_{x_i} h_1)(\partial_{y_j} h_2)$ in the right-hand side of (3.28). One can verify that each coefficient of $(\partial_{x_i} h_1)(y_j h_2)^\dagger, (x_i h_1)^\dagger(\partial_{y_j} h_2)$ and $(x_i h_1)^\dagger (y_j h_2)^\dagger$ in (3.29) equals the one of the corresponding terms in (3.28) similarly. This completes the proof. □

4. $(\mathfrak{g}, K)$-module associated with finite-dimensional $\mathfrak{sl}_2$-module

If a nonzero $f \in \mathcal{E}$ satisfies $Hf = \lambda f,\ X^+ f = 0$ and $(X^-)^{m+1} f = 0$ (resp. $Hf = \lambda f,\ X^- f = 0$ and $(X^+)^{m+1} f = 0$) for some $m \in \mathbb{N}$, then it follows from the representation theory of $\mathfrak{g}' = \mathfrak{sl}_2$ that $\lambda = m$ (resp. $\lambda = -m$). Thus, we introduce $(\mathfrak{g}, K)$-modules associated with the finite-dimensional $\mathfrak{sl}_2$-module F_m of dimension $m + 1$ as follows, which are the main objects of this paper.

Definition 4.1. Given $m \in \mathbb{N}$, we define $(\mathfrak{g}, K)$-modules $M^\pm(m)$ by

$$M^+(m) := \left\{ f \in \mathcal{E} \,\middle|\, Hf = mf, X^+ f = 0, (X^-)^{m+1} f = 0 \right\},$$

$$M^-(m) := \left\{ f \in \mathcal{E} \,\middle|\, Hf = -mf, X^- f = 0, (X^+)^{m+1} f = 0 \right\}.$$

The modules $M^\pm(m)$ were originally introduced in [12] without the condition of finite dimensionality. Note that $M^+(0)$ is identical to $M^-(0)$ by definition and that both $M^\pm(m)$ should correspond to the $\mathfrak{sl}_2$-module F_m under the Howe duality (cf. [5]).

If $M^+(m) \neq \{0\}$ (resp. $M^-(m) \neq \{0\}$), then one sees that $p \equiv q \bmod 2$; for, if one takes a nonzero $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in M^+(m)$ (resp. $f = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-} \in M^-(m)$) with $h_1 \in \mathcal{H}^k(\mathbb{R}^p), h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ and $\mu_\pm \in \mathbb{N}$, then

$$\pm m = -\kappa_+ + \kappa_- \pm 2\mu_\mp = -k + l - \frac{p-q}{2} \pm 2\mu_\mp \in \mathbb{Z} \tag{4.1}$$

by (3.15) (resp. (3.17)). Hence one obtains $(p - q)/2 \in \mathbb{Z}$. Therefore, we assume that $p \equiv q$ mod 2 in the rest of this paper.

Lemma 4.1. *For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$, $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$ and $m \in \mathbb{N}$, let*

$$v^+ = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in M^+(m) \quad \textit{and} \quad v^- = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-} \in M^-(m), \tag{4.2}$$

where $\mu_+, \mu_- \in \mathbb{N}$ such that $\mu_+ + \mu_- = m$. Then the $\mathfrak{g}'$-module generated by v^+ coincides with the one generated by v^-:

$$\langle v^+ \rangle_{\mathfrak{g}'} = \langle v^- \rangle_{\mathfrak{g}'}.$$

Proof. Both v^+ and $(X^+)^m v^-$ (resp. v^- and $(X^-)^m v^+$) are elements of $\mathcal{E} \subset \tilde{\mathcal{E}}$ that are highest (resp. lowest) weight vectors of weight m (resp. $-m$) under $\mathfrak{g}'$-action. Namely, they are solutions in $\tilde{\mathcal{E}}$ to the differential equation

$$Hf = \pm mf \quad \text{and} \quad X^\pm f = 0.$$

As we mentioned in Proposition 3.2, they are respectively equal to each other up to a constant multiple. This completes the proof. □

Proposition 4.1. *For $m \in \mathbb{N}$, $M^+(m)$ and $M^-(m)$ are isomorphic to each other.*

Proof. For $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, set

$$v^+ = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in M^+(m) \quad \text{and} \quad v^- = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-} \in M^-(m).$$

with $\mu_+ + \mu_- = m$ as in (4.2). Then, $(X^+)^m v^-$ (resp. $(X^-)^m v^+$) is equal to v^+ (resp. v^-) up to a constant multiple as we saw in Lemma 4.1, and thus, $(X^+)^m (X^-)^m v^+$ is equal to v^+ up to a constant multiple. In fact, $(X^+)^m (X^-)^m v^+ = (m!)^2 v^+$. Therefore,

$$(X^-)^m : M^+(m) \longrightarrow M^-(m)$$

provides an isomorphism of $(\mathfrak{g}, K)$-module. This completes the proof. □

Now we prepare two lemmas to prove our main result. Note that Lemma 4.2 below is just a special case of Proposition 3.4. However, we state it separately to highlight the rôle of the relation (3.23).

Lemma 4.2. *Let $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, and set $\kappa_+ = k + p/2$, $\kappa_- = l + q/2$.*

(1) For a highest weight vector $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in \mathcal{E}$, $\pi(X_{i,j}^-) f$ is given by

$$\begin{aligned} -\mathrm{i}\,\pi(X_{i,j}^-)\big(h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+}\big) = {} & \frac{\kappa_- + \mu_- - 1}{\kappa_- - 1} (\partial_{x_i} h_1)(\partial_{y_j} h_2) \rho_y^{\mu_-} \psi_{\kappa_+ - 1} \\ & + \mu_- (\partial_{x_i} h_1)(y_j h_2)^\dagger \rho_y^{\mu_- - 1} \psi_{\kappa_+ - 1} \\ & + \frac{\kappa_+ - \kappa_- - \mu_-}{\kappa_+(\kappa_- - 1)} (x_i h_1)^\dagger (\partial_{y_j} h_2) \rho_y^{\mu_- + 1} \psi_{\kappa_+ + 1} \\ & + \frac{\kappa_+ - \mu_- - 1}{\kappa_+} (x_i h_1)^\dagger (y_j h_2)^\dagger \rho_y^{\mu_-} \psi_{\kappa_+ + 1}. \end{aligned} \tag{4.3}$$

(2) For a lowest weight vector $f = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-}$, $\pi(X_{i,j}^-) f$ is given by

$$\begin{aligned} -\mathrm{i}\,\pi(X_{i,j}^-)\big(h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-}\big) = {} & \frac{\kappa_+ + \mu_+ - 1}{\kappa_+ - 1} (\partial_{x_i} h_1)(\partial_{y_j} h_2) \rho_x^{\mu_+} \psi_{\kappa_- - 1} \\ & + \frac{\kappa_- - \kappa_+ - \mu_+}{\kappa_-(\kappa_+ - 1)} (\partial_{x_i} h_1)(y_j h_2)^\dagger \rho_x^{\mu_+ + 1} \psi_{\kappa_- + 1} \\ & + \mu_+ (x_i h_1)^\dagger (\partial_{y_j} h_2) \rho_x^{\mu_+ - 1} \psi_{\kappa_- - 1} \\ & + \frac{\kappa_- - \mu_+ - 1}{\kappa_-} (x_i h_1)^\dagger (y_j h_2)^\dagger \rho_x^{\mu_+} \psi_{\kappa_- + 1}. \end{aligned} \tag{4.4}$$

Proof. We only show (4.3) here. The other formula (4.4) can be shown similarly.

Set $a = 0, b = \mu_-$ and $\alpha = \kappa_+$ in (3.28). Then, using the relation (3.23) with $\alpha = \kappa_+ - 1$, i.e.

$$\rho_x \rho_y \psi_{\kappa_+ + 1} = \kappa_+(\kappa_+ - 1)(\psi_{\kappa_+} - \psi_{\kappa_+ - 1}),$$

one sees that the coefficient of $(\partial_{x_i} h_1)(y_j h_2)^\dagger$ equals

$$\begin{aligned} & - \frac{\mu_-}{\kappa_+(\kappa_+ - 1)} \rho_x \rho_y^{\mu_-} \psi_{\kappa_+ + 1} + \mu_- \rho_y^{\mu_- - 1} \psi_{\kappa_+} \\ & = \mu_- \rho_y^{\mu_- - 1} \psi_{\kappa_+ - 1}. \end{aligned}$$

To show for the other coefficients is trivial and omitted. □

Lemma 4.3. *Let $h_1 \in \mathcal{H}^k(\mathbb{R}^p)$ and $h_2 \in \mathcal{H}^l(\mathbb{R}^q)$, and set $\kappa_+ = k + p/2$, $\kappa_- = l + q/2$.*

(1) For a highest weight vector $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+}$ of weight $\lambda = -\kappa_+ + \kappa_- + 2\mu_-$, one has

$$(X^-)^\nu f = h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(-\lambda + \nu - 1)_i^-(\mu_-)_{\nu-i}^-(\kappa_- + \mu_- - 1)_{\nu-i}^-}{(\kappa_+)_i} \rho_x^i \rho_y^{\mu_- - \nu + i} \psi_{\kappa_+ + i} \quad (4.5)$$

for $\nu = 0, 1, 2, \ldots$.

(2) For a lowest weight vector $f = h_1 h_2 \rho_x^{\mu_+} \psi_{\kappa_-}$ of weight $\lambda = -\kappa_+ + \kappa_- - 2\mu_+$, one has

$$(X^+)^\nu f = (-1)^\nu h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + \nu - 1)_i^-(\mu_+)_{\nu-i}^-(\kappa_+ + \mu_+ - 1)_{\nu-i}^-}{(\kappa_-)_i} \rho_x^{\mu_+ - \nu + i} \rho_y^i \psi_{\kappa_- + i} \quad (4.6)$$

for $\nu = 0, 1, 2, \ldots$.

Proof. We only show (4.6) by induction on ν here. The other case (4.5) can be shown similarly.

It is trivial if $\nu = 0$, and it is nothing but Propostion 3.3 if $\nu = 1$. Assume that it is true for $\nu \geqslant 1$, and apply X^+ to the both sides of (4.6). Then, one sees that the right-hand side equals

$$(-1)^\nu h_1 h_2 \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(\lambda + \nu - 1)_i^-(\mu_+)_{\nu-i}^-(\kappa_+ + \mu_+ - 1)_{\nu-i}^-}{(\kappa_-)_i}$$
$$\times \left(-(\mu_+ - \nu + i)(\kappa_+ + \mu_+ - \nu + i - 1) \rho_x^{\mu_+ - \nu + i - 1} \rho_y^i \psi_{\kappa_- + i} + \frac{-\lambda + i - 2\nu}{\kappa_- + i} \rho_x^{\mu_+ - \nu + i} \rho_y^{i+1} \psi_{\kappa_- + i + 1} \right). \quad (4.7)$$

The coefficient of $(-1)^\nu h_1 h_2 \rho_x^{\mu_+ - \nu + j - 1} \rho_y^j \psi_{\kappa_- + j}$ in (4.7), $j = 0, 1, \ldots, \nu + 1$, equals

$$\begin{aligned}
&\binom{\nu}{j} \frac{(\lambda + \nu - 1)_j^-(\mu_+)_{\nu-j}^-(\kappa_+ + \mu_+ - 1)_{\nu-j}^-}{(\kappa_-)_j} \cdot (-1)(\mu_+ - \nu + j)(\kappa_+ + \mu_+ - \nu + j - 1) \\
&+ \binom{\nu}{j-1} \frac{(\lambda + \nu - 1)_{j-1}^-(\mu_+)_{\nu-j+1}^-(\kappa_+ + \mu_+ - 1)_{\nu-j+1}^-}{(\kappa_-)_{j-1}} \cdot \frac{-(\lambda + 2\nu - j + 1)}{\kappa_- + j - 1} \\
&= -\left\{ \binom{\nu}{j} (\lambda + \nu - j) + \binom{\nu}{j-1} (\lambda + 2\nu - j + 1) \right\} \\
&\qquad \times \frac{(\mu_+)_{\nu-j+1}^-(\kappa_+ + \mu_+ - 1)_{\nu-j+1}^-(\lambda + \nu - 1)_{j-1}^-}{(\kappa_-)_j} \\
&= -\binom{\nu+1}{j} \frac{(\lambda + \nu)_j^-(\mu_+)_{\nu-j+1}^-(\kappa_+ + \mu_+ - 1)_{\nu-j+1}^-}{(\kappa_-)_j}.
\end{aligned}$$

This completes the proof. □

The following is our main result.

Theorem 4.1. *Assume that* $p \geqslant 1$, $q \geqslant 1$ *and* $p + q \in 2\mathbb{N}$. *Let* $m \in \mathbb{N}$ *be a nonnegative integer satisfying* $m + 3 \leqslant (p + q)/2$. *Then one has the following.*

The K*-type formula of* $M^{\pm}(m)$ *is given by*

$$M^{\pm}(m)\big|_K \simeq \bigoplus_{\substack{k,l \geqslant 0 \\ k-l+\frac{p-q}{2} \in \Lambda_m}} \mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q), \tag{4.8}$$

where $\Lambda_m = \{-m, -m+2, -m+4, \dots, m-2, m\}$, *the set of* H*-weights of* F_m;

(2) *Suppose further that* $p, q \geqslant 2$. *Then* $M^{\pm}(m)$ *are irreducible* $(\mathfrak{g}, K)$*-modules.*

Proof. It suffices to show the theorem for $M^+(m)$. Let $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \neq 0$ be an element of $M^+(m)$, where $h_1 \in \mathcal{H}^k(\mathbb{R}^p), h_2 \in \mathcal{H}^l(\mathbb{R}^q)$. Then by Lemma 4.3, one obtains

$$\begin{aligned}(X^-)^{m+1} f &= h_1 h_2 \sum_{i=0}^{m+1} \binom{m+1}{i} \frac{(0)_i^-(\mu_-)_{m+1-i}^-(\kappa_- + \mu_- - 1)_{m+1-i}^-}{(\kappa_+)_i} \rho_x^i \rho_y^{\mu_- - m - 1 + i} \psi_{\kappa_+ + i} \\ &= (\mu_-)_{m+1}^-(\kappa_- + \mu_- - 1)_{m+1}^- h_1 h_2 \rho_y^{\mu_- - m - 1} \psi_{\kappa_+}.\end{aligned}$$

Thus, $(X^-)^{m+1} f = 0$ implies that $(\mu_-)_{m+1}^- = 0$ or $(\kappa_- + \mu_- - 1)_{m+1}^- = 0$. Namely,

$$\mu_- = 0, 1, \dots, m, \quad \text{or} \tag{4.9}$$

$$\mu_- = -\kappa_- + 1, -\kappa_- + 2, \dots, -\kappa_- + m + 1. \tag{4.10}$$

The assumption that $m + 3 \leqslant (p + q)/2$, however, implies that (4.10) is impossible; if it holded true, then by (4.1), one would obtain

$$-m + 2 \leqslant \kappa_+ + \kappa_- \leqslant m + 2,$$

which contradicts $\kappa_+ + \kappa_- \geqslant (p + q)/2 \geqslant m + 3$. Therefore, it follows from (4.9) that

$$k - l + \frac{p - q}{2} = \kappa_+ - \kappa_- = -m + 2\mu_- \in \Lambda_m,$$

which proves (1).

Let us consider a closed subset $D_m \subset \mathbb{R}^2$ (with respect to the Euclidean topology of $\mathbb{R}^2$) given by

$$D_m = \left\{ (t_1, t_2) \in \mathbb{R}^2 \,\middle|\, t_1 \geqslant p/2,\, t_2 \geqslant q/2,\, |\, t_1 - t_2 \,| \leqslant m \right\}, \tag{4.11}$$

and the set of integral points of D_m given by

$$\Sigma_m = \left\{ (t_1, t_2) \in D_m \,\middle|\, \begin{matrix} t_1 - p/2 \in \mathbb{N},\, t_2 - q/2 \in \mathbb{N}, \\ t_1 - t_2 \in \Lambda_m \end{matrix} \right\}. \tag{4.12}$$

Note that the sum in the right-hand side of (4.8) can be written as the one with (κ_+, κ_-) running over the set Σ_m.

Now, applying (4.3) to $f = h_1 h_2 \rho_y^{\mu_-} \psi_{\kappa_+} \in M^+(m)$, we denote the coefficient of

$$(\partial_{x_i} h_1)(\partial_{y_j} h_2), \quad (x_i h_1)^\dagger (\partial_{y_j} h_2), \quad (\partial_{x_i} h_1)(y_j h_2)^\dagger \quad \text{and} \quad (x_i h_1)^\dagger (y_j h_2)^\dagger$$

in the right-hand side of (4.3) by C_{--}, C_{+-}, C_{-+} and C_{++} respectively, where $\mu_- = 0, 1, \ldots, m$. Namely,

$$C_{--} = \frac{\kappa_- + \mu_- - 1}{\kappa_- - 1} \rho_y^{\mu_-} \psi_{\kappa_+ - 1}, \qquad C_{-+} = \mu_- \, \rho_y^{\mu_- - 1} \psi_{\kappa_+ - 1},$$
$$C_{+-} = \frac{\kappa_+ - \mu_- - \kappa_-}{\kappa_+(\kappa_- - 1)} \rho_y^{\mu_- + 1} \psi_{\kappa_+ + 1}, \qquad C_{++} = \frac{\kappa_+ - \mu_- - 1}{\kappa_+} \rho_y^{\mu_-} \psi_{\kappa_+ + 1}.$$

(i) First, let us consider the case where $(\kappa_+, \kappa_-) \in \Sigma_m$ is an interior point of D_m. Note then that κ_+ and κ_- are > 1 by the assumption that $p, q \geqslant 2$. Then, one obtains $\mu_- = 1, 2, \ldots, m-1$ by (4.9). In particular, $C_{-+} \neq 0$. Now, $C_{--} = 0$ would imply $\mu_- = -\kappa_- + 1$, which contradicts $m + 3 \leqslant (p + q)/2$ as we saw above. It also follows from (4.1) that $\kappa_- - \kappa_+ + \mu_- = m - \mu_-$, and $C_{+-} \neq 0$. Finally, $C_{++} = 0$ would imply that $\kappa_+ + \kappa_- = m + 2$, which is absurd. Thus, all the coefficients in (4.3) never vanish.

(ii) Next, let us consider the case where $(\kappa_+, \kappa_-) \in \Sigma_m$ is in the boundary of D_m. Then there are three sub-cases:

(ii-a) $\mu_- = 0$,
(ii-b) $\mu_- = m$,
(ii-c) $0 < \mu_- < m$ and $k = 0$ or $l = 0$.

In Case (ii-a), $C_{-+} = 0$, and, C_{--}, C_{+-} and C_{++} are nonzero by the same reason as Case (i). In Case (ii-b), $C_{+-} = 0$ since $\kappa_+ - \kappa_- = m$, and all the other coefficients are nonzero. In Case (ii-c), all the coefficients are nonzero, but $\partial_{x_i} h_1 = 0$ or $\partial_{y_j} h_2 = 0$.

Therefore, by applying $\pi(X)$, $X \in \mathfrak{p}$, one can move from any K-type in $M^+(m)$ to any other K-type in $M^+(m)$, while $\pi(X)$, $X \in \mathfrak{k}$, preserves the K-type of each element of $M^+(m)$. This completes the proof of (2), and of the theorem. □

Example 4.1. Figure 1 below illustrates D_m in (4.11) and Σ_m in (4.12) in the case where $p = 14$, $q = 12$ and $m = 4$. The colored area and the dots sitting in the area indicates D_m and Σ_m respectively. Each K-type of $M^\pm(m)$ corresponds to a dot by the correspondence

$$\mathcal{H}^k(\mathbb{R}^p) \otimes \mathcal{H}^l(\mathbb{R}^q) \longleftrightarrow (k + p/2, l + q/2).$$

Let us apply $\pi(X^-_{i,j})$ to an element f of $M^\pm(m)$. Then, if the K-type of f corresponds to a white dot ∘ in Fig. 1, one can move to any adjacent dots in the north-east, north-west, south-east, and south-west direction; if it corresponds to a black dot • in Fig. 1, one can move only to adjacent dots in the interior or in the boundary of D_m.

Now, let us briefly recall the definitions of the Gelfand-Kirillov dimension and the Bernstein degree of a finitely generated $U(\mathfrak{g})$-module M, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. Namely, we choose a finite-dimensional subspace M_0 so that $M = U(\mathfrak{g})M_0$, and for each nonnegative integer n, we set $M_n := U_n(\mathfrak{g})M_0$, with $U_n(\mathfrak{g})$ denoting the subspace of $U(\mathfrak{g})$ spanned by products of at most n elements of $\mathfrak{g}$. Then there exists a polynomial $\psi_M(t) \in \mathbb{Q}[t]$ of degree $d - 1$ such that

$$\psi_M(n) = \dim(M_n/M_{n-1}) \quad \text{for all sufficiently large } n.$$

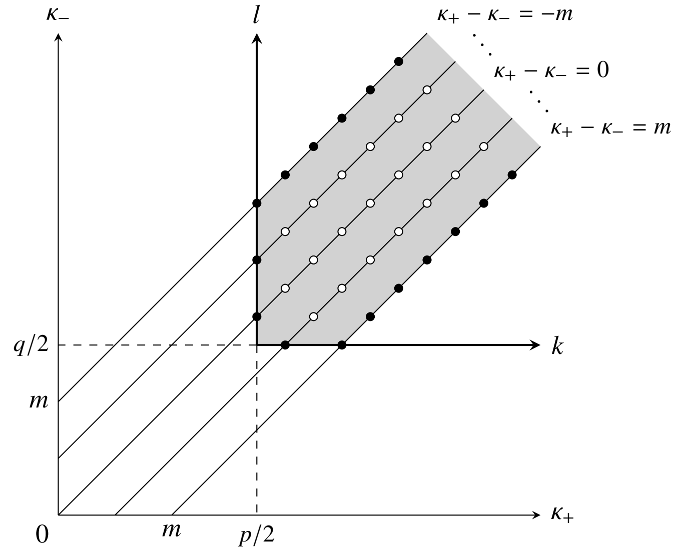

Fig. 1. Applying $\pi(X^-_{i,j})$, one can move from ∘ to dots in NE, NW, SE and SW directions, while from •, only to dots in the interior or in the boundary.

Moreover, the leading term of ψ_M is of the form

$$\frac{m}{(d-1)!}t^{d-1}$$

for a positive integer m. We call d the Gelfand-Kirillov dimension of M, and m its Bernstein degree, which we denote by $\operatorname{Dim} M$ and $\operatorname{Deg} M$ respectively (see [13] for more details).

Corollary 4.1. *If $p,q \geqslant 2, p+q \in 2\mathbb{N}$ and $m+3 \leqslant (p+q)/2$, then the Gelfand-Kirillov dimension and the Bernstein degree of $M^{\pm}(m)$ are given by*

$$\operatorname{Dim} M^{\pm}(m) = p+q-3, \tag{4.13}$$

$$\operatorname{Deg} M^{\pm}(m) = \frac{4(m+1)(p+q-4)!}{(p-2)!(q-2)!} \tag{4.14}$$

respectively.

Proof. Without loss of generality, one can assume that $p \geqslant q$. We will consider $M^+(m)$ here. Then, let $\ell(j)$ be a line in $\mathbb{R}^2$ given by

$$\ell(j) = \left\{(t_1,t_2) \in \mathbb{R}^2 \mid t_1+t_2 = j\right\}$$

with $j \in \mathbb{N}$, and set

$$c := \min\{j \in \mathbb{N} \mid \sharp(\ell(j) \cap \Sigma_m) = m+1\}. \tag{4.15}$$

As a generating (K-invariant) subspace of $M^+(m)$, we take

$$\mathsf{M}_0 := \bigoplus_{\substack{(\kappa_+,\kappa_-)\in\Sigma_m \\ \kappa_++\kappa_-\leqslant c}} \mathcal{H}^k(\mathbb{R}^p)\otimes\mathcal{H}^l(\mathbb{R}^q)\rho_y^{\mu_-}\psi_{\kappa_+}, \tag{4.16}$$

where, in each summand, μ_- is determined by $\mu_- = (1/2)(\kappa_+ - \kappa_- + m)$. If one sets $\mathsf{M}_n := U_n(\mathfrak{g})\mathsf{M}_0$ ($\mathsf{M}_{-1} := 0$), then it follows from (4.8) and (3.5) that

$$\begin{aligned}
&\dim(\mathsf{M}_n/\mathsf{M}_{n-1})\\
&= \sum_{j=0}^{m} \dim\left(\mathcal{H}^{n+j}(\mathbb{R}^p) \otimes \mathcal{H}^{n+m-j+\frac{p-q}{2}}(\mathbb{R}^q)\right)\\
&= 4\sum_{j=0}^{m} \frac{n+j+\frac{p}{2}-1}{(p-2)!}(n+j+1)(n+j+2)\cdots(n+j+p-3)\\
&\quad \times \frac{n+m-j+\frac{p-q}{2}+\frac{q}{2}-1}{(q-2)!}(n+m-j+\tfrac{p-q}{2}+1)(n+m-j+\tfrac{p-q}{2}+2)\\
&\qquad\qquad \cdots(n+m-j+\tfrac{p-q}{2}+q-3)\\
&= \frac{4(m+1)}{(p-2)!(q-2)!}\,n^{p+q-4} + (\text{lower order terms in } n) \qquad (4.17)
\end{aligned}$$

for all $n \in \mathbb{N}$, which implies (4.13). Furthermore, since the leading term of (4.17) can be rewritten as

$$\frac{4(m+1)}{(p-2)!(q-2)!}\,n^{p+q-4} = \frac{4(m+1)(p+q-4)!}{(p-2)!(q-2)!}\,\frac{n^{p+q-4}}{(p+q-4)!},$$

one obtains (4.14). This completes the proof. □

Remark 4.1. One can show that the nonnegative integer c in (4.15) is in fact equal to $\max\{m+p, m+q\}$.

It is well known that the Gelfand-Kirillov dimension of the minimal representation of $\mathrm{O}(p,q)$ is equal to $p+q-3$ (cf. [8–10, 16]). The K-type formula (4.8) of $M^+(0) = M^-(0)$ in Theorem 4.1 shows that it corresponds to the $(\mathfrak{g}, K)$-module of the minimal representation of $\mathrm{O}(p,q)$. However, as we have seen in Corollary 4.1, the Gelfand-Kirillov dimension of $M^\pm(m)$ is equal to $p+q-3$ for *any* $m \in \mathbb{N}$ satisfying $m+3 \leqslant (p+q)/2$. The Bernstein degree distinguishes the minimal representation from the others.

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